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D0–D6 States Counting and GW Invariants

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Abstract. We describe a correspondence between the Donaldson–Thomas invariants enumerating D0–D6 bound states on a Calabi–Yau 3-fold and certain Gromov–Witten invariants counting rational curves in a family of blowups of weighted projective planes. This is a variation on a correspondence found by Gross–Pandharipande, with D0–D6 bound states replacing representations of generalised Kronecker quivers. We build on a small part of the theories developed by Joyce–Song and Kontsevich–Soibelman for wall-crossing formulae and by Gross–Pandharipande–Siebert for factorisations in the tropical vertex group. Along the way we write down an explicit formula for the BPS state counts which arise up to rank 3 and prove their integrality. We also compare with previous “noncommutative DT invariants” computations in the physics literature.

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1. Introduction

1.1. A D0–D6/GW CORRESPONDENCE

Let X be a projective Calabi–Yau threefold with $H^1(\mathcal{O}_X) = 0$ and topological Euler characteristic χ. In this paper, we are concerned with either 0-dimensional or purely 3-dimensional (i.e. torsion free) coherent sheaves of $\mathcal{O}_X$-modules which are isomorphic to the trivial vector bundle of some rank outside a finite length subscheme. These are closely related to D0–D6 BPS bound states. We write their Chern character as

$$(a, r) := (r, 0, 0, -a) \in \bigoplus_{i=0}^{3} H^{2i}(X, \mathbb{Z}),$$

where $a = -\mathrm{ch}_3$, $r = \mathrm{ch}_0$. The key feature of these sheaves for us is that they can be “counted” in a suitable way.

For rank $r = 1$ these are the *ideal sheaves* of 0-dimensional subschemes of X, and a is the length of the subscheme. They are Gieseker stable with respect to every ample line bundle $\mathcal{O}_X(1)$ and have a fine moduli space $\mathcal{M}(a,1)\cong\mathrm{Hilb}^a(X)$ with a symmetric obstruction theory in the sense of [1]. Donaldson–Thomas theory [16] produces integral virtual counts $\#^{\mathrm{vir}}\,\mathrm{Hilb}^a(X)\in\mathbb{Z}$. The 0-dimensional Donaldson–Thomas partition function $\sum_{a\geq 1}\#^{\mathrm{vir}}\,\mathrm{Hilb}^a(X)t^a$ has been computed as $M(-t)^{\chi}$ in [1,11,12] (here $M(t)=\prod_{k\geq 1}(1-t^k)^{-k}$ is the MacMahon function, the generating series for 3-dimensional partitions), which was originally conjectured in [13].

For $r=0$ we are looking instead at direct sums of *structure sheaves* of 0-dimensional subschemes. Their Chern character is $(-a,0)$ where a is the length of the $\mathcal{O}_X$-module. Because of automorphisms their moduli space is an Artin stack $\mathcal{M}(-a,0)$ and it was not clear how to count them correctly until recently. However, as a very special case of the foundational work of Joyce and Song [9] we now have generalised Donaldson–Thomas invariants $\bar{\mathrm{DT}}(-a,0)\in\mathbb{Q}$; they are *not*, in general, the weighted Euler characteristic of the stack $\mathcal{M}(-a,0)$ with respect to its canonical Behrend function. It is shown in [9, Section 6.3] that $\bar{\mathrm{DT}}(-a,0)=-\chi\sum_{m|a}\frac{1}{m^2}$.

To generalise to higher rank r we follow closely [10, Section 6.5] and let $\mathcal{A}$ be the abelian category of coherent sheaves on X which are isomorphic to the trivial vector bundle of some rank in codimension 3 (by this we mean that they are trivial away from a codimension 3 closed subscheme. In the algebraic geometry literature these sheaves are called scheme-theoretically trivial in codimension 2; we thank B. Totaro for this observation). Notice that $\mathcal{A}$ embeds as the heart of a bounded t-structure in the triangulated category $\mathcal{D}:=\langle\mathcal{O}_X,\mathcal{O}_x : x\in X\rangle_{tr}\subset D^b(X)$ of D0–D6 states. The numerical Grothendieck group $K(\mathcal{A})$ is isomorphic to $\mathbb{Z}^2$ spanned by the classes $\mu=[\mathcal{O}_x],\gamma=[\mathcal{O}_X]$ (where $x\in X$ is any closed point). Consider the central charge $Z\colon K(\mathcal{A})\to\mathbb{C}$ given by $Z(\mu)=-1, Z(\gamma)=i$. Since Z maps the effective cone $K^+(\mathcal{A})$ into $\{\rho\exp(i\varphi):\rho>0, 0<\varphi\leq\pi\}\subset\mathbb{C}$ it determines a *stability condition* on $\mathcal{A}$ and so a Bridgeland stability condition on $\mathcal{D}$. The semistable objects $\mathcal{A}^{ss}\subset\mathcal{A}$ are in fact sheaves which are either 0-dimensional or torsion-free and isomorphic to the trivial vector bundle of some rank in codimension 3. There is an Artin stack of objects of $\mathcal{A}$ which as in [9, Section 5.1] is locally 2-isomorphic to the zero locus of the gradient of a regular function on a smooth scheme, and Z gives an admissible stability condition in the sense of [9, Section 3.2].

Joyce–Song theory then yields invariants $\bar{\mathrm{DT}}(a,r)\in\mathbb{Q}$ which count Z-semistable objects in a suitable way. We also refer to the very recent paper of Toda [17] for a number of foundational results on higher rank DT invariants in the sense of this paper. When (a,r) is a primitive class the $\bar{\mathrm{DT}}$ coincide with the DT invariants of [16]. In particular, we recover the numbers counting ideal sheaves and 0-dimensional subschemes. Notice also that one can show directly that $\bar{\mathrm{DT}}(0,r)=\frac{1}{r^2}$ and $\bar{\mathrm{DT}}(a,r)=0$ for $a=1,\dots,r-1$ (see [9, Example 6.1] and [10, Section 6.5]). Therefore, in the rest of this paper we concentrate on $\bar{\mathrm{DT}}(a,r)$ with $a\geq r$.

It is sometimes possible to compute higher rank D0–D6 numbers more or less directly, using Behrend functions. The reader can find an explicit calculation of

$\bar{\mathrm{DT}}(2,2) = -\frac{5}{4}\chi$, together with a brief introduction to Joyce–Song invariants in this context, in an appendix to the arXiv version of this paper.

A rather different take on the numbers $\bar{\mathrm{DT}}(a,r)$ is motivated by the work of Kontsevich and Soibelman [10] and Gross et al. [7]. The example of D0–D6 states is studied, in particular, in [10, Section 6.5]. According to their general theory, Kontsevich–Soibelman conjecture that one can extract *integers* Ω from the $\bar{\mathrm{DT}}$ (their underlying BPS state counts) by inverting the relation $\bar{\mathrm{DT}}(a,r) = \sum_{m\geq 1, m|(a,r)} \frac{1}{m^2}\Omega(\frac{a}{m}, \frac{r}{m})$ (i.e. by taking a Möbius transform). Moreover, they conjecture that these BPS state counts should be completely determined by a simple identity taking place in the so-called tropical vertex group, a Lie group of formal symplectomorphisms of the 2-dimensional algebraic torus.

The first main theme of this paper is a comparison of the Joyce–Song and Kontsevich–Soibelman wall-crossing formulae. We prove the KS identity for $\Omega(a,r)$ for rank $r\leq 3$ starting from the JS formula, use it to write down explicit formulae for the relevant BPS state counts and to prove their integrality in an elementary way (for a different situation in which one can show that the Joyce–Song invariants satisfy the relevant KS equation see [2]). Recently, Toda [17] also independently studied these $r=2$ DT invariants, in particular the partition function is computed in [17, Theorem 1.2] and integrality of BPS state counts is proved in loc. cit. Theorem 1.3.

The second point of view we will pursue is to regard the Kontsevich–Soibelman identity as a commutator expansion in the tropical vertex group. Gross, Pandharipande and Siebert [7] have developed a theory which interprets such commutators in the tropical vertex group in terms of genus zero Gromov–Witten invariants with a tangency condition. We will explain how "counting" (in the sense of Joyce–Song) the torsion-free sheaves on X which are isomorphic to the trivial vector bundle of some rank in codimension 3 becomes equivalent to computing the genus zero Gromov–Witten invariants (with a tangency condition) of some explicit 2-dimensional orbifolds, depending only on χ and the given K-theory class. The precise definitions and statement will be given in Section 4, but for now we express the result as follows.

The BPS state counts $\Omega(ha,hr)$ counting torsion-free sheaves with K-theory class a multiple of the primitive class given by coprime $\mathrm{ch}_0 = r$ and $\mathrm{ch}_3 = -a$, isomorphic to a trivial vector bundle in codimension 3 on a CY 3-fold with Euler characteristic χ, satisfy the identity in the ring of formal power series $\mathbb{C}[[x,y]]$

$$\prod_{h\geq 1}(1-(-1)^{h^2 ar}x^{ha}y^{hr})^{\Omega(ha,hr)} = \prod_{h\geq 1}\exp\left(\sum_{|\mathsf{P}_\chi|=ha}(-1)^{\mathsf{P}_\chi}hN[\mathsf{P}_\chi](-1)^{h(a+r)}x^{ha}y^{hr}\right), \tag{1.1}$$

where $N[\mathsf{P}_\chi]$ are Gromov–Witten invariants of a family of orbifold blowups of the toric surface given be the fan $\{(-1,0),(0,-1),(a,r)\}\subset\mathbb{R}^2$ (with some points removed and a tangency condition of order h along a smooth divisor), parameterised by a set of graded ordered partitions P_χ with length vector depending on the Euler characteristic χ (here $|\mathsf{P}_\chi|$ denotes the size and $(-1)^{\mathsf{P}_\chi}$ a certain sign attached to a graded ordered partition). In particular for coprime a, r, we find

$$\Omega(a,r) = -(-1)^{ar}(-1)^{(a+r)} \sum_{|\mathsf{P}_\chi|=a} (-1)^{\mathsf{P}_\chi} N[\mathsf{P}_\chi]. \tag{1.2}$$

The base toric surface is the weighted projective plane $\mathbb{P}(a,r,1)$ with some points removed. The index $h = \gcd(-\frac{1}{2}\mathrm{ch}_3, \mathrm{ch}_0)$ for sheaves corresponds to the order of tangency for holomorphic curves along the divisor D_{out} dual to (a,r). This result should be compared with the analogous formulae found by Gross and Pandharipande [6, Corollary 3], building on [7] and the work of Reineke [15]. In essence Gross–Pandharipande show that the Euler characteristics of the moduli spaces for semistable representations of the m-Kronecker quiver (with a suitable stability condition and framing) can be computed in terms of GW theory by a formula similar to (1.1). The above correspondence says that, in a different region of the tropical vertex group, m-Kronecker quiver representations are replaced by D0–D6 states. Notice that a physical context involving both the m-Kronecker quivers and D0–D6 states is described in [4, Sections 3 and 5]. Geometrically, starting with the same base orbifold $\mathbb{P}(a,r,1)$, the invariants for representations of the m-Kronecker quiver (with dimension vector proportional to (a,r)) are recovered for ordinary blowups along the divisors D_1, D_2 dual to $(-1,0),(0,-1)$, parametrised by suitable partitions $\mathsf{p}_1, \mathsf{p}_2$ of size proportional to a, r and length m. For D0–D6 states we blow up only once along D_2, and we also perform an ordinary blowup of D_1 along a partition; but the crucial difference is that now there are also higher order corrections, or more precisely *orbifold* blowups in addition to ordinary ones, and, in turn, these are parameterised by graded ordered partitions P_χ with length vector $(\chi, 2\chi, 3\chi \ldots)$ [so that the topological Euler characteristic χ plays the same role as the order of the quiver m (here we assume $\chi \geq 0$; for $\chi < 0$ there is a slight modification described in Section 4)].

Remark. We emphasise that we do *not* need to give an ad hoc proof that the numbers $N[\mathsf{P}_\chi]$ are given by suitable Gromov–Witten invariants in our case. What we will do is to show that the Kontsevich–Soibelman wall-crossing identity for the numbers $\Omega(a',r')$ (equation (1.4) below) can be recast precisely in the form covered by the general results of Gross–Pandharipande–Siebert, in particular their "full commutator formula", Theorem 4.1 below. As we will explain in Section 4, in principle, their theory allows to expand *every* such commutator in terms of Gromov–Witten invariants of orbifold blowups of weighted projective planes; in our special case, this becomes feasible essentially because of the Euler product formula for the MacMahon function.

