Stoppa, J., and E. Tenni. (2010) "A Simple Limit for Slope Instability,"
International Mathematics Research Notices, Vol. 2010, No. 10, pp. 1816–1830
Advance Access publication December 2, 2009
doi:10.1093/imrn/rnp195

A Simple Limit for Slope Instability

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Ross and Thomas have shown that subschemes can K-destabilize polarized varieties, yielding a notion known as slope (in)stability for varieties. Here, we describe a special situation in which slope instability for *varieties* (for example of general type) corresponds to a slope instability type condition for certain *bundles*, making the computations almost trivial. We illustrate this with a construction of new unstable classes on blowups of ruled surfaces.

1 Introduction

Slope stability for projective varieties is a special case of the K-stability of Tian and Donaldson. It has been introduced by Ross and Thomas in order to study the K-instability of projective bundles, algebraic surfaces, and special polarizations on varieties of general type. In this paper (Theorem 1.2), we find a class of varieties (including general-type ones) for which Ross–Thomas, nonlinear slope instability corresponds to the usual, linear slope instability of certain bundles, making the computations almost trivial.

Let X be a normal projective algebraic variety over $\mathbb{C}$. For any ample line bundle L on X and any closed subscheme $Z \subset X$, we can form the Hilbert–Samuel polynomial for $k \gg 1$

$$\chi(L^k \otimes \mathcal{I}_Z^{kx}) = \alpha_0(x)k^n + \alpha_1(x)k^{n-1} + o(k^{n-1}),$$

Received July 24, 2009; Revised October 13, 2009; Accepted October 19, 2009
Communicated by Prof. Simon Donaldson

where $\alpha_i(x), i = 0, 1$ are understood as polynomials in $x \in \mathbb{R}$ (while the left-hand side only makes sense for $kx \in \mathbb{N}$).

Recall that the Seshadri constant $\epsilon(Z, L)$ is defined as the supremum of rational c for which $L^k \otimes \mathcal{I}_Z^{kc}$ is globally generated for all large k such that $kc \in \mathbb{N}$.

For $c \in (0, \epsilon(Z, L)]$, Ross and Thomas [9] consider the integral

$$\mu_c(Z, L) = \frac{\int_0^c \left(\alpha_1(x) + \frac{\alpha_0'(x)}{2}\right) dx}{\int_0^c \alpha_0(x) dx}$$

and study the inequality

$$\mu_c(Z, L) > \mu(X, L) := \frac{\alpha_1(0)}{\alpha_0(0)} \tag{1}$$

reminiscent of slope instability for sheaves. If the latter inequality holds for some $c \in (0, \epsilon(Z, L)]$, we say that Z slope-destabilizes the polarized variety (X, L).

The main consequence of slope instability is the K-instability of the polarization L. This is most meaningful when X is smooth, since then the deep results of Chen and Tian [4] and Donaldson [5] imply a nonexistence result for Kähler metrics of constant scalar curvature representing the cohomology class $c_1(L)$ (for a differential-geometric interpretation of slope instability, see [12]). From an algebro-geometric point of view, the main consequence is asymptotic instability, see [10, Theorem 3.9].

Much progress has been made in understanding slope instability in special situations. For example, it has been proved recently by Ross and Panov that exceptional divisors of high genus always slope-destabilizes suitable polarizations on smooth algebraic surfaces [7] (a special case of this had been used previously by Ross to give the first example of an asymptotically unstable general-type surface [8]).

However, in general, the integral $\mu_c(Z, L)$ is hard to compute, making it difficult to find destabilizing subschemes in a systematic way. We are interested in special situations in which one can prove inequality (1) without having to evaluate the integral explicitly. In this paper, we describe one such situation. Our result can also be regarded as a K-instability result, independent of [9]; we relate to slope instability thanks to Lemma 2.2.

Let E be a locally free sheaf over a smooth projective curve C of genus g. We assume that the relative Serre line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ on the projective bundle $\pi : \mathbb{P}(E) \to C$

of *lines* is globally generated (i.e. E^* globally generated). For $m \geq 2$, let X be a normal element of the linear system $|\mathcal{O}_{\mathbb{P}(E)}(m)|$ (more generally, let X be a normal complete intersection of elements of $|\mathcal{O}_{\mathbb{P}(E)}(m_i)|$, $i = 1, \dots, r$). We will see that for any ample $\mathbb{Q}$-divisor A on C, the $\mathbb{Q}$-line bundle

$$\mathcal{L}_A = (\mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^* A)|_X$$

is ample. Let $F \subset E$ be a locally free subsheaf. We wish to study the slope (in)stability of the polarized variety $(X, \mathcal{L}_A)$ with respect to the closed subscheme $\mathbb{P}(F) \cap X \subset X$ (the scheme-theoretic intersection; we assume this is not all of X).

