Stability data, irregular connections and tropical curves

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Motivation from spaces of stability conditions

$\mathcal{C}$ = suitable triangulated category (e.g. $D^b(X), \text{Fuk}(Y)$...).
Stab($\mathcal{C}$) = Bridgeland's space of numerical stability conditions.
Stab($\mathcal{C}$) is a complex manifold locally modelled on $\text{Hom}(\Gamma, \mathbb{C})$ for $\Gamma = K(\mathcal{C})$.

Conjecturally Stab($\mathcal{C}$) should carry much more geometric structure.

- (Almost) Frobenius manifold.
- Should parametrise families of natural irregular connections, with geometric meaning.

(E.g. T.B. "Spaces of stability conditions" Sec. 7).

BTL's model case

Model studied by T. B. and V. Toledano-Laredo:

$A =$ finite dim. $\mathbb{C}$-algebra.
$\mathcal{A} = \mathrm{Mod}^{fd}(A)$, finite length Abelian category.

Example: fd reps of finite quiver Q without loops.

$\Gamma = K(\mathcal{A}) \cong \mathbb{Z}^N$.
$\mathrm{Stab}(\mathcal{A}) =$ *positive homs* $Z\colon \Gamma \to \mathbb{C}$, $Z(K_{>0}(\mathcal{A})) \subset \mathbb{H}$ ("central charges").

$\mathrm{Stab}(\mathcal{A}) \cong \mathbb{H}^N$.

(Semi)stable reps

$M \in \mathcal{A}$ is (semi)stable if

$phase_Z(N) < (\leq) phase_Z(M)$

for nontrivial $N < M$. Here $phase_Z(M) = phase(Z([M]))$.

$\mathcal{M}^{ss}_Z(\alpha) =$ (coarse, projective) moduli space of Z-sstables in class α (A. King).

BTL connections

BTL: Stab($\mathcal{A}$) **parametrises natural family of irregular connections** on $\mathbb{P}^1$, with values in (derivations of) Ringel-Hall algebra.

$\mathcal{H}(\mathcal{A})$ = Ringel-Hall algebra:
$\mathbb{C}$-vector space spanned by constr. $f \colon \mathrm{Ob}(\mathcal{A}) \to \mathbb{C}$.
Convolution: $f * g(M) = \int_{\{0 \to B \to M \to C \to 0\}} f(B)g(C)d\chi$.

$*$ is associative, noncommutative.

$\mathcal{H}(\mathcal{A})$ is Γ-graded Lie algebra with commutator bracket.
We will mostly work in Γ-completion.
Central charges are derivations via $[Z, g] = Z(\alpha)g$ for $\deg(g) = \alpha$.

Intermediate step: stability data on $\mathcal{H}(\mathcal{A})$

$\delta_\gamma =$ char. function of class γ, Z-sstables.
$\ell \subset \mathbb{H} =$ a ray.

Group elements $S_\ell = 1 + \sum_{Z(\gamma)\in\ell} \delta_\gamma$
(char. function of semistables with phase = $phase(\ell)$).

Remark. Can write $S_\ell = \exp\left(\sum_{Z(\alpha)\in\ell} \epsilon_\alpha\right)$
(Lie algebra exp), with
$\epsilon_\alpha = \sum_n \sum_{\gamma_1+\cdots+\gamma_n=\alpha} \frac{(-1)^{n-1}}{n} \delta_{\gamma_1} * \cdots * \delta_{\gamma_n}$
(effective decompositions).

Intermediate step: stability data on $\mathcal{H}(\mathcal{A})$

Lemma (**HN recursion:** Reineke, Kontsevich-Soibelman ...).

$V \subset \mathbb{H} =$ convex sector.

Then $\prod^{\rightarrow}_{\ell \subset V} S_{\ell}(Z)$ is constant in Z, as long as no rays cross ∂V.

Remark: in Kontsevich-Soibelman terminology, $\{\epsilon_{\alpha}(Z)\}$ is a **continuous family of stability data on graded Lie algebra** $\mathcal{H}(\mathcal{A})$.

Aside: general notion of stability data

The notion of continuous families of stability data makes sense for arbitrary Γ-graded Lie algebras $\mathfrak{g}$ over $\mathbb{Q}$ (KS).