1.2. COMPARISON WITH A PHYSICS RESULT

Even before their rigorous definition by Joyce–Song, Cirafici et al. [3] have addressed the problem of computing the punctual invariants $\mathrm{DT}(a,r)$ supported at the origin of the affine Calabi–Yau $\mathbb{C}^3$. However, as they explain in ibid (Section 7.2), their gauge-theoretic approach based on noncommutative deformation and localisation is only valid in the Coulomb phase, where they compute the partition function simply as $M((-1)^r t)^r$. As we will see this is very different from the result one would get by setting $\chi = 1$ in the correspondence (1.1). We will argue that the above result in the Coulomb phase (i.e. passing from nonabelian gauge group $U(r)$ to the abelian $U(1)^r$) can be recovered, up to a factor $\frac{1}{r^2}$, by "pretending" that certain operators associated with ranks $r \geq 1$ commute; this is explained more precisely in Section 2.2.

1.3. IDENTITY IN THE TROPICAL VERTEX GROUP

We follow the notation of [7]. The tropical vertex group G is a closed subgroup of $\mathrm{Aut}_{\mathbb{C}[[t]]}(\mathbb{C}[x,x^{-1},y,y^{-1}][[t]])$ in the (t)-adic topology. It is the (t)-adic completion of the subgroup generated by the automorphisms of the form

$$\theta_{(a,r),f}(x) = f^{-r} \cdot x, \quad \theta_{(a,r),f}(y) = f^{a} \cdot y$$

with $(a,r) \in \mathbb{Z}^2$ and f a formal power series in t of the form

$$f = 1 + tx^a y^r \cdot g(x^a y^r, t), \quad g(z,t) \in \mathbb{C}[z][[t]].$$

Alternatively, one can see G as a subgroup of the group of formal 1-parameter families of automorphisms of the algebraic 2-torus $\mathbb{C}^* \times \mathbb{C}^*$; by direct computation G preserves the standard holomorphic symplectic form $\frac{dx}{x} \wedge \frac{dy}{y}$.

A basic feature of G is that two elements $\theta_{(a,r),f}, \theta_{(a',r'),f'}$ with (a',r') a multiple of (a,r) commute.

The group G contains some special elements

$$T_{a,r} = \theta_{(a,r),1-(-1)^{ar}(tx)^a(ty)^r},$$

and for $\Omega \in \mathbb{Q}$ we define

$$T^{\Omega}_{a,r}(x,y) = \theta_{(a,r),(1-(-1)^{ar}(tx)^a(ty)^r)^{\Omega}}.$$

The notation makes sense from the point of view of Lie groups, since

$$T_{a,b} = \theta_{(a,r),f} = \exp(\log(f)\partial)$$

for $f = 1-(-1)^{ar}(tx)^a(ty)^r$ and some $\partial \in \mathbb{Z}dx \oplus \mathbb{Z}dy$ so $T^{\Omega}_{a,r}$ corresponds to $\exp(\Omega \log(f)\partial) = \exp(\log(f^{\Omega})\partial) = \theta_{(a,r),f^{\Omega}}$ (see [7, Section 1.1] and [8] for the general setup). Notice that in particular

$$(T_{a,r})^{-1} = T^{-1}_{a,r} = \theta_{(a,r),(1-(-1)^{ar}(tx)^a(ty)^r)^{-1}}.$$

By a fundamental result of Kontsevich–Soibelman every automorphism in the tropical vertex group admits a unique ordered product expansion

$$g = \prod^{\rightarrow}_{(a,r)\in\mathbb{Z}^2_+} T_{a,r}^{\Omega(a,r)}, \tag{1.3}$$

where $\mathbb{Z}^2_+ \subset \mathbb{Z}^2$ means $\{a, r \geq 0\} \setminus \{0\}$ (for a precise definition of the ordered product $\prod^{\rightarrow}$ see [10, Section 2.2] and for the proof of an equivalent statement see, e.g. [7, Theorem 1.4]).

Let us now go back to the category $\mathcal{A}$. Kontsevich–Soibelman introduce BPS invariants associated to the $\bar{\mathrm{DT}}$ as their Möbius transform

$$\Omega(a,r) = \sum_{m\geq 1, m|(a,r)} \frac{\mu(m)}{m^2} \bar{\mathrm{DT}}\left(\frac{a}{m}, \frac{r}{m}\right),$$

where $\mu(m)$ is the Möbius function (with $\mu(1) = 1, \mu(2) = -1, \mu(3) = -1, \ldots$).

For rank one we get simply $\Omega(a,1) = \bar{\mathrm{DT}}(a,1)$ for $a \geq 1$ since these classes are primitive. On the other hand, one can compute $\Omega(-a,0) = -\chi$ for $a \geq 1$, see [9, Section 6.3]. Another example is $\Omega(2,2) = -\chi$ and can be found in appendix to the arXiv version of this paper.

In [10, Section 6.5], Kontsevich–Soibelman write down an identity in the tropical vertex group which should be satisfied by the BPS invariants Ω, namely

$$\prod_{a\geq 1} T_{a,0}^{-\chi} \cdot T_{0,1} = \prod^{\rightarrow}_{a\geq 0, r\geq 1} T_{a,r}^{\Omega(a,r)} \cdot \prod_{a\geq 1} T_{a,0}^{-\chi}. \tag{1.4}$$

According to the factorisation theorem recalled above this formula would determine the $\Omega(a,r)$ uniquely.

Let us explain the origin of the formula (1.4), referring to loc. cit. for a detailed discussion (see also [17, Section 2]). Kontsevich–Soibelman [10] propose an alternative approach to generalised Donaldson–Thomas invariants counting semistable objects in suitable triangulated categories with respect to a Bridgeland stability condition. In particular, they propose universal formulae on how the BPS invariants change as the stability condition moves in the space *Stab*. Locally these remain constant, but there are *walls* in *Stab* on crossing which the Ω change according to formulae of the form of (1.4). This theory is still conjectural in parts. However, we can apply it formally to $\mathcal{A}$. For this embed $\mathcal{A}$ as the heart of a t-structure in the triangulated category $\mathcal{D} := \langle \mathcal{O}_X, \mathcal{O}_x : x \in X\rangle_{tr} \subset D^b(X)$ of D0–D6 states, as before. The slope function Z becomes a central charge on $\mathcal{D}$ defining a Bridgeland stability condition, and the Joyce–Song BPS invariants Ω conjecturally coincide with the Kontsevich–Soibelman invariants counting Z-semistable objects with phases in some fixed sector. Kontsevich–Soibelman deform Z by prescribing $Z_\tau([\mathcal{O}_X]) = i$, $Z_\tau([\mathcal{O}_x]) = \exp(i\tau\pi)$ for $\tau \in [1, \frac{3}{2} + \epsilon)$ for sufficiently small $\epsilon > 0$. As soon as $t > 1$ the heart $\mathcal{A}$ jumps to $\mathcal{A}' := \langle \mathcal{O}_X, \mathrm{Coh}_0(X)[-1]\rangle_{ext}$, the intersection of the tilting of $\mathcal{A}$ inside $D^b(X)$ with respect to 0-dimensional sheaves with the subcategory $\mathcal{D}$ (i.e. $-[\mathcal{O}_x]$ becomes effective in $K(\mathcal{A}')$ as it is the class of $\mathcal{O}_x[-1]$). The BPS invariants, however, remain unchanged until $\tau \geq \frac{3}{2}$. For $\tau > \frac{3}{2}$ the heart remains the same, $\mathcal{A}'$, but now the only semistable objects have unmixed classes which are multiples of either $[\mathcal{O}_X]$ or $-[\mathcal{O}_x]$. The latter objects have BPS invariants $-\chi$ as we already discussed. By assumption $\mathcal{O}_X$ is rigid and so according to [9, Section 6.1], we have $\Omega(r\mathcal{O}_X) = \delta_{r,1}$. The equality of factorisations (1.4) then becomes the Kontsevich–Soibelman wall-crossing formula for this particular wall-crossing.

1.4. THE TROPICAL VERTEX FOR GW INVARIANTS

Clearly, the wall-crossing formula (1.4) can be rewritten as an expansion for a commutator in the group G,

$$(T_{0,1})^{-1} \cdot \left(\prod_{a\geq 1} T_{a,0}^{-\chi} \right) \cdot T_{0,1} \cdot \left(\prod_{a\geq 1} T_{a,0}^{-\chi} \right)^{-1} = \prod^{\rightarrow}_{a\geq 1, r\geq 1} T_{a,r}^{\Omega(a,r)}.$$

Using the definition of $T_{a,0}$ and the well-known product formula for the MacMahon function $M(x) = \prod_{a\geq 1}(1-x^a)^{-a}$ one can check that this is equivalent to

$$(\theta_{0,1})^{-1} \cdot \theta_{(1,0),M(-tx)^{\chi}} \cdot \theta_{0,1} \cdot \theta_{(1,0),M(-tx)^{-\chi}} = \prod^{\rightarrow}_{a\geq 1, r\geq 1, \gcd(a,r)=1} \theta_{(a,r),f_{(a,r)}} \tag{1.5}$$

with

$$f_{(a,r)} = \prod_{k\geq 1} (1-(-1)^{k^2ar}(tx)^{ka}(ty)^{kr})^{\Omega(ka,kr)}.$$

This is precisely the kind of commutator expansions studied in [7] by Gross–Pandharipande–Siebert. They have shown that for the ordered product factorisation

$$[\theta_{(a',r'),f'}, \theta_{(a'',r''),f''}] = \prod^{\rightarrow}_{a\geq 1, r\geq 1, \gcd(a,r)=1} \theta_{(a,r),f_{(a,r)}}$$

of the commutator of two generators of G one can write the coefficients of the power series $\log f_{(a,r)}$ in terms of the GW invariants of orbifold blowups of a toric surface $X_{a,r}$ (with a tangency condition). We will describe explicitly how this result applies to our case, relating D0–D6 states to GW invariants of orbifolds.

1.5. SYMMETRY

Explicit computation with the rank 2 and 3 formulae for Ω which we establish in Section 2 suggests the identity

$$\Omega(a, a-i) = \Omega(a, i) \quad \text{for } i = 1, \ldots, a$$

(so, in particular, $\Omega(a,a)=-\chi$ for $a \geq 1$). One can in fact prove this symmetry using GW invariants and the correspondence (1.1). Indeed the method of [6, Section 5] (curve-counting symmetries induced by elementary transformations of ruled surfaces) yields an equality

$$f_{(a,r)}|_{ty=-1} = f_{(a,a-r)}|_{ty=-1}$$

($\gcd(a,r)=1$). A direct proof at the level of sheaves is also possible and has been suggested to the author by Y. Toda. It is proved in [17, Proposition 2.2] that the tilted category $\mathcal{A}'$ above is equivalent to the category of two-term complexes $[\mathcal{O}_X^{\oplus r} \xrightarrow{s} F]$ for a 0-dimensional sheaf F (say with length a). Each such complex has a dual $[(H^0(F)/s(\mathcal{O}_X^{\oplus r}))^* \otimes \mathcal{O}_X \longrightarrow \mathcal{E}xt^3_{\mathcal{O}_X}(F, \mathcal{O}_X)]$. The vector space $H^0(F)/s(\mathcal{O}_X^{\oplus r})$ has dimension $a-r$, while $\mathcal{E}xt^3_{\mathcal{O}_X}(F, \mathcal{O}_X)$ is 0-dimensional with length a. A complex in $\mathcal{A}'$ is (semi)stable if and only if its dual is inducing the equality $\Omega(a,r)=\Omega(a,a-r)$ for $a>r$. There are analogous symmetries in the GW/Kronecker quivers correspondence given by the reflection functors of [6, Section 5.3].

1.6. PLAN OF THE PAPER

In Sections 2 and 3 we show that the Joyce–Song invariants satisfy the relevant Kontsevich–Soibelman identities for rank up to 3. Section 2 also contains explicit formulae for the BPS state counts which arise up to rank $r=3$ and an elementary proof of their integrality (as we mentioned analogous $r=2$ formulae and integrality results have been found by Toda [17]). Finally in Section 4, we briefly review the theory of Gross–Pandharipande–Siebert and apply it to our special case, thus obtaining the required formulae for D0–D6 states counts in terms of GW invariants of orbifold blowups of weighted projective planes.