We will do this by considering a test configuration $\mathcal{X}$ for X and proving it is an equivariant contraction of the degeneration to the normal cone $\widehat{\mathcal{X}}$ of $\mathbb{P}(F) \cap X$ inside X. We will then compute the (sign of the) Donaldson–Futaki invariant for $\mathcal{X}$ with $A \approx 0$ and relate it to the integral $\mu_c(Z, \mathcal{L}_A)$ for $c \approx 1$, thanks to an important estimate of Ross and Thomas. In the last step, we take asymptotics for $g = g(C) \gg 1$ keeping $|\mu(F)|, |\mu(E)|$ bounded and explain how this corresponds to taking the limit near C "infinitely special" (precise definitions of terminology will be given in Section 2).

Let $Q = E/F$. The degeneration $\mathcal{X}$ takes $X \subset \mathbb{P}(E)$ to a cone over $\mathbb{P}(F) \cap X$ inside $\mathbb{P}(F \oplus Q)$ and can be described as follows. Let $\xi \in \mathrm{Ext}^1(Q, F)$ define the extension class of E. Scaling the fibers of F with weight 1 gives a family of extension classes $t \cdot \xi$ for $t \in \mathbb{C}$ and defines a flat family $\mathcal{E} \to \mathbb{C}$ with $\mathcal{E}_t \cong E$ for $t \neq 0$, $\mathcal{E}_0 \cong F \oplus Q$. Projectivizing this gives a family $\mathbb{P}(\mathcal{E}) \to \mathbb{C}$ with a natural action of $\mathbb{C}^*$ which preserves the projection to $\mathbb{C}$. Our variety X can be embedded in the fiber $\mathbb{P}(\mathcal{E})_1$ by choosing a form $\varphi \in H^0(S^m \mathcal{E}_1^*)$. We then define $\mathcal{X}$ as the flat closure of the family over $\mathbb{C} \setminus \{0\}$ given by the images of $X \subset \mathbb{P}(\mathcal{E})_1$ under $\mathbb{C}^*$. The central fiber $\mathcal{X}_0$ is defined by the form $\varphi|_F := \mathrm{im}(\varphi)$ under the map $S^m E^* \to S^m F^*$ but regarding now $S^m F^*$ as a direct summand of $S^m(F \oplus Q)^*$. Therefore, fiberwise over C, it is the cone from $\mathbb{P}(0 \oplus Q)$ to $\mathbb{P}(F) \cap X$ regarded as a closed subscheme of $\mathbb{P}(\mathcal{E})_0 \cong \mathbb{P}(F \oplus Q)$. The same construction goes through when X is a complete intersection; when X is sufficiently general (with respect to F) the central fiber $\mathcal{X}_0$ is again a complete intersection.

We give a criterion for the Donaldson–Futaki invariant of $\mathcal{X}$ to be negative. In turn, this gives a criterion for the subscheme $\mathbb{P}(F) \cap X$ to slope-destabilize. The main point here is that computing the Donaldson–Futaki invariant for $\mathcal{X}$ is considerably easier than computing the integral $\mu_c(\mathbb{P}(F) \cap X, L)$, but we can use a result of Ross and Thomas to give a lower bound on the latter.

Definition 1.1. Let F be a locally free sheaf on the curve C and r a positive integer with $\operatorname{rank}(F) > r$. We define a modified slope function by

$$\mu^r(F) = \frac{\deg(F)}{\operatorname{rank}(F) - r}. \qquad \square$$

Theorem 1.2. Let E be a locally free sheaf on the curve C with $\mathcal{O}_{\mathbb{P}(E)}(1)$ globally generated, $F \subset E$ a subbundle with $\operatorname{rank}(F) > r$.

Let X be a general codimension r normal complete intersection of elements of $\mathcal{O}_{\mathbb{P}(E)}(m_i)$ for $i = 1, \ldots, r$. Suppose moreover that the genus g of the curve C is large enough (with respect to r, the m_is and the numerical invariants of E and F).

If the inequality

$$\mu^r(F) > \mu^r(E)$$

holds, then the subscheme $\mathbb{P}(F) \cap X$ slope-destabilizes $(X, \mathcal{L}_A)$ for $A \approx 0$, $c \approx 1$. $\square$

More precisely, g must be larger than a lower bound depending only on the numbers r, $\max_i m_i$, $\operatorname{rank}(F)$, $\operatorname{rank}(E)$, $|\mu(E)|$, and $|\mu(F)|$. In particular, we should be allowed to let $g \to \infty$ while keeping $\mathcal{O}_{\mathbb{P}(E)}(1)$ globally generated with a uniform bound on $|\deg(E)|$.