$(Z, \{\epsilon_{\alpha}(Z)\})$ is C^0 as Z varies if $\prod^{\to}_{\ell \subset V} S_{\ell}(Z)$ is constant in Z, as long as no rays cross ∂V.

So the space of pairs $(Z, \{\epsilon_{\alpha}(Z)\})$ (with extra "support condition") becomes a topological space Stab($\mathfrak{g}$).

KS: Stab($\mathfrak{g}$) is a complex manifold locally modelled on Hom($\Gamma, \mathbb{C}$).

Back to BTL connections

Theorem (BTL). There exist connections on $\mathbb{P}^1$ of the form

$$\nabla^{BTL}(Z) = d - \left(\frac{Z}{t^2} + \frac{f(Z)}{t}\right) dz$$

whose (generalised) **monodromy at** 0 **(Stokes data) is given precisely by the group elements** $S_\ell(Z)$**.**

There are **explicit formulae for residue** $f(Z) \in \mathcal{H}(\mathcal{A})$ (inner derivation).

By C^0 **property of stability data**, the $\nabla^{BTL}(Z)$ have constant (generalised) monodromy (**"isomonodromic family"**).

Remark. $\nabla^{BTL}(Z)$ have double pole at 0, simple pole at ∞.

Explicit formulae for residues

The residue $f(Z)$ of $\nabla^{BTL}(Z)$ is *positively graded*,
$f(Z) = \sum_{\alpha \in K_{>0}(\mathcal{A})} f_\alpha$
and
$f_\alpha(Z) = \sum_{n \geq 1} \sum_{\alpha_1 + \cdots + \alpha_n = \alpha} J_n(Z(\alpha_1), \ldots, Z(\alpha_n)) \epsilon_{\alpha_1} * \cdots * \epsilon_{\alpha_n}$
for certain sectionally holomorphic, **universal** special functions J_n.

$J_n =$ a sum over graphs (**trees**) of iterated integrals (**multilogarithms**).

Remark BTL prove that $f_\alpha(Z)$ coincides with Joyce's holomorphic generating functions for invariants counting semistable objects in class α.

BTL mysteries

- What is the special role of $\mathbb{P}^1$?
- Do $\nabla^{BTL}(Z)$ have some geometric content? E.g. what about their flat sections?
- How do sums over graphs (trees) arise? Do these trees have a special combinatorial/geometric meaning?

Insights from mathematical physics

C^0 stability data appear naturally in the physics of $\mathcal{N} = 2$ four-dimensional gauge theories on $\mathbb{R}^3 \times S^1$ (Gaiotto, Moore, Neitzke).

$(\Gamma, \langle -, - \rangle) =$ lattice with $\mathbb{Z}$-valued “symplectic" form.
$\mathfrak{g} =$ derivations of the Poisson algebra $C^\infty(\Gamma \otimes \mathbb{R}/\mathbb{Z}, \mathbb{C})$.

$\mathcal{B} \subset \mathrm{Hom}(\Gamma, \mathbb{C}) =$ suitable Lagrangian submanifold.

The gauge theory should provide natural C^0 map
$\mathcal{B} \to \mathrm{Stab}(\mathfrak{g})$, $Z \mapsto \{\log S_\ell(Z)\}$,
$S_\ell(Z) =$ (explicit) Poisson automorphisms of $C^\infty(\Gamma \otimes \mathbb{R}/\mathbb{Z}, \mathbb{C})$.

GMN connections

GMN construction. There should exist family of connections $\nabla^{GMN}(Z)$ on $\mathbb{P}^1$, for $Z \in \mathcal{B}$, of the form

$$\nabla^{GMN}(Z) = d - \left(\frac{\mathcal{A}^{(-1)}(Z)}{z^2} + \frac{\mathcal{A}^{(0)}(Z)}{z} + \mathcal{A}^{(1)} \right) dz$$

$\mathcal{A}^{(i)}(Z) =$ complex vector field on $\Gamma \otimes \mathbb{R}/\mathbb{Z}$, with generalized monodromy at 0 and ∞ (Stokes data) given by $\{S_\ell(Z)\}$.