After the first appearance of this and Toda's papers, Nagao [14] has proved the integrality of D0–D6 BPS counts, in general, using representations of certain bipartite quivers.

2. Kontsevich–Soibelman Side

In this section, we use a small part of the theory developed by Kontsevich and Soibelman [10, Sections 1–6]. An introduction from a physical perspective can be found in [5, Section 2].

2.1. BAKER–CAMPBELL–HAUSDORFF FORMULA

Write Γ for the lattice $\mathbb{Z}^2$ with basis $\gamma=(0,1), \mu=(1,0)$ and anti-symmetric bilinear form $\langle\gamma,\mu\rangle=-1$ (in the categorical picture above this corresponds to $\gamma=$

$[\mathcal{O}_X], \mu = [\mathcal{O}_x[-1]]$). The positive cone $\Gamma_+ \subset \Gamma$ is given by those elements with non-negative components $(a, r), a + r \geq 1$.

Consider the Γ_+-graded Lie algebra $\mathfrak{g}$ generated over $\mathbb{C}$ by symbols $e_\eta, \eta \in \Gamma_+$ with bracket

$$[e_\xi, e_\eta] = (-1)^{\langle \xi, \eta \rangle} \langle \xi, \eta \rangle e_{\xi+\eta}. \tag{2.1}$$

Then writing $\eta = (a, r)$ for an element of Γ there is a natural identification

$$T_{(a,r)} = T_\eta = \exp\left(-\sum_{n \geq 1} \frac{e_{n\eta}}{n^2} \right) \tag{2.2}$$

seeing the automorphism T_η as an element of the exponential of the completion of $\mathfrak{g}$ (see [10, Section 1.4] for this identification, and notice that here we are replacing the t-grading with the finer Γ_+-grading). We rewrite the KS formula (1.4) as

$$\prod_{n \geq 1} T_{n\mu}^{-\chi} \cdot T_\gamma \cdot \left(\prod_{n \geq 1} T_{n\mu}^{-\chi} \right)^{-1} = \prod_{n \geq 0, r \geq 1}^{\rightarrow} T_{n\mu + r\gamma}^{\Omega(n\mu + r\gamma)}. \tag{2.3}$$

Let us define elements in the completion

$$A = \chi \sum_{n \geq 1} \sum_{i \geq 1} \frac{e_{in\mu}}{i^2}, \quad B = -\sum_{j \geq 1} \frac{e_{j\gamma}}{j^2}.$$

In what follows we will denote the left and right hand sides of (2.3) simply by lhs, rhs. Using repeatedly (2.1) and (2.2) the left hand side of (2.3) can be rewritten as

$$\mathrm{lhs} = \exp(A) \exp(B) \exp(-A).$$

We will use the following form of the Baker–Campbell–Hausdorff (BCH) formula,

$$\begin{aligned} \exp(A) \exp(B) \exp(-A) &= \exp\left(B + \sum_{k \geq 1} \frac{\mathrm{Ad}_A^k(B)}{k!} \right) \\ &= \exp\left(B + [A, B] + \frac{1}{2}[A, [A, B]] + \cdots\right). \end{aligned} \tag{2.4}$$

Let us write n, i for multi-indexes of length $k \geq 1$ with integer entries $n_l, i_l \geq 1$, and $\mathrm{n} \cdot \mathrm{i} = \sum_{l=1}^k n_l i_l$ for their ordinary scalar product. For $k \geq 1$ we can compute

$$\mathrm{Ad}_A^k(B) = -\chi^k \sum_{\mathrm{n},\mathrm{i}} \sum_{j \geq 1} (-1)^{j\,\mathrm{n}\cdot\mathrm{i}} j^{k-2} \frac{\prod n_l}{\prod i_l} e_{\mathrm{n}\cdot\mathrm{i}\mu + j\gamma}.$$

Thus, we find

$$\log(\mathrm{lhs}) = -\sum_{j \geq 1} \frac{e_{j\gamma}}{j^2} - \sum_{k \geq 1} \frac{\chi^k}{k!} \sum_{\mathrm{len(n)} = \mathrm{len(i)} = k} \sum_{j \geq 1} (-1)^{j\,\mathrm{n}\cdot\mathrm{i}} j^{k-2} \frac{\prod n_l}{\prod i_l} e_{\mathrm{n}\cdot\mathrm{i}\mu + j\gamma}. \tag{2.5}$$

2.2. RANK $r = 1$

Consider the subspace $\mathfrak{g}_{>m}$ of $\mathfrak{g}$ generated by e_η with $\langle \mu, \eta \rangle > m$. By (2.1) this is an ideal $\mathfrak{g}_{>m} < \mathfrak{g}$, so, in particular, we can form the quotient Lie algebra $\mathfrak{g}/\mathfrak{g}_{>1}$. The right hand side rhs of (2.3) can be projected via

$$\pi_{\leq 1} \colon \exp(\mathfrak{g}) \to \exp(\mathfrak{g}/\mathfrak{g}_{>1})$$

taking the form

$$\pi_{\leq 1}(\mathrm{rhs}) = \pi_{\leq 1}(T_{0,1}) \prod_{a \geq 1} \pi_{\leq 1}(T_{a,1}^{\Omega(a,1)}).$$

Now

$$\pi_{\leq 1}(T_{0,1}) = \exp(-e_\gamma), \quad \pi_{\leq 1}(T_{a,1}^{\Omega(a,1)}) = \exp(-\Omega(a,1) e_{a\mu+\gamma}),$$

and *in the quotient* we have $[e_{a\mu+\gamma}, e_{a'\mu+\gamma}] = 0$, so

$$\log(\mathrm{rhs}) = -e_\gamma - \sum_{a \geq 1} \Omega(a,1) e_{a\mu+\gamma}.$$

Comparing with the left hand side gives the rank $r = 1$ KS formula

$$\Omega(a,1) = (-1)^a \sum_{k \geq 1} \frac{\chi^k}{k!} \sum_{\mathrm{len(n)=len(i)}=k, \mathrm{n} \cdot \mathrm{i} = a} \frac{\prod n_l}{\prod i_l}. \tag{2.6}$$

These are the usual 0-dimensional DT invariants, but this particular way to represent them turns out to be very useful for the generalisation to higher rank.

Remark. The $r = 1$ formula thus gives

$$\sum_{a \geq 0} t^a (-1)^a \sum_{k \geq 0} \frac{\chi^k}{k!} \sum_{\mathrm{len(n)=len(i)}=k, \mathrm{n} \cdot \mathrm{i} = a} \frac{\prod n_l}{\prod i_l} = M(-t)^\chi = \prod_{k \geq 1} (1 - (-t)^k)^{-\chi k}.$$

In general, comparing with (2.5) above we find

$$\sum_{a \geq 0} t^a (-1)^{ra} \sum_{k \geq 0} r^{k-2} \frac{\chi^k}{k!} \sum_{\mathrm{len(n)=len(i)}=k, \mathrm{n} \cdot \mathrm{i} = a} \frac{\prod n_l}{\prod i_l} = \frac{1}{r^2} M((-1)^r t)^{r\chi}. \tag{2.7}$$

In the light of (2.3) we see that if we could assume formally that the operators $T_{a,r}$ for $r \geq 1$ commute (but retaining, crucially, that they do *not* commute with rank 0 operators) we would get the partition function $\frac{1}{r^2} M((-1)^r t)^{r\chi}$ for rank r DT invariants. We believe that this is related to the physical computation in [3] mentioned in the introduction; in particular, assuming that the $T_{a,r}$ commute for $r \geq 1$ should be similar to the breaking of the gauge group from $U(r)$ to $U(1)^r$ (which only makes sense for $r \geq 1$).

Remark. In a previous version of this paper it was stated erroneously that $\frac{1}{r^2}M((-1)^r t)^{r\chi}$ is also the partition function for DT invariants on the wall (say DT^0) where the phases of $\mathcal{O}_X, \mathcal{O}_x[-1]$ coincide. This is wrong by a factor of $\left(\prod_{n\geq 1} T_{n\mu}^{-\chi}\right)^{-1}$ in the wall-crossing formula, as pointed out by Y. Toda. Indeed when the phases of $\mathcal{O}_X, \mathcal{O}_x[-1]$ coincide there is a single Kontsevich–Soibelman operator collecting the BPS contributions of all the semistable objects (which all have the same phase), $\exp(-\sum_\alpha \Omega^0(\alpha)e_\alpha)$. According to the general theory of [10, Section 2.3], and, in particular, the identity at the bottom of page 27, the relevant wall-crossing (or rather wall-hitting) identity becomes

$$\prod_{n\geq 1} T_{n\mu}^{-\chi}\cdot T_\gamma = \exp\left(-\sum_\alpha \Omega^0(\alpha)e_\alpha\right).$$

The underlying BPS invariants Ω^0 are not expected to be integers since the degenerate stability condition is not generic in the sense of [9] Section 1.4.

Computing the first few terms in the BCH expansion we find, for example $\mathrm{DT}^0(1,1) = -\frac{\chi}{2}$, $\mathrm{DT}^0(2,1) = \frac{5}{4}\chi + \frac{1}{12}\chi^2$, $\mathrm{DT}^0(3,1) = -\frac{10}{6}\chi - \frac{5}{12}\chi^2$. As we will briefly remark later, these results are also in perfect agreement with the Joyce–Song formula.

2.3. RANK $r=2$

Similarly, we can work out a formula for $\Omega(a,2)$. Let us consider the quotient $\mathfrak{g}/\mathfrak{g}_{>2}$ with projection $\pi_{\leq 2}\colon \exp(\mathfrak{g}) \to \exp(\mathfrak{g}/\mathfrak{g}_{>2})$; the projection of the right hand side is

$$\pi_{\leq 2}(\mathrm{rhs}) = \pi_{\leq 2}(T_{0,1}) \overrightarrow{\prod_{a\geq 1, 1\leq r\leq 2}} \pi_{\leq 2}(T_{(a,r)}^{\Omega(a,r)}).$$

Explicitly,

$$\pi_{\leq 2}(T_{0,1}^{\Omega(0,1)}) = \exp\left(-e_\gamma - \frac{1}{4}e_{2\gamma}\right),$$
$$\pi_{\leq 2}(T_{a,1}^{\Omega(a,1)}) = \exp\left(-\Omega(a,1)\left(e_{a\mu+\gamma} + \frac{1}{4}e_{2a\mu+2\gamma}\right)\right),$$
$$\pi_{\leq 2}(T_{a,2}^{\Omega(a,2)}) = \exp(-\Omega(a,2)e_{a\mu+2\gamma}).$$

Since by (2.1) $[\mathfrak{g}/\mathfrak{g}_{>2}, [\mathfrak{g}/\mathfrak{g}_{>2}, \mathfrak{g}/\mathfrak{g}_{>2}]] = 0$ the BCH formula gives

$$\log(\mathrm{rhs}) = -e_\gamma - \frac{1}{4}e_{2\gamma} - \sum_{a\geq 1}\Omega(a,1)(e_{a\mu+\gamma} + \frac{1}{4}e_{2a\mu+2\gamma}) - \sum_{a\geq 1}\Omega(a,2)(e_{a\mu+2\gamma})$$
$$+\frac{1}{2}\sum_{a'<a''}(-1)^{a'-a''}(a'-a'')\Omega(a',1)\Omega(a'',1)e_{(a'+a'')\mu+2\gamma}.$$

When a is odd (i.e. in the primitive case) comparing with lhs gives

$$\begin{aligned}\Omega(a,2)=&\sum_{k\geq 1}2^{k-2}\frac{\chi^k}{k!}\sum_{\mathrm{len(n)=len(i)}=k,\mathrm{n}\cdot\mathrm{i}=a}\frac{\prod n_l}{\prod i_l}\\&+\frac{(-1)^a}{2}\sum_{a'<a'',a'+a''=a}(a'-a'')\Omega(a',1)\Omega(a'',1),\end{aligned}\tag{2.8}$$

while for $a>0$ and even there is an additional term,

$$\begin{aligned}\Omega(a,2)=&\sum_{k\geq 1}2^{k-2}\frac{\chi^k}{k!}\sum_{\mathrm{len(n)=len(i)}=k,\mathrm{n}\cdot\mathrm{i}=a}\frac{\prod n_l}{\prod i_l}\\&+\frac{(-1)^a}{2}\sum_{a'<a'',a'+a''=a}(a'-a'')\Omega(a',1)\Omega(a'',1)\\&-\frac{1}{4}\Omega(a/2,1).\end{aligned}\tag{2.9}$$