Thus, the geometric meaning of this condition is that we take $C = C(g)$ very special. The best way to see this is to think of the simple case $E(g) = \mathcal{O}_{C(g)}^{\oplus e-1} \oplus L(g)$. We require that $L^*(g)$ is globally generated with a fixed degree which does *not* depend on g. A solution is obtained by taking $C = C(g)$ to be a curve of fixed gonality k, independent of the genus, and L^* to be its g^1_k. Such curves always exist, but the codimension of their locus in the moduli space M_g diverges as $g \to \infty$. In any case, choosing $F(g) = \mathcal{O}_{C(g)}^{\oplus e-1}$ and g sufficiently large gives a typical example for Theorem 1.2 (we can also take $E(g)$ to be one of the extensions parameterized by the large group $\operatorname{Ext}^1(L(g), \mathcal{O}_{C(g)}^{e-1}) \cong H^1(L^*(g))^{\oplus(e-1)}$).

It is straightforward to use this result to produce many examples of slope unstable polarized manifolds. In particular, it becomes almost trivial to produce a wealth of K-unstable general-type polarized manifolds (compare with the "folklore conjecture" disproved in [8]).

In the last section, we concentrate on one example of a different kind: we will use Theorem 1.2 to describe a class of slope unstable blowups of ruled surfaces (without holomorphic vector fields) for polarizations which differ from those of [9, Corollary 5.29]. Our trick is to regard these surfaces as conic bundles.

Finally, we observe that the only known examples of Kähler classes without constant scalar curvature representatives on a general-type manifold are obtained via slope instability. We believe it would be quite interesting to find K-unstable, slope stable polarizations on a general-type manifold. Note that while proving K-stability directly is essentially impossible, by the work of Ross and Panov [7], proving slope stability on a surface of general type seems almost within reach, since we should only test against rather special divisors (we must mention in this connection the fundamental inequalities of Chen [3] and Song and Weinkove [11]).

Notation. Throughout the paper, we mix the additive and multiplicative notation for line bundles. We denote by tr the total weight of a representation of $\mathbb{C}^*$. A typical space of sections over a scheme Y is denoted by H^0_Y and its dimension by h^0_Y.

2 Main Result

The basic object in algebraic K-stability is a *test configuration*. This is roughly a $\mathbb{C}^*$-equivariant flat family $\mathcal{Y}$ over $\mathbb{C}$ with a fiberwise ample line bundle $\mathcal{L}$ on it, and so a one-parameter degeneration of the polarized variety $(Y, L) = (\mathcal{Y}_1, \mathcal{L}_1)$ to the polarized scheme $(\mathcal{Y}_0, \mathcal{L}_0)$ (we will not give a precise definition here but see e.g. [10]).

A test configuration $\mathcal{Y}$ induces an action of $\mathbb{C}^*$ on the central fiber $\mathcal{Y}_0$, and by (equivariant) Riemann–Roch, one can expand the dimension and total $\mathbb{C}^*$-weight of the spaces of sections as

$$h^0_{\mathcal{Y}_0}(\mathcal{L}_0^k) = b_0 k^n + b_1 k^{n-1} + o(k^{n-1}),$$
$$\operatorname{tr} H^0_{\mathcal{Y}_0}(\mathcal{L}_0^k) = a_0 k^{n+1} + a_1 k^n + o(k^{n-1}),$$

where $n = \dim Y$. The rational number

$$\mathcal{F}(\mathcal{Y}) = a_0 b_1 - a_1 b_0$$

is called the *Donaldson–Futaki invariant* of $\mathcal{Y}$, and $\mathcal{Y}$ *K-destabilizes* if $\mathcal{F}(\mathcal{Y}) < 0$.

Let $Z \subset Y$ be a closed subscheme. The *degeneration to the normal cone* is a special test configuration $\mathcal{Y}$ for Y given by the blowup $\mathrm{Bl}_{Z\times\{0\}}\, Y \times \mathbb{C}$ lifting the natural action of $\mathbb{C}^*$ on the second factor of $Y \times \mathbb{C}$. This is polarized by $L - cE$ where E denotes the exceptional divisor and c is positive, rational, and less than the Seshadri constant of

Z with respect to the ample line bundle L on Y (which we also identify with its pullback). Ross and Thomas proved

Theorem 2.1. ([9] Theorem 4.3) The inequality (1) holds if and only if the degeneration to the normal cone K-destabilizes (namely its Donaldson–Futaki invariant $\mathcal{F}_c(\mathcal{Y})$ is negative). □

Let us now go back to the setup described in the Introduction. Recall, we are assuming $\mathcal{O}_{\mathbb{P}(E)}(1)$ is globally generated, hence nef. A standard application of the Nakai–Moishezon criterion and Kleiman's theorem then shows that for any ample $\mathbb{Q}$-divisor A on C, the $\mathbb{Q}$-line bundle $\mathcal{L}_A$ is ample. A general complete intersection X of powers of $\mathcal{O}_{\mathbb{P}(E)}(1)$ is smooth; it is enough for our purposes to take X normal. Its codimension will be denoted by r.