In particular $\nabla^{GMN}(Z)$ would be **isomonodromic.**

GMN connections: geometric meaning

$\nabla^{GMN}(Z)$ should have basis of local flat sections $\mathcal{X}_i(z; Z)$ (fixed by $z \to 0$, $z \to \infty$ asymptotics), such that:

$\Omega(z; Z) = -\frac{1}{8\pi R}\langle -, -\rangle^{ij} d\log\mathcal{X}_i \wedge d\log\mathcal{X}_j$

is the family of holomorphic symplectic forms
for a Hyperkähler metric g on $\Gamma \otimes \mathbb{R}/\mathbb{Z}$-local system on $\mathcal{B}$.

Here $z \in \mathbb{P}^1 =$ twistor sphere.

For “theories in class $\mathcal{S}$", g should extend to Hitchin’s metric on a class of moduli spaces of meromorphic connections $\mathcal{M}$.

GMN connections: poles

$\nabla^{GMN}(Z)$ has **double poles** at 0, ∞.

Monodromy at 0 and ∞ are "complex conjugate".

There should also by symmetry $\overline{\mathcal{A}^{(-1)}}(Z) = \mathcal{A}^{(1)}(Z)$.

Our setup

$\mathfrak{g} = \mathbb{C}[\Gamma]$, the group algebra gen. by e_α with $e_\alpha e_\beta = e_{\alpha+\beta}$.

$\mathfrak{g} =$ Poisson with $[e_\alpha, e_\beta] = \langle \alpha, \beta \rangle e_{\alpha+\beta}$ (KS algebra).

$Z \mapsto \{a_\gamma(Z)\}$ any *positive* C^0 family $\mathcal{U} \to \mathrm{Stab}(\mathfrak{g})$.

$\widehat{\mathfrak{g}} =$ completion of $\mathfrak{g}^+ \subset \mathfrak{g}$.

$S_\ell(Z) = \exp(\sum_{Z(\alpha)\in\ell} a_\alpha(Z)) \in \exp(\widehat{\mathfrak{g}})$.

$\widehat{\mathfrak{g}} \subset D^*(\widehat{\mathfrak{g}}) =$ comm. algebra derivations.

Existence result

Theorem (F. G.-F. S.). There exists a family of connections $\nabla(Z)$ on $\mathbb{P}^1$, for $Z \in \mathcal{U}$, of the form

$$\nabla(Z) = d - \left(\frac{\mathcal{A}^{(-1)}(Z)}{z^2} + \frac{\mathcal{A}^{(0)}(Z)}{z} + \mathcal{A}^{(1)}\right) dz,$$

$\mathcal{A}^{(i)}(Z) \in D^*(\widehat{\mathfrak{g}})$,
with generalized monodromy at 0 and ∞ (Stokes data) given by $\{S_\ell(Z)\}$.

In particular $\nabla(Z)$ is **isomonodromic.**

Remark: proof very different from BTL and much closer to physical approach.

Aside: stability data on $\mathfrak{g}$

Examples coming from GMN setup produce positive $\mathcal{U} \to \mathrm{Stab}(\mathfrak{g})$.

For many quivers Q (ext. Dynkin, m-Kronecker...)
BTL families in $\mathcal{H}(\mathbb{C}[Q])$ produce positive $\mathcal{U} \to \mathrm{Stab}(\mathbb{C}[K(Q)])$ after **integration map** and **quasi-classical limit**.

(More generally we would only get families in $D^*(\mathbb{C}\widehat{[K(Q)}]))$.

Explicit formulae

We have explicit formulae for $\nabla(Z)$ via $\mathrm{Aut}(\widehat{\mathfrak{g}})$-valued flat sections in Stokes sectors.

$X(z; Z)e_\alpha =$
$e_\alpha \exp_* \left(z^{-1}Z(\alpha) + z\bar{Z}(\alpha) - \langle \alpha, \sum_T W_T(Z)G_T(z; Z)\rangle\right).$

$T =$ Γ-decorated graphs (trees).

$W_T(Z) =$ combinatorial weights in $\mathbb{Q}$.

$G_T(z; Z) =$ iterated integrals (resembling multilogs).

Trees and iterated integrals appear naturally by iteration of a single integral operator (as in GMN).

BTL limit

Theorem (F. G.-F. S.).