According to the definition of BPS invariants in each case we get

$$\begin{aligned}\bar{\mathrm{DT}}(a,2)=&\sum_{k\geq 1}2^{k-2}\frac{\chi^k}{k!}\sum_{\mathrm{len(n)=len(i)}=k,\mathrm{n}\cdot\mathrm{i}=a}\frac{\prod n_l}{\prod i_l}\\&+\frac{(-1)^a}{2}\sum_{a'<a'',a'+a''=a}(a'-a'')\Omega(a',1)\Omega(a'',1).\end{aligned}\tag{2.10}$$

EXAMPLE. The first few terms of the partition function for rank $r=2$ and $\chi=1$ BPS states are

$$Z_r^{\mathrm{BPS}}(t)=-t^2(1+6t+21t^2+61t^3+165t^4+426t^5+\cdots)$$

The physics result from [3] gives instead

$$Z_{r,\mathrm{Coulomb}}^{\mathrm{BPS}}(t)=1-2t+7t^2-18t^3+47t^4-110t^5+258t^6-568t^7+\cdots$$

2.4. RANK $r=3$

Under the projection $\pi_{\leq 3}\colon \exp(\mathfrak{g})\to\exp(\mathfrak{g}/\mathfrak{g}_{>2})$ we find

$$\pi_{\leq 3}(\mathrm{rhs})=\pi_{\leq 3}(T_{0,1})\prod_{a\geq 1,1\leq r\leq 3}^{\rightarrow}\pi_{\leq 3}(T_{(a,r)}^{\Omega(a,r)})$$

with

$$\begin{aligned}
\pi_{\leq 3}(T_{0,1}^{\Omega(0,1)}) &= \exp\left(-e_{\gamma} - \frac{1}{4}e_{2\gamma} - \frac{1}{9}e_{3\gamma}\right), \\
\pi_{\leq 3}(T_{a,1}^{\Omega(a,1)}) &= \exp\left(-\Omega(a,1)\left(e_{a\mu+\gamma} + \frac{1}{4}e_{2a\mu+2\gamma} + \frac{1}{9}e_{3a\mu+3\gamma}\right)\right), \\
\pi_{\leq 3}(T_{a,2}^{\Omega(a,2)}) &= \exp(-\Omega(a,2)e_{a\mu+2\gamma}) \\
\pi_{\leq 3}(T_{a,3}^{\Omega(a,3)}) &= \exp(-\Omega(a,3)e_{a\mu+3\gamma}).
\end{aligned}$$

We can compute which terms x with $\langle \mu, x\rangle = 3$ appear in log(rhs) in the Lie algebra $\mathfrak{g}/\mathfrak{g}_{>3}$. These terms have a different form according to an ordered partition for the rank $r=3$, namely $3, 2+1, 1+2, 1+1+1$, corresponding to the order of the Lie brackets involved. The type 3 term is

$$-\frac{1}{9}\sum_{a\geq 0}\Omega(a,1)e_{3a\mu+3\gamma} - \sum_{a\geq 1}\Omega(a,3)e_{a\mu+3\gamma}.$$

The type $2+1$ comprises

$$\begin{aligned}
&\frac{1}{2}\sum_{a_1<a_2}\left[-\frac{1}{4}\Omega(a_1,1)e_{2a_1\mu+2\gamma}, -\Omega(a_2,1)e_{a_2\mu+\gamma}\right] \\
&\quad = \frac{1}{4}\sum_{a_1<a_2}(a_1-a_2)\Omega(a_1,1)\Omega(a_2,1)e_{(2a_1+a_2)\mu+3\gamma}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2}\sum_{a_1<2a_2}[-\Omega(a_1,2)e_{a_1\mu+2\gamma}, -\Omega(a_2,1)e_{a_2\mu+\gamma}] \\
&\quad = \frac{1}{2}\sum_{a_1<2a_2}(-1)^{a_1-2a_2}(a_1-2a_2)\Omega(a_1,2)\Omega(a_2,1)e_{(a_1+a_2)\mu+3\gamma}.
\end{aligned}$$

Similarly for type $1+2$ there are terms

$$\begin{aligned}
&\frac{1}{2}\sum_{a_1<a_2}\left[-\Omega(a_1,1)e_{a_1\mu+\gamma}, -\frac{1}{4}\Omega(a_2,1)e_{2a_2\mu+2\gamma}\right] \\
&\quad = \frac{1}{4}\sum_{a_1<a_2}(a_1-a_2)\Omega(a_1,1)\Omega(a_2,1)e_{(a_1+2a_2)\mu+3\gamma}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2}\sum_{2a_1<a_2}[-\Omega(a_1,1)e_{a_1\mu+\gamma}, -\Omega(a_2,2)e_{a_2\mu+2\gamma}] \\
&\quad = \frac{1}{2}\sum_{2a_1<a_2}(-1)^{2a_1-a_2}(2a_1-a_2)\Omega(a_1,1)\Omega(a_2,2)e_{(a_1+a_2)\mu+3\gamma}.
\end{aligned}$$

For the type $1+1+1$ term recall the BCH formula up to order 3 Lie brackets,

$$\log(\exp(X)\exp(Y)) = X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}([X,[X,Y]]+[Y,[Y,X]]),$$

and applying this inductively to the ordered product factorisation we find contributions

$$\begin{aligned}&\frac{1}{4}\sum_{a_1<a_2<a_3}[[-\Omega(a_1,1)e_{a_1\mu+\gamma},-\Omega(a_2,1)e_{a_2\mu+\gamma}],-\Omega(a_3,1)e_{a_3\mu+\gamma}]\\ &\quad=-\frac{1}{4}\sum_{a_1<a_2<a_3}(-1)^{a_1-a_2}(-1)^{a_1+a_2-2a_3}(a_1-a_2)(a_1+a_2-2a_3)\\ &\qquad\times\Omega(a_1,1)\Omega(a_2,1)\Omega(a_3,1)e_{(a_1+a_2+a_3)\mu+3\gamma}\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{12}\sum_{a_1<a_3,a_2<a_3}[-\Omega(a_1,1)e_{a_1\mu+\gamma},[-\Omega(a_2,1)e_{a_2\mu+\gamma},-\Omega(a_3,1)e_{a_3\mu+\gamma}]]\\ &\quad=-\frac{1}{12}\sum_{a_1<a_3,a_2<a_3}(a_2-a_3)(2a_1-a_2-a_3)\Omega(a_1,1)\Omega(a_2,1)\Omega(a_3,1)\\ &\qquad\times e_{(a_1+a_2+a_3)\mu+3\gamma},\\ &\frac{1}{12}\sum_{a_2<a_1}[-\Omega(a_2,1)e_{a_2\mu+\gamma},[-\Omega(a_2,1)e_{a_2\mu+\gamma},-\Omega(a_1,1)e_{a_1\mu+\gamma}]]\\ &\quad=-\frac{1}{12}\sum_{a_2<a_1}(a_1-a_2)^2(\Omega(a_1,1))^2\Omega(a_2,1)e_{(2a_1+a_2)\mu+3\gamma}.\end{aligned}$$

Summing over the previous terms we find the lengthy $r=3$ KS identity

$$\begin{aligned}\Omega(a,3)=(-1)^a\sum_{k\geq 1}3^{k-2}\frac{\chi^k}{k!}&\sum_{\mathrm{len(n)=len(i)}=k,\mathrm{n}\cdot\mathrm{i}=a}\frac{\prod n_l}{\prod i_l}\\ &+\frac{1}{4}\sum_{a_1<a_2,2a_1+a_2=a}(a_1-a_2)\Omega(a_1,1)\Omega(a_2,1)\\ &+\frac{1}{2}\sum_{a_1<2a_2,a_1+a_2=a}(-1)^{a_1-2a_2}(a_1-2a_2)\Omega(a_1,2)\Omega(a_2,1)\\ &+\frac{1}{4}\sum_{a_1<a_2,a_1+2a_2=a}(a_1-a_2)\Omega(a_1,1)\Omega(a_2,1)\\ &+\frac{1}{2}\sum_{2a_1<a_2,a_1+a_2=a}(-1)^{2a_1-a_2}(2a_1-a_2)\Omega(a_1,1)\Omega(a_2,2)\\ &-\frac{1}{4}\sum_{a_1<a_2<a_3,a_1+a_2+a_3=a}(a_1-a_2)(a_1+a_2-2a_3)\Omega(a_1,1)\Omega(a_2,1)\Omega(a_3,1)\\ &-\frac{1}{12}\sum_{a_1<a_3,a_2<a_3,a_1+a_2+a_3=a}(a_2-a_3)(2a_1-a_2-a_3)\Omega(a_1,1)\Omega(a_2,1)\Omega(a_3,1)\end{aligned}$$

$$-\frac{1}{12}\sum_{a_2<a_1,2a_1+a_2=a}(a_1-a_2)^2(\Omega(a_1,1))^2\Omega(a_2,1)$$
$$-\frac{1}{9}\Omega(a/3,1) \tag{2.11}$$

where it is understood that the last term only appears when $a>0$ and $3\,|\,a$. As in the case of rank $r=2$ this gives an identity for $\bar{\mathrm{DT}}(a,3)=\Omega(a,3)+\frac{1}{9}\Omega(a/3,1)$ where the last term only appears if $a>0$ and $3\,|\,a$.

2.5. APPLICATION TO INTEGRALITY

In the next section we will prove that the Joyce–Song invariants $\bar{\mathrm{DT}}(a,r)$ for $r\le 3$ satisfy the above KS identities. Here, we show how to deduce integrality of the BPS state counts for $r\le 3$ from these identities in an elementary way. A different proof for $r=2$ has also been found by Toda [17]. The best result towards integrality, in general, has been proved by Reineke [15].

Consider first the case $r=2$. When $2\nmid a$ we have $\Omega(a,2)=\bar{\mathrm{DT}}(a,2)$ which is integral by Joyce–Song theory since the class $(a,2)$ is primitive. Therefore, we assume $2\,|\,a$. Going back to the $r=2$ KS identity, notice that

$$\sum_{a'<a'',a'+a''=a}(a'-a'')\Omega(a',1)\Omega(a'',1)=\sum_{a'<a'',a'+a''=a}(a-2a'')\Omega(a',1)\Omega(a'',1)$$
$$\equiv 0 \mod 2.$$

So integrality of $\Omega(a,2)$ follows if we can prove that

$$\sum_{k\ge 1}2^{k-2}\frac{\chi^k}{k!}\sum_{\mathrm{len(n)=len(i)}=k,\mathrm{n}\cdot\mathrm{i}=a}\frac{\prod n_l}{\prod i_l}-\frac{1}{4}\Omega(a/2,1)$$

is an integer. This, in turn, is equivalent to

$$\sum_{k\ge 1}\frac{(2\chi)^k}{k!}\sum_{\mathrm{len(n)=len(i)}=k,\mathrm{n}\cdot\mathrm{i}=a}\frac{\prod n_l}{\prod i_l}\equiv\Omega(a/2,1)\mod 4.$$

But notice that we can use the $r=1$ KS identity to relate the left hand side of the above congruence to the MacMahon function $M(t)$, namely the left hand side is just the coefficient of t^a in the formal power series $M(t)^{2\chi}=\prod_{n\ge 1}(1-t^n)^{-2n\chi}$. The right hand side is the coefficient of $t^{a/2}$ in the formal power series $M(-t)^{\chi}=\prod_{n\ge 1}(1-(-t)^n)^{-n\chi}$. So the $r=1,2$ KS identities together reduce integrality to the following computation.