Starting with $F \subset E$, a subbundle, we form the degeneration $\mathcal{X}$ described in the Introduction. We wish to relate its Donaldson–Futaki invariant to that of the degeneration to the normal cone of $\mathbb{P}(F) \cap X \subset X$, which we denote by $\widehat{\mathcal{X}}$.

Lemma 2.2. Suppose that $\mathcal{F}(\mathcal{X})$ is negative. Then $\mathcal{F}_c(\widehat{\mathcal{X}})$ is also negative for $c \approx 1$. □

Proof. The test configuration $\mathcal{X}$ (with the line bundle $\mathcal{O}_{\mathcal{X}}(1)$ given by $\mathcal{X} \subset \mathbb{P}(\mathcal{E})$ as in the Introduction) is an equivariant contraction of the degeneration to the normal cone $\widehat{\mathcal{X}}$ of $\mathbb{P}(F) \cap X$. To see this, let

$$p\colon \mathrm{Bl}_{\mathbb{P}(F)\times\{0\}}\, \mathbb{P}(E) \times \mathbb{C} \to \mathbb{P}(E) \times \mathbb{C},$$

and notice that the semi-ample line bundle $\mathcal{O}_{\widehat{\mathcal{X}}}(1) := \mathcal{O}_{\mathbb{P}(E)}(1) - p^*[\mathbb{P}(F)]$ defines a morphism which is an equivariant contraction to the test configuration splitting $\mathbb{P}(E)$ into $\mathbb{P}(F \oplus Q)$ with weight 1 action on F (see [9, Remark 5.14]). Then the claim for $\mathcal{X}$ follows since it is the image of the flat closure of $X \subset \mathbb{P}(E) \times \{1\}$ under the $\mathbb{C}^*$-action, which in turn coincides with degeneration to the normal cone of $\mathbb{P}(F) \cap X$.

Suppose now that $\mathcal{F}(\mathcal{X}) < 0$. Then since the general fiber X of $\mathcal{X}$ is normal [10, Proposition 5.1] shows that there exists $a \geq 0$ such that (denoting by t a coordinate on the base $\mathbb{C}$)

$$\mathrm{tr}\left(H^0_{\mathcal{X}_0}(\mathcal{O}_{\mathcal{X}}(k))\right) = \mathrm{tr}\left(H^0_{\widehat{\mathcal{X}}}(\mathcal{O}_{\widehat{\mathcal{X}}}(k))/tH^0_{\widehat{\mathcal{X}}}(\mathcal{O}_{\widehat{\mathcal{X}}}(k))\right) - ak^n + o(k^n).$$

This implies in particular

$$\lim_{c\to 1} \mathcal{F}_c(\widehat{\mathcal{X}}) \leq \mathcal{F}(\mathcal{X}) < 0,$$

and by continuity of the Donaldson–Futaki invariant in the parameter c, we obtain $\mathcal{F}_c(\widehat{\mathcal{X}}) < 0$ for $c \approx 1$ as required. ∎

The rest of this section is devoted to the proof of Theorem 1.2. More precisely, we prove that $\mathcal{F}(\mathcal{X})$ is a linear function of g and its leading coefficient equals a positive constant times by the difference of slopes $\mu^r(F) - \mu^r(E)$. In view of Lemma 2.2 and Theorem 2.1, this will complete the proof of our main result.

Note that by the definition of the Donaldson–Futaki invariant, we may replace X with the central fiber of $\mathcal{X}_0$ and assume that E splits as $F \oplus Q$ to start with (this new X will not usually be normal, but generically for a fixed F, it will still be a codimension r complete intersection). Moreover, we will assume that $\mathcal{O}_{\mathbb{P}(E)}(1)$ is ample to start with and set $A = 0$. When $\mathcal{O}_{\mathbb{P}(E)}(1)$ is not ample, our computations below only give the limit $\lim_{A\to 0} \mathcal{F}(\mathcal{X})$; by continuity of the Donaldson–Futaki invariant in the parameter A, this is clearly enough for our purposes.