1. The BTL construction goes through for C^0 families in $\mathrm{Stab}(\mathfrak{g})$, yielding $\nabla^{BTL}_{\mathfrak{g}}(Z)$. (For ext. Dynkin etc. this coincides with “semi-classical limit" via adjoint).
2. There exists a family of gauge transformations $g(R)$, $R \in \mathbb{R}_{>0}$, such that

$$\lim_{R\to 0} g(R) \cdot \nabla(z = Rt; RZ) = \nabla^{BTL}_{\mathfrak{g}}(t; Z).$$

Thus BTL connection $\nabla^{BTL}_{\mathfrak{g}}(t; Z)$ is the fixed point of GMN type connection $\nabla(z; Z)$ under scaling limit $z \to Rt, Z \to RZ$, $R \to 0$.

Comparison with GMN

By its very costruction, $\nabla(Z)$ should compare to $\nabla^{GMN}(Z)$ under "reduction of structure algebra"

$\mathfrak{X}(\Gamma \otimes \mathbb{R}/\mathbb{Z}) \otimes \mathbb{C} \to D^*(\widehat{\mathfrak{g}})$

We check that this actually works in the main example when $\nabla^{GMN}(Z)$ is well-defined: the Ooguri-Vafa hyperkähler metrics.

Remark. The scaling limit $z \to Rt, Z \to RZ$, $R \to 0$ is called "conformal limit" in recent work of Gaiotto, where is is related to the oper submanifold in $\mathcal{M}$.

Semiflat limit

Because of the specific form

$$\nabla(Z) = d - \left(\frac{\mathcal{A}^{(-1)}(Z)}{z^2} + \frac{\mathcal{A}^{(0)}(Z)}{z} + \mathcal{A}^{(1)} \right) dz$$

it makes sense to study a different scaling limit:

$\nabla(z; RZ), R \to \infty$.

This is called the “semiflat" limit by GMN, and in the geometric context of $\nabla^{GMN}(Z)$ it is (conjecturally) mirror dual to a large complex structure limit.

Does our rough approximation $\nabla(Z)$ display some interesting behaviour in the sf limit?

Model case

$Q = \kappa$-Kronecker quiver.

$\mathrm{Stab}(\mathrm{Rep}(Q)) = \mathbb{H}^2$.

$\mathcal{U}^+ \subset \mathrm{Stab}(\mathrm{Rep}(Q)) =$ trivial chamber (only simples are stable).

$\Rightarrow$ get family $\nabla(z; Z)$ over Stab(Rep(Q)).

Fix generic $z^* \in \mathbb{C}^*$ and recall expansion
$X(z; Z)e_\alpha =$
$e_\alpha \exp_* \left(z^{-1}Z(\alpha) + z\bar{Z}(\alpha) - \langle \alpha, \sum_T W_T(Z) G_T(z; Z)\rangle\right)$.

Role of plane tropical curves

Theorem (F. G.-F. S.). As Z crosses the boundary of $\mathcal{U}^+$ from the interior, a special function $G_T(z^*; Z, R)$ appearing in the expansion for the flat section $X(z^*; Z, R)$ is replaced by a linear combination of the form

$$\sum_{T'} \pm G_{T'}(z^*; Z, R),$$

where we sum over a finite set of trees (not necessarily distinct). The terms corresponding to a single-vertex tree in the sum above are uniquely characterised by their asymptotic behaviour as $R \to \infty$. These leading order terms are in bijection with a finite set of weighted *trivalent* graphs C_i, which have a natural interpretation as combinatorial types of rational tropical curves immersed in $\mathbb{R}^2$. They come with a sign $\varepsilon(C_i(T)) = \pm 1$ determined by residue theorem.

Connection to GPS and tropical invariants

Theorem (F. G.-F. S.). The sum of contributions $\varepsilon(C_i(T)) = \pm 1$ over tropical types C_i, weighted by the coefficients W_T in the expansion for flat section in $\mathcal{U}^+$,

$$\sum_{\deg(T)=\mathbf{w}} W_T \sum_i \varepsilon(C_i(T))$$

equals a tropical invariant $N^{\mathrm{trop}}(\mathbf{w})$ enumerating plane rational tropical curves, times a simple combinatorial factor in $\Gamma \otimes \mathbb{Q}$.

Proof: based on work of Gross-Pandharipande-Siebert.

The tropical invariants $N^{\mathrm{trop}}(\mathbf{w})$ equal in fact certain relative Gromov-Witten invariants of weighted projective planes, and play a crucial role in GPS.