LEMMA 2.12. *For $2\,|\,a$*

$$[t^a]M(t)^{2\chi}\equiv(-1)^{a/2}[t^{a/2}]M(t)^{\chi}\mod 4.$$

Proof. We use the identity for Euler products

$$[t^a]\prod_{n\geq 1}(1-t^n)^{-c_n}=\sum_{\mathsf{p}\vdash a}\prod_{i\geq 1}\binom{c_n-1+p_i-p_{i+1}}{p_i-p_{i+1}}, \tag{2.13}$$

where (in contrast to the rest of the paper) the sum is over partitions rather than ordered partitions. We learned of this representation from [15, Lemma 5.3]. In our case this gives

$$[t^a]M(t)^{2\chi}=\sum_{\mathsf{p}\vdash a}\prod_{i\geq 1}\binom{2i\chi-1+p_i-p_{i+1}}{p_i-p_{i+1}}$$

and

$$[t^{a/2}]M(t)^{\chi}=\sum_{\mathsf{q}\vdash a/2}\prod_{j\geq 1}\binom{j\chi-1+q_i-q_{i+1}}{q_i-q_{i+1}}.$$

Note that

$$\binom{2i\chi-1+\xi}{\xi}\equiv 0 \mod 2 \quad \text{for } \xi\equiv 1 \mod 2,$$

so the restriction of the first sum to partitions which contain parts of each parity is $\equiv 0 \mod 4$. On the other hand, if the partition only contains odd parts, there must be an even number of them since (as a is even) and then the sum is still $\equiv 0 \mod 4$ by the congruence

$$\binom{2i\chi-1+\xi}{\xi}\equiv 0 \mod 4 \quad \text{for } i\equiv 0 \mod 2 \text{ and } \xi\equiv 1 \mod 2$$

applied when ξ is the last part of the partition. It remains to show that for a partition p with even parts

$$\prod_{i\geq 1}\binom{2i\chi-1+p_i-p_{i+1}}{p_i-p_{i+1}}\equiv(-1)^{a/2}\prod_{i\geq 1}\binom{i\chi-1+p_i/2-p_{i+1}/2}{p_i/2-p_{i+1}/2} \mod 4.$$

But this follows from

$$\binom{2i\chi-1+\xi}{\xi}\equiv(-1)^{\xi/2}\prod_{i\geq 1}\binom{i\chi-1+\xi/2}{\xi/2} \mod 4 \quad \text{for } \xi\equiv 0 \mod 2$$

which can be proved by induction.

Similarly in the $r=3$ case we already know that the Joyce–Song invariants are integral for primitive classes, so we assume $3\,|\,a$. We need an analogue of the above lemma.

LEMMA 2.14. *For* $3\,|\,a$

$$[t^a]M(-t)^{3\chi}\equiv[t^{a/3}]M(-t)^{\chi} \mod 9.$$

For the proof see the arXiv version of this paper. One can then show that the result would follow from the integrality of

$$-\frac{1}{12}\sum_{a_1<a_3,a_2<a_3,a_1+a_2+a_3=a}(a_2-a_3)(2a_1-a_2-a_3)\Omega(a_1,1)\Omega(a_2,1)\Omega(a_3,1)$$

and

$$-\frac{1}{12}\sum_{a_2<a_1,2a_1+a_2=a}(a_1-a_2)^2(\Omega(a_1,1))^2\Omega(a_2,1).$$

But this holds since we are assuming $3\,|\,a$ so for the first term

$$2a_1-a_2-a_3=3a_1-a=3\left(a_1-\frac{a}{3}\right),$$

and similarly for the second

$$a_1-a_2=3a_1-a=3\left(a_1-\frac{a}{2}\right).$$

3. Joyce–Song Side

In this section, we use Joyce–Song theory for precisely the same wall-crossing described in introduction. The tilted category $\mathcal{A}'$ satisfies again the assumptions of the theory and the Joyce–Song invariants do not change until the phase of μ crosses that of γ. One can check directly that for $\phi(\mu)>\phi(\gamma)$ the Joyce–Song invariants, which we call $\bar{\mathrm{DT}}^{-}$, vanish for all mixed classes. The general wall-crossing formula in JS theory (see [9, Section 5]) is

$$\begin{aligned}\bar{\mathrm{DT}}(\alpha)=\sum_{n\geq 1}\sum_{\alpha_1,\dots,\alpha_n}&\frac{(-1)^{n-1}}{2^{n-1}}\mathsf{U}(\alpha_1,\dots,\alpha_n;\phi_{\mp})\\ &\cdot\sum_{\Upsilon}\prod_{\{i\to j\}\subset\Upsilon}(-1)^{\langle\alpha_i,\alpha_j\rangle}\langle\alpha_i,\alpha_j\rangle\prod_k\bar{\mathrm{DT}}^{-}(\alpha_k),\end{aligned}$$

where we are summing over effective decompositions $\sum_i\alpha_i$ of the K-theory class α (weighted by certain combinatorial coefficients U, recalled below) and ordered trees Υ, respectively (more precisely the relevant trees are labelled by $\{1,2,\dots,n\}$, and satisfy $i\to j\Rightarrow i<j$). The bracket above denotes the Euler form. In general, it is not known if the $\bar{\mathrm{DT}}$ invariants satisfy the Kontsevich–Soibelman wall-crossing formula and only the more complicated Joyce–Song identity has been rigorously established.

3.1. RANK $r=1$

3.1.1. *Decompositions and Partitions*

Fix a K-theory class $\alpha=\gamma+a\mu$ and let

$$\alpha=\alpha_1+\cdots+\alpha_n \tag{3.1}$$

be an ordered *decomposition* into effective classes; this corresponds to a 2D ordered partition of the integer vector $(a, 1)$. This decomposition a priori gives a contribution to $\bar{\mathrm{DT}}(\alpha)$ via Joyce–Song wall-crossing, which is given by a multiple of the $\bar{\mathrm{DT}}^-$ invariant of the 2D partition, $\prod_k \bar{\mathrm{DT}}^-(\alpha_k)$. However, $\bar{\mathrm{DT}}^-(\beta)$ vanishes for "mixed classes" $\langle\beta,\gamma\rangle, \langle\beta,\mu\rangle \neq 0$. Thus, we can effectively restrict to summing over pairs (p, i) given by an *ordered partition* p for a of length $n-1$ and an integer $i = 1, \ldots, n$ denoting the place of the (unique) summand γ in the decomposition, so that the decomposition of α above looks like

$$\gamma + a\mu = p_1\mu + \cdots + p_{i-1}\mu + \gamma + p_i\mu + \cdots + p_{n-1}\mu \tag{3.2}$$

(writing $\mathsf{p} = (p_1, \ldots, p_{n-1})$). We write $\mathsf{p} \vdash a$ for an ordered partition of a.

3.1.2. S *Symbols*

Let us denote by $\phi_\mp = \arg \circ Z_\mp$ the phase functions with respect to two different central charges $Z_\mp$ with $\tau > \frac{3}{2}\pi$ and $\tau < \frac{3}{2}\pi$, respectively. We need to compute Joyce's S symbol (see, e.g. [9, Definition 3.12])

$$\mathsf{S}(\mathsf{p}, i) = \mathsf{S}(p_1\mu, \ldots, p_{i-1}\mu, \gamma, p_i\mu, \ldots, p_{n-1}\mu; \phi_\mp).$$

Its value is determined by a set of "seesaw" inequalities (the inequalities (a) and (b) in [9] Definition 3.12), which say roughly that S is an ordering operator. More precisely let $\alpha_1, \ldots, \alpha_n$ be a collection of charges (with $n \geq 1$). If for all $i = 1, \ldots, n-1$ we have either

1. $\phi_-(\alpha_i) < \phi_-(\alpha_{i+1})$ and $\phi_+(\alpha_1 + \cdots + \alpha_i) \geq \phi_+\alpha_{i+1} + \cdots + \alpha_(n)$, or
2. $\phi_-(\alpha_i) \geq \phi_-(\alpha_{i+1})$ and $\phi_+(\alpha_1 + \cdots + \alpha_i) < \phi_+\alpha_{i+1} + \cdots + \alpha_(n)$

then one defines $\mathsf{S}(\alpha_1, \ldots, \alpha_n; s, w)$ to be $(-1)^{\#\{\text{indices satisfying (1)}\}}$. Otherwise $\mathsf{S}(\alpha_1, \ldots, \alpha_n; s, w)$ vanishes.

Suppose $i > 2$. Then since

$$\begin{aligned} &\phi_-(p_1\mu) = \phi_-(p_2\mu), \\ &\phi_+(p_1\mu) < \phi_+(\gamma + p_2\mu + \cdots + p_{n-1}\mu) \end{aligned}$$

the seesaw inequalities do not hold and $\mathsf{S} = 0$. For $i = 2$ the inequalities do hold since

$$\begin{aligned} \phi_-(p_1\mu) &> \phi_-(\gamma), \\ \phi_+(p_1\mu) &\leq \phi_+(\gamma + p_1\mu + \cdots + p_{n-1}\mu); \\ \phi_-(\gamma) &< \phi_-(p_2\mu), \\ \phi_+(\gamma + p_1\mu) &> \phi_+(p_2\mu + \cdots + p_{n-1}\mu); \\ \phi_-(p_k\mu) &= \phi_-(p_{k+1}\mu), \\ \phi_+(\gamma + p_1\mu + \cdots + p_k\mu) &> \phi_+(p_{k+1}\mu + \cdots + p_{n-2}\mu) \end{aligned}$$

for $k=2,\ldots,n-2$. When the seesaw inequalities hold S is $(-1)^{\#\text{adjacent}(\le,>)\text{pairs}}$, which gives

$$\mathsf{S}(\mathsf{p},2)=(-1)^{n-2}. \tag{3.3}$$

Similarly for $i=1$ the seesaw inequalities hold since

$$\begin{aligned}\phi_-(\gamma)&<\phi_-(p_1\mu),\\ \phi_+(\gamma)&>\phi_+(p_1\mu+\cdots+p_{n-1}\mu);\\ \phi_-(p_k\mu)&=\phi_-(p_{k+1}\mu),\\ \phi_+(\gamma+p_1\mu+\cdots+p_k\mu)&<\phi_+(p_{k+1}\mu+\cdots+p_{n-1}\mu)\end{aligned}$$

for $k=1,\ldots,n-2$, giving

$$\mathsf{S}(\mathsf{p},1)=(-1)^{n-1}. \tag{3.4}$$

Remark. Let ϕ_0 denote the degenerate phase function of the wall (i.e. for $\tau=\frac{3}{2}\pi$). Then one can easily compute a few S symbols when hitting the wall from $\tau>\frac{3}{2}\pi$, e.g.

$$\begin{aligned}&\mathsf{S}(\gamma,\mu;\phi_0^-)=\mathsf{S}(\gamma,2\mu;\phi_0^-)=0,\quad \mathsf{S}(\mu,\gamma;\phi_0^-)=\mathsf{S}(2\mu,\gamma;\phi_0^-)=1,\\ &\mathsf{S}(\gamma,\mu,\mu;\phi_0^-)=\mathsf{S}(\mu,\gamma,\mu;\phi_0^-)=\mathsf{S}(\mu,\mu;\phi_0^-)=0.\end{aligned}$$

3.1.3. *U Symbols*

Let $\alpha_1,\ldots,\alpha_n$ be a collection of charges (with $n\ge 1$). Joyce introduced certain combinatorial coefficients (see, e.g. [9, Definition 3.12]), which play a fundamental role in the wall-crossing formula, and whose general definition is rather involved,

$$\begin{aligned}&\mathsf{U}(\alpha_1,\ldots,\alpha_n;\phi_\mp)\\ &=\sum_{\substack{1\le l\le m\le n,\ 0=a_0<a_1<\cdots<a_m=n,\ 0=b_0<b_1<\cdots<b_l=m:\\ \text{Define } \beta_1,\ldots,\beta_m \text{ by } \beta_i=\alpha_{a_{i-1}+1}+\cdots+\alpha_{a_i}.\\ \text{Define } \gamma_1,\ldots,\gamma_l \text{ by } \gamma_i=\beta_{b_{i-1}+1}+\cdots+\beta_{b_i}.\\ \text{Then } \phi_-(\beta_i)=\phi_-(\alpha_j),\ i=1,\ldots,m,\ a_{i-1}<j\le a_i,\\ \text{and } \phi_+(\gamma_i)=\phi_+(\alpha_1+\cdots+\alpha_n),\ i=1,\ldots,l}} \frac{(-1)^{l-1}}{l}\cdot\prod_{i=1}^{l}\mathsf{S}(\beta_{b_{i-1}+1},\beta_{b_{i-1}+2},\ldots,\beta_{b_i};s,w)\cdot\prod_{i=1}^{m}\frac{1}{(a_i-a_{i-1})!}.\end{aligned}$$

We need to compute these symbols in our case. Consider again the decomposition (3.2). We can obtain a new one of the same form by partitioning the *head* and *tail* sets

$$\{p_1\mu,\ldots,p_{i-1}\mu\},\{p_i\mu,\ldots,p_{n-1}\mu\}$$

according to partitions $\mathsf{q}', \mathsf{q}''$ of $i-1, n-i$ and taking the partial sums of $\mathsf{q}', \mathsf{q}''$. We call this a *contraction* (p', i) of the decomposition (p, i). A contraction carries a *weight*

$$\frac{1}{\prod_k q_k'! \prod_l q_l''!}.$$

The ideal sheaves K-theory classes $(a, 1)$ are primitive. In this situation Joyce's $\mathsf{U}(\mathsf{p}, i)$ symbol reduces to the weighted sum over all contractions of (p, i) with non-vanishing S symbol.