Write e, f, and q for the ranks of the bundles E, F, and Q, respectively. According to the definition, we need to expand

$$\begin{aligned} h^0_X(k) &:= h^0_X(\mathcal{O}_X(k)) = b_0 k^{e-r} + b_1 k^{e-r-1} + o(k^{e-r-1}),\\ \mathrm{tr}_X(k) &:= \mathrm{tr}\, H^0_X(\mathcal{O}_X(k)) = a_0 k^{e-r+1} + a_1 k^{e-r} + o(k^{e-r}). \end{aligned}$$

To do this, we will first find the corresponding expansions for $h^0_{\mathbb{P}(E)}(\mathcal{O}_{\mathbb{P}(E)}(k))$ and $\mathrm{tr}\, H^0_{\mathbb{P}(E)}(\mathcal{O}_{\mathbb{P}(E)}(k))$. We will then use the Koszul resolution of $\mathcal{O}_X$ to write the above polynomials for X as certain linear combinations of shifts of the analogous polynomials for $\mathbb{P}(E)$.

We start by pushing forward to C, so by Riemann–Roch, we compute

$$\begin{aligned} h^0_{\mathbb{P}}(k) &:= h^0(\mathcal{O}_{\mathbb{P}(E)}(k)) = h^0(C, \mathrm{Sym}^k E^*)\\ &= \binom{e-1+k}{e-1}(-k\mu(E) + 1 - g)\\ &\sim -\frac{\mu(E)}{(e-1)!}k^e + \left(\frac{1-g}{(e-1)!} - \frac{e\mu(E)}{2(e-2)!}\right)k^{e-1} + o(k^{e-1}). \end{aligned} \tag{2}$$

Writing

$$H^0(\mathcal{O}_{\mathbb{P}(E)}(k)) = \bigoplus_{i=0}^{k} W_{-i}$$

for the decomposition into semi-invariant spaces of weight $-i$, we have

$$\mathrm{tr}_{\mathbb{P}}(k) := \mathrm{tr}\, H^0(\mathcal{O}_{\mathbb{P}(E)}(k)) = -\sum_{i=0}^{k} i \dim W_{-i},$$

where

$$W_{-i} = H^0(S^i F^* \otimes S^{k-i} Q^*),$$

and as before, Riemann–Roch gives

$$\dim W_{-i} = \binom{f-1+i}{f-1}\binom{q-1+k-i}{q-1}(-i\mu(F) - (k-i)\mu(Q) + 1 - g).$$

An elementary way to compute the trace is by absorption, starting from

$$\begin{aligned}\sum_{i=0}^{k} i \binom{f-1+i}{f-1}\binom{q-1+k-i}{q-1} &= f\sum_{j=0}^{k-1}\binom{f+j}{f}\binom{q-1+k-1-j}{q-1}\\ &= f\binom{e+k-1}{e} \sim \frac{f}{e!}k^e + \frac{f}{2(e-2)!}k^{e-1} + o(k^{e-1}),\end{aligned}$$

which gives in turn

$$\begin{aligned}&\sum_{i=0}^{k} i^2 \binom{f-1+i}{f-1}\binom{q-1+k-i}{q-1}\\ &\quad= \sum_{i=0}^{k} i(i-1)\binom{f-1+i}{f-1}\binom{q-1+k-i}{q-1} + f\binom{e+k-1}{e}\\ &\quad= f(f+1)\sum_{j=0}^{k-2}\binom{f+1+j}{f+1}\binom{q-1+k-2-j}{q-1} + f\binom{e+k-1}{e}\\ &\quad= f(f+1)\binom{e+1+k-2}{e+1} + f\binom{e+k-1}{e}\\ &\quad\sim \frac{f(f+1)}{(e+1)!}k^{e+1} + \frac{f(f+1)(e-2)+2f}{2e!}k^e + o(k^e).\end{aligned}$$

Similarly,

$$\sum_{i=0}^{k} i(k-i)\binom{f-1+i}{f-1}\binom{q-1+k-i}{q-1} = fq\binom{e+1+k-2}{e+1}$$
$$\sim \frac{fq}{(e+1)!}k^{e+1} + \frac{fq(e-2)}{2e!}k^{e} + o(k^{e}).$$

Thus, we find

$$\mathrm{tr}_{\mathbb{P}}(k) \sim f\left(\frac{(f+1)\mu(F)}{(e+1)!} + \frac{q\mu(Q)}{(e+1)!}\right)k^{e+1}$$
$$+ f\left(\frac{(e(f+1)-2f)\mu(F)}{2e!} + \frac{q(e-2)\mu(Q)}{2e!} - \frac{(1-g)}{e!}\right)k^{e} + o(k^{e}). \tag{3}$$

Suppose now that X is a complete intersection of divisors in $|\mathcal{O}_{\mathbb{P}(E)}(m_i)|$ for $i = 1, \ldots, r$. By the Koszul resolution of $\mathcal{O}_X$, we can compute

$$h^0_X(k) = \sum_{s=0}^{r}(-1)^s \sum_{1\le i_1<i_2<\cdots<i_s\le r} h^0_{\mathbb{P}}(k-m_{i_1}-\cdots-m_{i_s}), \tag{4}$$

and similarly, taking into account the shift in the $\mathbb{Z}$-grading given by the $\mathbb{C}^*$-action,

$$\mathrm{tr}_X(k) = \sum_{s=0}^{r}(-1)^s \sum_{1\le i_1<i_2<\cdots<i_s\le r} \mathrm{tr}_{\mathbb{P}}(k-m_{i_1}-\cdots-m_{i_s})$$
$$-\sum_{s=0}^{r}(-1)^s \sum_{1\le i_1<i_2<\cdots<i_s\le r} (m_{i_1}+\cdots+m_{i_s})h^0_{\mathbb{P}}(k-m_{i_1}-\cdots-m_{i_s}). \tag{5}$$

We will need the following combinatorial identities (easily proved by induction) for a polynomial $p(x) = \alpha x^n + o(x^n)$,

$$\sum_{s=0}^{r}(-1)^s \sum_{1\le i_1<i_2<\cdots<i_s\le r} p(k-m_{i_1}-\cdots-m_{i_s}) = \binom{n}{r}r!\alpha\prod_{i=1}^{r} m_i x^{n-r} + o(x^{n-r}),$$

$$\sum_{s=0}^{r}(-1)^s \sum_{1\le i_1<i_2<\cdots<i_s\le r} (m_{i_1}+\cdots+m_{i_s})\,p(k-m_{i_1}-\cdots-m_{i_s})$$
$$= -\binom{n}{r-1}r!\alpha\prod_{i=1}^{r} m_i\, x^{n-r+1} + o(x^{n-r+1}).$$

We apply these identities to (4) and (5) to compute the leading terms with respect to k:

$$b_0 = -\binom{e}{r} r! \prod_{i=1}^{r} m_i \frac{\mu(E)}{(e-1)!} = -\prod_{i=1}^{r} m_i \frac{e\mu(E)}{(e-r)!},$$

$$\begin{aligned} a_0 &= \binom{e+1}{r} r! \prod_{i=1}^{r} m_i\, f\left(\frac{(f+1)\mu(F)}{(e+1)!} + \frac{q\mu(Q)}{(e+1)!}\right) - \binom{e}{r-1} r! \prod_{i=1}^{r} m_i \frac{\mu(E)}{(e-1)!} \\ &= \prod_{i=1}^{r} m_i \frac{f\mu(f) + e(f-r)\mu(E)}{(e+1-r)!}. \end{aligned}$$

It turns out that the same combinatorial identities can also be applied to (4) and (5) to compute the top coefficients of b_1, a_1 as functions of g. This is because a_0, b_0 do not depend on g. We find

$$\begin{aligned} b_1 &= -\left(\binom{e-1}{r} r! \prod_{i=1}^{r} m_i \frac{1}{(e-1)!}\right) g + O(g^0) \\ &= -\left(\prod_{i=1}^{r} m_i \frac{1}{(e-1-r)!}\right) g + O(g^0), \\ a_1 &= \left(\binom{e}{r} r! \prod_{i=1}^{r} m_i \frac{1}{e!} - \binom{e-1}{r-1} r! \prod_{i=1}^{r} m_i \frac{1}{(e-1)!}\right) g + O(g^0) \\ &= \left(\prod_{i=1}^{r} m_i \frac{f-r}{(e-r)!}\right) g + O(g^0). \end{aligned}$$

By definition, $\mathcal{F}(\mathcal{X}) = a_0 b_1 - a_1 b_0$, so

$$\begin{aligned} \mathcal{F}(\mathcal{X}) &= \left(\left(\prod_{i=1}^{r} m_i\right)^2 \frac{(f-r)e\mu(E) - (e-r)f\mu(f)}{(e-r)!(e-r+1)!}\right) g + O(g^0) \\ &= \left(\left(\prod_{i=1}^{r} m_i\right)^2 \frac{\mu^r(E) - \mu^r(F)}{(e-r)!(e-r+1)!}\right) g + O(g^0). \end{aligned}$$

This completes the proof of Theorem 1.2.

3 An Example

In this section, we give new examples of slope unstable blowups of ruled surfaces. Our polarizations differ from those of [9, Corollary 5.29] as the exceptional divisors of the morphism to the ruled surface all have the same large area. Our trick is to regard these blowups as conic bundles.