Suppose $i > 1$. Then the only choice for q' is the trivial partition of $i-1$ (i.e. we must contract all of $\{p_1\mu, \dots, p_{i-1}\mu\}$ to the single class $(p_1 + \cdots + p_{i-1})\mu$) with weight $(i-1)!^{-1}$. On the other hand, we can contract the tail with an arbitrary $\mathsf{q} \vdash n-i$ with weight $(\prod_k q_k!)^{-1}$. The contracted decomposition is of type $(\mathsf{p}', 2)$, has length $2 + \operatorname{len}(\mathsf{q})$ and thus S symbol $(-1)^{\operatorname{len}(\mathsf{q})}$.

For $i = 1$ instead the head is empty and the q-contracted decomposition has type $(\mathsf{p}', 1)$, length $1 + \operatorname{len}(\mathsf{q})$ and thus $\mathsf{S} = (-1)^{\operatorname{len}(\mathsf{q})}$. So we see that for $i \geq 1$

$$\mathsf{U}(\mathsf{p}, i) = \frac{1}{(i-1)!} \sum_{\mathsf{q} \vdash n-i} \frac{(-1)^{\operatorname{len}(q)}}{\prod_k q_k!}. \tag{3.5}$$

The result is *independent* of p. Next notice the identity

$$\sum_{\mathsf{q} \vdash s} \frac{(-1)^{\operatorname{len}(\mathsf{q})}}{\prod_l q_l!} = \frac{(-1)^s}{s!}$$

which is easily proved by induction,

$$\begin{aligned} \sum_{\mathsf{q} \vdash s} \frac{(-1)^{\operatorname{len}(\mathsf{q})}}{\prod_l q_l!} &= -\sum_{q_1=1}^{s} \frac{1}{q_1!} \sum_{\mathsf{q}' \vdash s-q_1} \frac{(-1)^{\operatorname{len}(\mathsf{q}')}}{\prod_l q_l'!} \\ &= -\sum_{q_1=1}^{s} \frac{(-1)^{s-q_1}}{q_1!(s-q_1)!} = \frac{(-1)^s}{s!}. \end{aligned}$$

Using this identity we find for $i \geq 1$

$$\mathsf{U}(\mathsf{p}, i) = \frac{(-1)^{n-i}}{(i-1)!(n-i)!}. \tag{3.6}$$

Notice that, in particular

$$\sum_{i=1}^{n} (-1)^i \mathsf{U}(\mathsf{p}, i) = (-1)^n \frac{2^{n-1}}{(n-1)!}.$$

Remark. As an example of the general definition we compute a few U symbols when hitting the wall, e.g.

$$\mathsf{U}(\gamma, \mu; \phi_0^-) = \mathsf{U}(\gamma, 2\mu; \phi_0^-) = -\frac{1}{2}, \quad \mathsf{U}(\mu, \gamma; \phi_0^-) = \mathsf{U}(2\mu, \gamma; \phi_0^-) = \frac{1}{2},$$
$$\mathsf{U}(\gamma, \mu, \mu; \phi_0^-) = \frac{1}{12}, \quad \mathsf{S}(\mu, \gamma, \mu; \phi_0^-) = -\frac{1}{6}, \quad \mathsf{U}(\mu, \mu; \phi_0^-) = \frac{1}{12}.$$

3.1.4. *Sums over Trees*

The wall-crossing for the decomposition (3.1) carries a sum over trees factor

$$\sum_{\Upsilon} \prod_{\{k\to l\}\subset \Upsilon^1} (-1)^{\langle \alpha_k, \alpha_l\rangle} \langle \alpha_k, \alpha_l\rangle$$

which is especially simple for $r=1$. Since $\langle p_k\mu, p_l\mu\rangle = 0$ the only ordered tree which gives a nonvanishing factor is the unique ordered tree rooted at i with leaves labelled by $1,\ldots i-1, i+1,\ldots n$. The factor is then

$$\prod_k (-1)^{p_k} \prod_{l=1}^{i-1} \langle p_l\mu, \gamma\rangle \prod_{l=i}^{n-1} \langle \gamma, p_l\mu\rangle = (-1)^{n+i} \prod_k (-1)^{p_k} p_k = (-1)^a (-1)^{n+i} \prod_k p_k.$$

3.1.5. $\bar{\mathrm{DT}}^-$ *of a Partition*

Since $\bar{\mathrm{DT}}^-(\gamma)=1$ this is simply the product

$$\prod_k \mathrm{DT}^-(p_k\mu),$$

in particular, it only depends on the unordered partition underlying p. Thus, we compute

$$\bar{\mathrm{DT}}^-(\mathsf{p}, i) = (-1)^{n-1}\chi^{n-1} \prod_{k=1}^{n-1} \left(\sum_{m|p_k} \frac{1}{m^2} \right).$$

3.1.6. $r=1$ *Wall-Crossing*

We can now write down the rank $r=1$ wall-crossing formula explicitly in terms of ordered partitions for integers,

$$\begin{aligned} \bar{\mathrm{DT}}(a,1) &= (-1)^a \sum_{n\geq 2} (-1)^n (-\chi)^{n-1} \frac{(-1)^{n-1}}{2^{n-1}} \left(\sum_{i=1}^{n} (-1)^i \mathsf{U}(\mathsf{p}, i) \right) \\ &\quad \times \left(\sum_{\mathsf{p}\vdash a, \mathrm{len}(p)=n-1} \prod_k p_k \left(\sum_{m|p_k} \frac{1}{m^2} \right) \right) \\ &= (-1)^a \sum_{n\geq 2} \frac{\chi^{n-1}}{(n-1)!} \left(\sum_{\mathsf{p}\vdash a, \mathrm{len}(p)=n-1} \prod_k p_k \left(\sum_{m|p_k} \frac{1}{m^2} \right) \right). \end{aligned}$$

This can be compared directly with the KS wall-crossing. Rearranging we find

$$\sum_{\mathsf{p}\vdash a, \mathrm{len}(\mathsf{p})=k} \prod_l p_l \left(\sum_{m|p_l} \frac{1}{m^2} \right) = \sum_{\mathrm{len}(\mathrm{n})=\mathrm{len}(\mathrm{i})=k, \mathrm{n}\cdot\mathrm{i}=a} \frac{\prod n_l}{\prod i_l}. \tag{3.7}$$

which proves the required equivalence.

Remark. Similarly we can compute on the wall (emphasising the contribution of each partition),

$$\begin{aligned}
\bar{\mathrm{DT}}^0(1,1) &= -\frac{1}{2}\mathsf{U}(\gamma,\mu)(-1)^2(1)(-\chi) - \frac{1}{2}\mathsf{U}(\mu,\gamma)(-1)(1)(1)(-\chi)\\
&= -\frac{\chi}{4} - \frac{\chi}{4} = -\frac{\chi}{2},\\
\bar{\mathrm{DT}}^0(2,1) &= -\frac{1}{2}\mathsf{U}(\gamma,2\mu)(-2)(1)(-\chi) - \frac{1}{2}\mathsf{U}(2\mu,\gamma)(2)(1)(-\chi)\\
&\quad + \frac{1}{4}\mathsf{U}(\gamma,\mu,\mu)(-1)^2(1)(-\chi)^2 + \frac{1}{4}\mathsf{U}(\mu,\gamma,\mu)(-1)(1)(-\chi)^2\\
&\quad + \frac{1}{4}\mathsf{U}(\mu,\mu,\gamma)(-1)^2(1)(-\chi)^2 = \frac{5}{4}\chi + \frac{1}{12}\chi^2,
\end{aligned}$$

in agreement with our previous KS computations on the wall.

3.2. $r=2$

3.2.1. *Decompositions*

The rank $r=2$ wall-crossing formula contains a copy of the $r=1$ case, up to scale, given by ordered decompositions of the form

$$2\gamma + a\mu = p_1\mu + \cdots + p_{i-1}\mu + 2\gamma + p_i\mu + \cdots p_{n-1}\mu. \tag{3.8}$$

This is because for decompositions of the form above it makes no difference if the K-theory class is not primitive: U remains the sum of S over all possible contractions. The factor

$$\prod_k (-1)^{p_k} p_k \left(\sum_{m|p_k} \frac{1}{m^2}\right) = (-1)^a \prod_k p_k \left(\sum_{m|p_k}\frac{1}{m^2}\right)$$

in the χ^{n-1} coefficient of the $r=1$ formula must be replaced by

$$2^{n-1}\prod_k p_k \frac{1}{4}\left(\sum_{m|p_k}\frac{1}{m^2}\right)$$

accounting for products $\pm(-1)^{\langle p_k\mu, 2\gamma\rangle}\langle p_k\mu, 2\gamma\rangle$ and $\bar{\mathrm{DT}}^-(2\gamma) = \frac{1}{4}$, giving

$$\begin{aligned}
\bar{\mathrm{DT}}(a,2) = \sum_{n\geq 2} 2^{n-3}\frac{\chi^{n-1}}{(n-1)!}\left(\sum_{\mathsf{p}\vdash a,\mathrm{len}(p)=n-1}\prod_k p_k\left(\sum_{m|p_k}\frac{1}{m^2}\right)\right)\\
+\text{contribution of new decompositions.}
\end{aligned}$$

The first term coincides precisely with the first term of the rank $r=2$ KS formula (we may call this the "‘scaling" behaviour of both the KS and the JS formulae in the D0–D6 case). The residual contribution comes from decompositions of the form

$$\begin{aligned} 2\gamma + a\mu = p_1\mu + \cdots + p_{i-1}\mu + \gamma + p_i\mu \\ + \cdots + p_{j-2}\mu + \gamma + p_{j-1}\mu + \cdots + p_{n-2}\mu \end{aligned} \tag{3.9}$$

with copies of γ sitting at places $1 \leq i < j \leq n$, which we denote by (p, i, j), where p is a length $n-2 \geq 1$ ordered partition of a. In the rest of this section we compute this residual contribution.

3.2.2. *Sum over Trees*

For fixed values of indexes i, j, $1 \leq i < j \leq n$ choose a special integer $l \in \{1, \ldots, n\} \setminus \{i, j\}$; then choose possibly empty subsets of

$$\{1, \ldots, i-1\} \setminus \{l\}, \{i+1, \ldots, j-1\} \setminus \{l\}, \{j+1, \ldots, n\} \setminus \{l\}$$

with cardinality h, m, t, respectively. These choices give rise to a well-defined ordered tree rooted at i, j by connecting the chosen sets to the vertex labelled i, the special vertex l to both i, j and the remaining edges to j. Two such trees can be distinguished by their Prüfer code, and all admissible trees for (3.9) are of this form.

A fixed tree contributes to the wall-crossing formula by a common factor $\prod_k (-1)^{p_k} p_k = (-1)^a \prod_k (-1)^{p_k} p_k$ times by a factor specific to the tree. Suppose first $l \in \{1, \ldots, i-1\}$; then this factor is

$$(-1)^{\#\{\text{edges outgoing from } i \text{ or } j\}} p_l = (-1)^m (-1)^{n-j} p_l.$$

There are $2^{i-2} 2^{n-j} \binom{j-i-1}{m}$ trees for such fixed l. For $l \in \{i+1, j-1\}$ the factor is

$$(-1)^{\#\{\text{edges outgoing from } i \text{ or } j\}} p_{l-1} = (-1)^{m+1} (-1)^{n-j} p_{l-1},$$

and there are $2^{i-1} 2^{n-j} \binom{j-i-2}{m}$ trees for such fixed l. Finally, $l \in \{j+1, \ldots, n\}$ gives a factor

$$(-1)^{\#\{\text{edges outgoing from } i \text{ or } j\}} p_{l-2} = (-1)^m (-1)^{n-j+1} p_{l-2}$$

for $2^{i-1} 2^{n-j-1} \binom{j-i-1}{m}$ trees. Thus, the sum over graphs turns out to be

$$\begin{aligned} &\sum_{\Upsilon} \prod_{\{k \to l\} \subset \Upsilon^1} (-1)^{\langle \alpha_k, \alpha_l \rangle} \langle \alpha_k, \alpha_l \rangle \\ &= \prod_k (-1)^{p_k} p_k \left(2^{i-2} 2^{n-j} (-1)^{n-j} \sum_{l=1}^{i-1} p_l \sum_{m=0}^{j-i-1} (-1)^m \binom{j-i-1}{m} \right. \end{aligned}$$

$$+2^{i-1}2^{n-j}(-1)^{n-j}\sum_{l=i+1}^{j-1}p_{l-1}\sum_{m=0}^{j-i-2}(-1)^{m+1}\binom{j-i-2}{m}$$
$$\left.+2^{i-1}2^{n-j-1}(-1)^{n-j+1}\sum_{l=j+1}^{n}p_{l-2}\sum_{m=0}^{j-i-1}(-1)^{m}\binom{j-i-1}{m}\right).$$

By the binomial theorem this equals

$$\begin{cases} (-1)^n(-1)^{i+1}2^{n-3}(-1)^a\prod_k p_k(\sum_{l=1}^{i-1}p_l-\sum_{l=i}^{n-2}p_l) & \text{if } j=i+1,\\ (-1)^n(-1)^{i+1}2^{n-3}(-1)^a\prod_k p_k p_i & \text{if } j=i+2,\\ 0 & \text{otherwise.}\end{cases}$$

(recall $n\geq 3$). The upshot of this is that among decompositions (3.9) the only that can possibly contribute to the wall-crossing are those with (i, j) as above.

3.2.3. S *and* U

We only need to compute U of the decompositions with nonvanishing $\sum_\Upsilon$ factor. Notice first that as in the $r=1$ case the S symbol of a partition can only be nonvanishing if the first copy of γ lies in the first or second place. As in the primitive case $\mathsf{U}(\mathsf{p}, i, j)$ contains a "first order" term which is the weighted sum of S over admissible contractions of p. For an arbitrary p we must contract the head $\{p_1\mu,\dots,p_{i-1}\mu\}$ to the singleton $\{(p_1+\cdots+p_{i-1})\mu\}$.

Suppose first $j=i+1$. Then contracting the head to a singleton plus contracting the tail using a partition q has S symbol

$$(-1)^{\mathrm{len}(\mathsf{q})+1}\delta_{p_1+\cdots+p_{i-1}<p_i+\cdots+p_{n-2}}.$$

If we also contract the couple $\{\gamma,\gamma\}$ (with weight 1/2) the S symbol becomes $(-1)^{\mathrm{len}(\mathsf{q})}$. The first order U symbol for $j=i+1$ is therefore

$$\frac{1}{(i-1)!}\sum_{\mathsf{q}\vdash n-i-1}\frac{(-1)^{\mathrm{len}(\mathsf{q})}}{\prod_l q_l!}\left(-\delta_{p_1+\cdots+p_{i-1}<p_i+\cdots+p_{n-2}}+\frac{1}{2}\right)$$
$$=\frac{1}{2}\frac{(-1)^{n-i-1}}{(i-1)!(n-i-1)!}(\delta_{p_1+\cdots+p_{i-1}\geq p_i+\cdots+p_{n-2}}-\delta_{p_1+\cdots+p_{i-1}<p_i+\cdots+p_{n-2}}).$$

For $j=i+2$ the corresponding first order term is

$$\frac{1}{(i-1)!}\sum_{\mathsf{q}\vdash n-i-1}\frac{(-1)^{\mathrm{len}(\mathsf{q})}}{\prod_l q_l!}(\delta_{p_1+\cdots+p_{i-1}<p_i+\cdots+p_{n-2}}\cdot\delta_{p_1+\cdots+p_i\geq p_{i+1}+\cdots+p_{n-2}})$$
$$=\frac{(-1)^{n-i-1}}{(i-1)!(n-i-1)!}(\delta_{p_1+\cdots+p_{i-1}<p_i+\cdots+p_{n-2}}\cdot\delta_{p_1+\cdots+p_i\geq p_{i+1}+\cdots+p_{n-2}}).$$

In both cases when $2|a$ there is also a "second order" term. For $j=i+1$ it is

$$-\frac{1}{2}\frac{1}{(i-1)!}\sum_{\mathsf{q}\vdash n-i-1}\frac{(-1)^{\mathrm{len}(\mathsf{q})}}{\prod_l q_l!}\delta_{p_1+\cdots+p_{i-1}=p_i+\cdots+p_{n-2}}$$
$$=-\frac{1}{2}\frac{(-1)^{n-i-1}}{(i-1)!(n-i-1)!}\delta_{p_1+\cdots+p_{i-1}=p_i+\cdots+p_{n-2}},$$

while for $j=i+2$ we get

$$-\frac{1}{2}\frac{1}{(i-1)!}\sum_{\mathsf{q}\vdash n-i-1}\frac{(-1)^{\mathrm{len}(\mathsf{q})}}{\prod_l q_l!}(-\delta_{p_1+\cdots+p_i=p_{i+1}+\cdots+p_{n-2}}+\delta_{p_1+\cdots+p_{i-1}=p_i+\cdots+p_{n-2}})$$
$$=\frac{1}{2}\frac{(-1)^{n-i-1}}{(i-1)!(n-i-1)!}(\delta_{p_1+\cdots+p_i=p_{i+1}+\cdots+p_{n-2}}-\delta_{p_1+\cdots+p_{i-1}=p_i+\cdots+p_{n-2}}).$$

3.2.4. $r=2$ *Wall-Crossing*

Recall that our aim is to compare the residual contribution given in Joyce–Song theory by decompositions of the form (3.9) with the corresponding term in the $r=2$ KS formula, namely

$$\frac{(-1)^a}{2}\sum_{a'<a'',a'+a''=a}(a'-a'')\Omega(a',1)\Omega(a'',1).$$

By the above discussion it is enough to sum over p and i since j is either $i+1$ or $i+2$, and the $\frac{(-1)^{n-1}}{2^{n-1}}\mathsf{U}$ factor over such a sum over decomposition equals

$$\frac{(-1)^a}{2}\frac{(-1)^n}{4}\frac{1}{(i-1)!(n-i-1)!}\prod_k p_k\left(\sum_{m|p_k}\frac{1}{m^2}\right)$$
$$\cdot\left(\left(\sum_{l=1}^{i-1}p_l-\sum_{l=i}^{n-2}p_l\right)(\delta_{p_1+\cdots+p_{i-1}<p_i+\cdots+p_{n-2}}-\delta_{p_1+\cdots+p_{i-1}\geq p_i+\cdots+p_{n-2}})\right.$$
$$-p_i(2\delta_{p_1+\cdots+p_{i-1}<p_i+\cdots+p_{n-2}}\cdot\delta_{p_1+\cdots+p_i\geq p_{i+1}+\cdots+p_{n-2}}$$
$$\left.-\delta_{p_1+\cdots+p_{i-1}=p_i+\cdots+p_{n-2}}+\delta_{p_1+\cdots+p_i=p_{i+1}+\cdots+p_{n-2}})\right).$$

Notice that the second order term for U when $j=i+1$ is only nonzero when $\sum_{l=1}^{i-1}p_l=\sum_{l=i}^{n-2}p_l$, hence it gives no contribution in the formula above.

Now sum over all p,i and compare to the KS term. The second factor in the formula above acts as on ordering operator, giving the sum over $a'<a''$. This can be seen using the fact that for a fixed partition p there exists a unique i with

$$\delta_{p_1+\cdots+p_{i-1}<p_i+\cdots+p_{n-2}}\cdot\delta_{p_1+\cdots+p_i\geq p_{i+1}+\cdots+p_{n-2}}=1.$$

The first factor equals the sum over all products $\Omega(a',1)\Omega(a'',1)$ by the usual rearrangement

$$\sum_{\mathsf{p}\vdash a',\mathrm{len}(\mathsf{p})=k}\prod_l p_l\left(\sum_{m|p_l}\frac{1}{m^2}\right)=\sum_{\mathrm{len(n)=len(i)}=k,\mathrm{n}\cdot\mathrm{i}=a'}\frac{\prod n_l}{\prod i_l},$$

(same for a''), and the $r=1$ KS wall-crossing, i.e.

$$\Omega(a',1)=(-1)^{a'}\sum_{k\geq 1}\frac{\chi^k}{k!}\sum_{\mathrm{len(n)=len(i)}=k,\mathrm{n}\cdot\mathrm{i}=a'}\frac{\prod n_l}{\prod i_l},$$

(same for a'').

3.3. $r=3$

Exactly as for $r=2$ case there is a copy of the rank $r=1$ KS formula, up to scaling γ to 3γ, contributing

$$(-1)^a\sum_{n\geq 2}3^{n-1}\frac{\chi^{n-1}}{(n-1)!}\left(\sum_{\mathsf{p}\vdash a,\mathrm{len}(p)=n-1}\prod_k p_k\frac{1}{9}\left(\sum_{m|p_k}\frac{1}{m^2}\right)\right)$$

which can be identified with the term

$$(-1)^a\sum_{k\geq 1}3^{k-2}\frac{\chi^k}{k!}\sum_{\mathrm{len(n)=len(i)}=k,\mathrm{n}\cdot\mathrm{i}=a}\frac{\prod n_l}{\prod i_l}$$

in the $r=3$ KS formula.

Let us now consider the case when exactly 2 copies of γ appear in the decomposition, or in other words decompositions $(\mathsf{p},2_i+1_j),(\mathsf{p},1_i+2_j)$ for $i<j$. Both cases are very close, up to scale, to the decompositions studied for the $r=2$ case. One can go through all of the previous subsection, treating the first or second copy of γ as a "variable" which can be rescaled to 2γ, without any additional changes, until we reach the very last paragraph where the $r=1$ formula for $\Omega(a',1),\Omega(a'',1)$ is used. This must now be replaced with the corresponding $r=2$ formula for $\Omega(a',2)$ (respectively, $\Omega(a'',2)$), which gives a contribution

$$\begin{aligned}&\frac{1}{2}\sum_{a_1<2a_2,a_1+a_2=a}(-1)^{a_1-2a_2}(a_1-2a_2)\Omega(a_1,2)\Omega(a_2,1)\\&\quad+\frac{1}{2}\sum_{2a_1<a_2,a_1+a_2=a}(-1)^{2a_1-a_2}(2a_1-a_2)\Omega(a_1,1)\Omega(a_2,2)\\&\quad-\frac{1}{4}\sum_{a_1<a_2<a_3,a_1+a_2+a_3=a}(a_1-a_2)(a_1+a_2-2a_3)\Omega(a_1,1)\Omega(a_2,1)\Omega(a_3,1)\end{aligned}$$

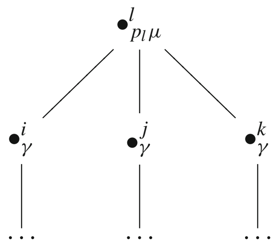
$$+\frac{1}{4}\sum_{a_1<a_2,2a_1+a_2=a}(a_1-a_2)\Omega(a_1,1)\Omega(a_2,1)$$
$$+\frac{1}{4}\sum_{a_1<a_2,a_1+2a_2=a}(a_1-a_2)\Omega(a_1,1)\Omega(a_2,1)$$

in the $r=3$ KS identity.

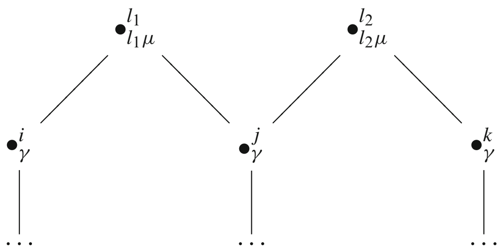
It remains to consider the "genuine" new decompositions, i.e. those of the form $(\mathsf{p}, 1_i+1_j+1_k)$ for $i<j<k$. We expect that these contribute

$$-\frac{1}{12}\sum_{a_2<a_1,2a_1+a_2=a}(a_1-a_2)^2(\Omega(a_1,1))^2\Omega(a_2,1)$$
$$-\frac{1}{12}\sum_{a_1<a_3,a_2<a_3,a_1+a_2+a_3=a}(a_2-a_3)(2a_1-a_2-a_3)\Omega(a_1,1)\Omega(a_2,1)\Omega(a_3,1).$$

This can be shown be summing over graphs of the form



and



for $l_1<l_2$.

Remark. In the arXiv version of this paper we outline an inductive argument for the KS identities starting from the JS identities, for arbitrary rank r.

4. From D0–D6 to GW

In this section we explain how the theory of Gross–Pandharipande–Siebert, in particular the main result from [7], the "full commutator formula" Theorem 5.6, applies to the case of D0–D6 states. From this point of view, the link between D0–D6 states and GW invariants is given by the product formula for the

MacMahon function. We briefly recollect the full commutator formula in the form we will need.