Let C be a genus g *hyperelliptic* Riemann surface with hyperelliptic divisor H and $D \neq H$ a divisor on C such that $\mathcal{O}_C(D)$ is non trivial and globally generated (the same argument would also apply to any Riemann surface with a nontrivial globally generated divisor H of degree less than $\frac{g+2}{3}$, but we only write it down in the hyperelliptic case for simplicity). In particular (since the hyperelliptic divisor is unique), we have $\deg(D) > \deg(H) = 2$. Consider the vector bundle

$$E = \mathcal{O}_C \oplus \mathcal{O}_C(-H) \oplus \mathcal{O}_C(-D) \tag{6}$$

and the projective bundle $\mathbb{P}(E)$ over C. The linear system $|\mathcal{O}_{\mathbb{P}(E)}(2)|$ contains a smooth irreducible surface S, and the induced map $\pi : S \to C$ is a fibration by degree 2 plane curves, i.e. a conic bundle.

It is clear that $F = \mathcal{O}_C \oplus \mathcal{O}_C(-H) \subset E$ satisfies $\mu^1(F) > \mu^1(E)$. According to Theorem 1.2, then the divisor $\mathbb{P}(F) \cap S$ slope-destabilizes S for $c \approx 1$, $A \approx 0$ and all large g. Note that in this particular case, a more careful computation shows that taking $g > 16$ is enough to give instability for all D.

In the rest of this section, we prove that S is a blowup of some ruled surface and that it has no holomorphic vector fields.

Lemma 3.1. A generic surface $S \in |\mathcal{O}_{\mathbb{P}(E)}(2)|$ is the blowup of some ruled surface $\overline{S} \to C$ in $2\deg(D) + 4$ points. □

Proof. Let us first show that $\chi(\mathcal{O}_S) = 1 - g$ and $K_S^2 = 8(1 - g) - 2\deg(D) - 4$. Consider the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}(E)}(-2) \to \mathcal{O}_{\mathbb{P}(E)} \to \mathcal{O}_S \to 0. \tag{7}$$

It follows from [6, Exercises III.8.1 and III.8.4] that $\chi(\mathcal{O}_{\mathbb{P}(E)}) = \chi(\mathcal{O}_C)$ and $\chi(\mathcal{O}_{\mathbb{P}(E)}(-2)) = 0$, thus $\chi(\mathcal{O}_S) = \chi(\mathcal{O}_C) = 1 - g$.

We also get $K_{\mathbb{P}(E)} = \mathcal{O}_{\mathbb{P}(E)}(-3) \otimes \pi^*(\Lambda^3 E^* + K_C)$ and, by adjunction,

$$K_S = \mathcal{O}_S(-1) \otimes \pi^*(\Lambda^3 E^* + K_C) = \mathcal{O}_S(-1) \otimes \pi^*(\mathcal{O}_C(H + D) + K_C)$$

By [2, Theorem I.4], we find

$$K_S^2 = \chi(\mathcal{O}_S(-2K_S)) - 2\chi(\mathcal{O}_S(-K_S)) + \chi(\mathcal{O}_S)$$

Similarly to $\chi(\mathcal{O}_{\mathbb{P}(E)})$, we can then compute

$$\chi(\mathcal{O}_S(-2K_S)) = \chi(S^2E^*(-2D-2H-2K_C)) - \chi(\mathcal{O}_C(-2D-2H-2K_{\mathbb{C}}))$$

and

$$\chi(\mathcal{O}_S(-K_S)) = \chi(E^*(-D-K_C)).$$

By Riemann–Roch on C, we conclude that

$$K_S^2 = 8(1-g) - 2\deg(D+H) = 8(1-g) - 2\deg(D) - 4.$$

Let us now show that $\pi : S \to C$ has exactly $2\deg(D)+4$ nodal fibers. The topological Euler characteristic is given by

$$e(S) = 2e(C) + \sum_{s\in C}(e(F_s) - 2),$$

where F_s is the fiber over the point $s \in C$ (see [2, Proposition X.10]). Since S is smooth, a local computation shows that there are no double fibers, and the only singular fibers that can occur are the union of two irreducible rational components meeting in one point. Such fibers have topological Euler number equal to 3; thus, the number $e(S) - 2e(C)$ is precisely the number of singular fibers. We can compute this by Noether's formula:

$$e(S) = 12\chi(\mathcal{O}_S) - K_S^2 = 4(1-g) + 2\deg(D) + 4;$$

thus, we have precisely $2\deg(D)+4$ singular fibers.