4.1. ORBIFOLD BLOWUPS

Let $D \subset S$ be a divisor in a smooth surface and $x \in D$ a smooth point. Smoothness implies that for each $j \geq 1$ there is a unique subscheme of D of length j with reduced scheme x. We view this nonreduced scheme as a subscheme $x_D^j \subset S$. For $j \geq 2$ the scheme-theoretic blowup S_j of S along x_D^j has a unique singular point of type A_{j-1} lying in the exceptional divisor E. Thus, we can put on $\mathcal{S}_j$ the structure of a smooth orbifold over S_j. For example, the blowup of $\mathbb{C}^2$ along a length 2 subscheme Z supported at the origin has an ordinary double point at the point of E corresponding to the direction cut out by Z, and so is locally the smooth orbifold $\mathbb{C}^2/\mathbb{Z}_2$. In this case one can check directly that, on the smooth orbifold, $E^2 = -\frac{1}{2}$, and, in general, one can prove that, on $\mathcal{S}_j$, $E^2 = -\frac{1}{j}$.

4.2. GRADED ORDERED PARTITIONS

A graded ordered partition P is a d-tuple $\mathsf{P} = (\mathsf{p}^1, \dots, \mathsf{p}^d)$ of ordered partitions such that every part of p^j is divisible by j. Its parts are labelled by p_k^j for $j = 1, \dots, d$ and $k = 1, \dots, \ell^j = \mathrm{len}(\mathsf{p}^j)$. We set $\mathrm{len}(\mathsf{P}) = (\ell^1, \dots, \ell^d)$ and $|\mathsf{P}| = \sum_{j,k} p_k^j$.

4.3. TORIC ORBIFOLDS

Let (a,r) denote a *primitive* vector. The fan given by $(-1,0), (0,-1)$ and (a,r) defines a toric surface $X = X_{(a,r)}$, the weighted projective plane $\mathbb{P}(a,r,1)$. The faces then correspond to toric divisors D_1, D_2, D_{out}. Removing the 3 torus fixed points $[1:0:0], [0:1:0], [0:0:1]$ we obtain a quasi-projective toric orbifold $X^o_{(a,r)}$ with divisors D_1^o, D_2^o, D_{out}^o.

Let now $\mathsf{G} = (\mathsf{P}_1, \mathsf{P}_2)$ be pair of graded ordered partitions $\mathsf{P}_1 = (\mathsf{p}_1^1, \dots, \mathsf{p}_1^{d_1})$, $\mathsf{P}_2 = (\mathsf{p}_2^1, \dots, \mathsf{p}_2^{d_2})$. For $i = 1, 2$ we choose distinct points $x_{ik}^j \in D_i^o$ corresponding to the parts p_{ik}^j of p_i^j. We pick a toric resolution $\widetilde{X} \to X$ whose corresponding divisors $\widetilde{D}_1, \widetilde{D}_2, \widetilde{D}_{out}$ are disjoint and we define a smooth orbifold $\widetilde{X}[\mathsf{G}]$ over $\widetilde{X}$ as the orbifold blowup of $\widetilde{X}$ at the points x_{ik}^j, with weights p_{ik}^j. The underlying singularities become worse as j increases. We denote by $X^o[\mathsf{G}] \subset \widetilde{X}[\mathsf{G}]$ the preimage of X^o. The exceptional divisors are E_{ik}^j, and we can define a class $\beta \in H_2(\widetilde{X}, \mathbb{Z})$ by prescribing the intersection numbers $\beta \cdot \widetilde{D}_i = |P_i|$ for $i = 1, 2$, $\beta \cdot \widetilde{D}_{out} = \gcd(|\mathsf{P}_1|, |\mathsf{P}_2|)$ (the index of a possibly nonprimitive vector) and $\beta \cdot D = 0$ for all other generators of the Picard group. From β we obtain a natural class β_{G} on the blowup, i.e. in orbifold cohomology $H_2(\widetilde{X}[\mathsf{G}])$, by pulling back and subtracting the weighted exceptional divisors, namely

$$\beta_{\mathsf{G}} = \pi^* \beta - \sum_{i=1,2} \sum_{j=1}^{d_i} \sum_{k=1}^{\ell_i^j} p_{ik}^j [E_{ik}^j].$$

4.4. MODULI SPACES OF RELATIVE STABLE MAPS

Gross–Pandharipande–Siebert consider the moduli stack $\overline{M}(\widetilde{X}[\mathsf{G}]/\widetilde{D}_{out})$ of genus 0 stable relative maps in the class β_{G} with full tangency of order $\gcd(|\mathsf{P}_1|, |\mathsf{P}_2|)$ at an unspecified point of the divisor $\widetilde{D}_{out}$, and the open substack $\overline{M}(X^o[\mathsf{G}]/D^o_{out})$ given by maps which avoid $\widetilde{X}[\mathsf{G}] \setminus X^o[\mathsf{G}]$. One of their main technical results ([7, Proposition 5.5]) proves that $\overline{M}(X^o[\mathsf{G}]/D^o_{out})$ is proper with a deformation-obstruction theory of virtual dimension 0, so for all G one has well-defined GW invariants

$$N[\mathsf{G}] = \int_{[\overline{M}(X^o[\mathsf{G}]/D^o_{out})]^{\mathrm{vir}}} 1 \in \mathbb{Q}.$$

4.5. FULL COMMUTATOR FORMULA

For $d_1, d_2 \gg 1$ consider the functions

$$\sigma = \prod_{j=1}^{d_1} \prod_{k=1}^{\ell_1^j} \left(1 + s_k^j x^j\right), \quad \tau = \prod_{j=1}^{d_2} \prod_{k}^{\ell_2^j} \left(1 + t_k^j y^j\right)$$

as elements of the ring of formal power series $\mathbb{C}[[x, y, s_{\bullet}^{\bullet}, t_{\bullet}^{\bullet}]]$ in as many variables as necessary. We define monomials

$$s^{\mathsf{P}_1} = \prod_{j=1}^{d_1} \prod_{k=1}^{\ell_1^j} \left(s_k^j\right)^{\frac{p^j_{1k}}{j}}, \quad t^{\mathsf{P}_2} = \prod_{j=1}^{d_2} \prod_{k=1}^{\ell_2^j} \left(t_k^j\right)^{\frac{p^j_{2k}}{j}}.$$

Let $(a, r) \in \mathbb{Z}^2$ be a *primitive* vector. Gross–Pandharipande–Siebert prove a formula for the formal power series $\log f_{(a,r)}$ attached to (a, r) in the ordered product factorisation for the commutator $\tau^{-1}\sigma^{-1}\tau\sigma$.

THEOREM 4.1 (Gross–Pandharipande–Siebert [7] Theorem 5.6). *There is an identity of formal power series*

$$\log f_{(a,r)} = \sum_{h \geq 1} \sum_{\mathsf{G}=(\mathsf{P}_1,\mathsf{P}_2)} h N[\mathsf{G}] s^{\mathsf{P}_1} t^{\mathsf{P}_2} x^{ha} y^{hr}$$

where the sum is over all graded ordered partitions P_1 of length $(\ell_1^1, \dots, \ell_1^{d_1})$ *and* P_2 *of length* $(\ell_2^1, \dots, \ell_2^{d_2})$ *such that* $(|\mathsf{P}_1|, |\mathsf{P}_2|) = h \cdot (a, r)$.

4.6. APPLICATION TO D0–D6 STATES

We wish to apply the above result to study the wall-crossing identity (1.5). Thus, we should consider elements of $\mathbb{C}[x, y][[u]]$ given by

$$\widetilde{\sigma}=\prod_{n\geq 1}(1-(-u)^n x^n)^{n\chi},\quad \tau=(1-uy).$$

We can fit τ in the setup for Theorem 4.1 just described by choosing $d_2=1, \ell_2^1=1$ and $t_1^1=-u$. As for $\widetilde{\sigma}$, suppose to start with that $\chi\geq 0$. Then we can truncate σ to a fixed $d_1\gg 1$ and write

$$\sigma=\prod_{j=1}^{d_1}\prod_{k=1}^{j\chi}(1-(-u)^j x^j),$$

which we can fit in the notation for Theorem 4.1 by choosing

$$\ell_1^j=j\chi,\ s_k^j=-(-1)^j u^j \quad \text{for } k=1,\ldots,j\chi.$$

Now fix a primitive vector (a,r). The admissible ordered partitions $(\mathsf{P}_1,\mathsf{P}_2)$ actually have the form (P_1,hr) for some $h\geq 1$, so $t^{\mathsf{P}_2}=(-1)^{hr}u^{hr}$. On the other hand, $\mathsf{P}_1=(\mathsf{p}_1^1,\mathsf{p}_1^2,\ldots,\mathsf{p}_1^{d_1})$ is a d_1-tuple of ordered partitions, with $\operatorname{len}(\mathsf{p}_1^j)=j\chi$ and $|\mathsf{P}_1|=ha$, and where each part of p_1^j is divisible by j. It follows that

$$s^{\mathsf{P}_1}=\prod_{j=1}^{d_1}\prod_{k=1}^{j\chi}(-(-1)^j u^j)^{\frac{p_{1k}^j}{j}}=(-1)^{\mathsf{P}_1}(-1)^{ha}u^{ha}$$

where we define the sign of a graded ordered partition as $(-1)^{\mathsf{P}_1}=(-1)^{\sum_{j,k}\frac{p_{1k}^j}{j}}$. By the full commutator formula then

$$\log f_{(a,r)}=\sum_{h\geq 1}\sum_{|\mathsf{P}_\chi|=ha}(-1)^{\mathsf{P}_1}hN[\mathsf{P}_\chi](-1)^{h(a+r)}(ux)^{ha}(uy)^{hr}$$

where the sum is over all graded ordered partitions P_χ with length vector

$$\operatorname{len}(\mathsf{P}_\chi)=(\chi,2\chi,\ldots,d_1\chi)$$

for $d_1\gg 1$ (by which we mean that we only need to choose a finite d_1 large enough for fixed (a,r) and h). So for $\chi\geq 0$, we have obtained the required D0–D6/GW duality in the ring $\mathbb{C}[[x,y]]$

$$\prod_{h\geq 1}(1-(-1)^{h^2ar}x^{ha}y^{hr})^{\Omega(ha,hr)}$$
$$=\prod_{h\geq 1}\exp\left(\sum_{|\mathsf{P}_\chi|=ha}(-1)^{\mathsf{P}_1}hN[\mathsf{P}_\chi](-1)^{h(a+r)}x^{ha}y^{hr}\right). \tag{4.2}$$

The two sets of invariants are completely determined through each other.

When $\chi<0$ we introduce one more truncation parameter $N\gg 1$ (which we can assume for convenience to be odd) and factor the d_1,N-truncation of $\widetilde{\sigma}$ as

$$\widetilde{\sigma} = \prod_{j\geq 1}^{d_1} \prod_{k=1}^{j|\chi|} \prod_{\xi^N=1, \xi\neq 1} (1 - \bar{\xi}(-u)^j x^j),$$

or in other words we set

$$\ell_1^j = j|\chi|(N-1), \;\; s_{k,\xi}^j = -\bar{\xi}(-u)^j.$$

Then we compute

$$s^{\mathsf{P}_1} = \prod_{j=1}^{d_1} \prod_{k=1}^{j\chi} \prod_{\xi^N=1, \xi\neq 1} (-\bar{\xi}(-u)^j)^{\frac{p^j_{1,(k,\xi)}}{j}} = (-1)^{ha} u^{ha}.$$

Accordingly, the D0–D6/GW formula has to be modified by dropping the $(-1)^{\mathsf{P}_\chi}$ sign, and by enlarging the set of partitions over which the GW sum extends to those graded ordered partitions P_χ with $|\mathsf{P}_\chi| = ha$ and length vector

$$\mathrm{len}(\mathsf{P}_\chi) = ((N-1)|\chi|, 2(N-1)|\chi|, \ldots, d_1(N-1)|\chi|)$$

for $d_1, N \gg 1$ (once again, this means that fixing d_1, N large enough will suffice for fixed (a, r) and h).

We point out that the difference in our treatment of the $\chi \geq 0$ and $\chi < 0$ cases reflects the different kind of quivers appearing in the work of Nagao [14] according to the sign of the Euler characteristic, in particular the appearance of loops for $\chi > 0$.

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