The proof will be complete if we can show that any component of a singular fiber is a -1 curve, since then we can contract one component for any singular fiber to obtain a ruled surface $\overline{S} \to C$. Let $F = D_1 + D_2$ be a singular fiber (so D_i, $i = 1, 2$ are its rational irreducible components). By [2, Proposition VIII.3], we have $D_i^2 < 0$, and $F^2 = 0$ implies $D_1^2 + D_2^2 = -2D_1.D_2 = -2$, so $D_i^2 = -1$ for $i = 1, 2$. ■

Proposition 3.2. A generic $S \in |\mathcal{O}_{\mathbb{P}(E)}(2)|$ has no holomorphic vector fields. □

Proof. Let us first show that any holomorphic vector field on S extends to one on $\mathbb{P}(E)$. Regarding $H^0(S, T_S)$ naturally as a subspace of $H^0(S, (T_{\mathbb{P}(E)})_{|S})$, it is enough to prove that the restriction map $H^0(\mathbb{P}(E), T_{\mathbb{P}(E)}) \to H^0(S, (T_{\mathbb{P}(E)})_{|S})$ is onto. Its cokernel is contained in $H^1(\mathbb{P}(E), T_{\mathbb{P}(E)}(-S))$. We will prove that the latter group vanishes. By the exact sequence defining the relative tangent bundle

$$0 \to T_{\mathbb{P}(E)|C} \to T_{\mathbb{P}(E)} \to \pi^* T_C \to 0,$$

it is enough to prove the vanishing of the groups $H^1(\mathbb{P}(E), T_{\mathbb{P}(E)|C}(-S))$, and $H^1(\mathbb{P}(E), \pi^* T_C(-S))$. This can be verified again using [6, Exercises III.8.1, III.8.4].

It remains to be checked that there are no holomorphic vector fields on $\mathbb{P}(E)$ preserving S. The infinitesimal action of vector fields on $\mathbb{P}(E)$ on S is given by a map $\varphi\colon \mathrm{End}(E)/\mathbb{C}I \to H^0(S^2 E^*)$ (writing $\mathrm{End}/\mathbb{C}I$ for dividing out by the addition of multiples of the identity). To see this, we can represent S by $Q \in H^0(S^2 E^*)$ uniquely up to rescaling; then φ is given on $A \in \mathrm{End}(E)/\mathbb{C}I$ by $\varphi(A) = A^* Q + QA$ (regarding A, Q as matrices in a natural way). We write

$$A = \begin{pmatrix} A_0 & v_A \\ 0 & 0 \end{pmatrix}, \tag{8}$$

where $A_0 \in \mathrm{End}(\mathcal{O}_C \oplus \mathcal{O}_C(-H))$, ${}^t v_A \in H^0(\mathcal{O}_C(D) \oplus \mathcal{O}_C(D-H))$. The first 0 in the lower row means that $\mathcal{O}_C(-D)$ and $\mathcal{O}_C(H-D)$ have no global sections; the second is only the choice of a representative modulo $\mathbb{C}I$. Similarly, we can write

$$Q = \begin{pmatrix} Q_0 & v_Q \\ {}^t v_Q & q \end{pmatrix}, \tag{9}$$

where $Q_0 \in H^0(S^2(\mathcal{O}_C \oplus \mathcal{O}_C(H)))$, ${}^t v_Q \in H^0(\mathcal{O}_C(D) \oplus \mathcal{O}_C(D+H))$, and $q \in H^0(\mathcal{O}_C(2D))$. Therefore,

$$\varphi(A) = \begin{pmatrix} A_0^* Q_0 + Q_0 A_0 & A_0^* v_Q + Q_0 v_A \\ {}^t v_A Q_0 + {}^t v_Q A_0 & 2\,{}^t v_A v_Q \end{pmatrix}. \tag{10}$$

The condition $\varphi(A) = 0$ implies ${}^t v_A v_Q = 0$. A slight twist of the "free pencil trick" (see, e.g. [1, p. 126]) shows that for generic v_Q, we must have $v_A = 0$. In turn, this implies $A_0^* v_Q = 0$.

The proof will be complete if we can show that for generic v_Q, this forces $A_0 = 0$ since then $\varphi(A) = 0$ implies $A = 0$. We compute

$$A_0^* v_Q = \begin{pmatrix} \lambda & 0 \\ t & \mu \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} \lambda s_1 \\ t s_1 + \mu s_2 \end{pmatrix}, \tag{11}$$

where $t \in H^0(\mathcal{O}_C(H))$, $s_1 \in H^0(\mathcal{O}_C(D))$, and $s_2 \in H^0(\mathcal{O}_C(D+H))$ (the upper right 0 in A_0^* expressing that $\mathcal{O}_C(-H)$ has no global sections). We would then have $\lambda = 0$, and μs_2 would lie in the image of the multiplication map $H^0(\mathcal{O}_C(H)) \xrightarrow{\otimes s_1} H^0(\mathcal{O}_C(D+H))$. For generic s_1 and s_2, this is impossible for dimension reasons. ■

Acknowledgements

We thank J. Ross, R. Thomas, and G. Székelyhidi for their comments on the preliminary version of this paper. The first author is grateful to the Mathematics Department of Pavia University for kind hospitality.

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