Some recent developments in Kähler geometry

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Complex geometry

X compact complex manifold.

Local holomorphic coordinates $z_i = x_i + \sqrt{-1}y_i$.

Get tensor $J \in \mathrm{End}(TX)$ with $J(\partial_{x_i}) = \partial_{y_i}, J(\partial_{y_i}) = -\partial_{x_i}$. Note $J^2 = -I$.

Given J, split $TX \otimes \mathbb{C} = T^{1,0}X \oplus T^{0,1}X$, $\pm\sqrt{-1}$ J-eigenspaces.

Similarly split forms $\mathcal{A}^k(X) = \oplus_{p,q}\mathcal{A}^{p,q}(X)$.

Get $\partial_J = \Pi_{p+1,q}d|_{\mathcal{A}^{p,q}(X)}$, $\bar{\partial}_J = \Pi_{p,q+1}d|_{\mathcal{A}^{p,q}(X)}$.

Integrability: J comes from complex structure iff $\bar{\partial}_J^2 = 0$.

Hermitian and Kähler metrics

g Riemannian metric on X.

g Hermitian if J is g-isometry.

Then $\omega_g = g(J-, -)$ is a 2-form.

Strongest compatibility: $\nabla^g(J) = 0$.

It holds iff $d\omega_g = 0$. This is the Kähler condition.

E.g.: $\omega_{FS} = \sqrt{-1}\partial\bar{\partial}\log(\sum_i |Z_i|^2)$ on $\mathbb{P}^n$.

E.g.: $\iota^*_X\omega_{FS}$ for $\iota\colon X \hookrightarrow \mathbb{P}^n$ a smooth projective variety.

Curvature of Kähler metrics

g Riemannian so get curvature tensor Riem(g).

Ricci curvature is equivalent to Ricci form

$$\begin{aligned}\mathrm{Ric}(\omega_g) &= -\sqrt{-1}\partial\bar{\partial}\log\det(g)\\ &= -\sqrt{-1}\partial_i\partial_{\bar{j}}\log\det(g_{k\bar{l}})dz_i \wedge d\bar{z}_j.\end{aligned}$$

Scalar curvature is given by

$$s(g) = -\sqrt{-1}g^{i\bar{j}}\partial_i\partial_{\bar{j}}\log\det(g).$$

Kähler-Einstein metrics

g is Riemannian Einstein iff

$$\mathrm{Ric}(\omega_g) = \lambda \omega_g.$$

Can assume $\lambda \in \{-1, 0, 1\}$.
Taking cohomology

$$H^2(M, \mathbb{Z}) \ni c_1(X) = (2\pi)^{-1}[\mathrm{Ric}(\omega_g)] = (2\pi)^{-1}\lambda[\omega_g].$$

So X must be general type ($\lambda = -1$), Calabi-Yau ($\lambda = 0$) or Fano ($\lambda = 1$).

Complex Monge-Ampere

Use $\partial\bar{\partial}$-Lemma.

KE equation is equivalent to the CMA for Kähler potential $\varphi \in C^\infty(X)$:

$$(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{F-\lambda\varphi}\omega_0^n$$

where $\mathrm{Ric}(\omega_0) - \lambda\omega_0 = \sqrt{-1}\partial\bar{\partial}F$.

Continuity methods

If $\lambda = -1$ we consider the continuity method

$$(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_t)^n = e^{tF+\varphi_t}\omega_0^n,$$
$$\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_t > 0,\ t \in [0,1].$$

If $\lambda = 0$ we consider

$$(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_t)^n = e^{tF+c_t}\omega_0^n,$$
$$\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_t > 0,\ t \in [0,1].$$

for $t \in [0,1]$ and uniquely defined constants c_t.

Continuity methods and Yau's theorems

If $\lambda = 1$ we consider

$$(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_t)^n = e^{F - t\varphi_t}\omega_0^n,$$
$$\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_t > 0,\ t \in [0,1].$$

Theorem (Yau, 80s)

In all cases the set of times for which there is a solution is open and contains $t = 0$. It is closed iff a C^0 estimate on φ_t holds along the continuity path.

Theorem (Yau, 80s)

For $\lambda = -1, 0$, the C^0 estimate on φ_t holds along the continuity path.

The Fano case

The C^0 estimate can fail when $\lambda = 1$!

E.g.: Take $X = \mathrm{Bl}_p \mathbb{P}^2$. Then φ_t blows up for some explicit $0 < \bar{t} < 1$ along the continuity path. In fact there is no KE metric.

Theorem (Chen-Donaldson-Sun 2012; Datar-Szekelyhidi 2014)

In the Fano case, the C^0 estimate estimate along the continuity path (and so the existence of a KE metric) is a purely algebro-geometric condition, known from previous work, namely K-polystability.

Test-configurations

Fix a complex polarised variety (X, L).
(In the Fano case $(X, -K_X)$).
Let $\mathbb{C}^*$ act in the standard way on $\mathbb{C}$.

Definition

A test-configuration $(\mathcal{X}, \mathcal{L})$ for (X, L) with exponent r is a $\mathbb{C}^*$-equivariant flat morphism $\pi\colon \mathcal{X} \to \mathbb{C}$, together with a π-ample line bundle $\mathcal{L}$ and a linearisation of the action of $\mathbb{C}^*$ on $\mathcal{L}$, such that the fibre over 1 is isomorphic to $(X, L^{\otimes r})$.

Properties of test-configurations

We say that $(\mathcal{X}, \mathcal{L})$ is

- *very ample*, if $\mathcal{L}$ is π-very ample;
- a *product*, if it is isomorphic to $(X \times \mathbb{C}, L^{\otimes r} \boxtimes \mathcal{O}_{\mathbb{C}})$, where the action of $\mathbb{C}^*$ on $X \times \mathbb{C}$ is induced by a one-parameter subgroup λ of $\mathrm{Aut}(X, L)$ by $\lambda(\tau) \cdot (x, z) = (\lambda(\tau) \cdot x, \tau z)$;
- *trivial*, if it is a product and, moreover, λ is trivial;
- *normal*, if the total space $\mathcal{X}$ is normal;
- *equivariant with respect to a subgroup* $H \subset \mathrm{Aut}(X, L)$, if the action of $\mathbb{C}^*$ can be extended to an action of $\mathbb{C}^* \times H$ such that the action of $\{1\} \times H$ is the natural action of H on $(X, L^{\otimes r})$;
- in the Fano case, a test-configuration is a *special degeneration* if $\mathcal{X}$ is normal, all the fibres are klt and a positive rational multiple of $\mathcal{L}$ equals $-K_{\mathcal{X}}$.

DF invariant and L^2 norm

Let $h(k) = h^0(\mathcal{X}_0, \mathcal{L}_0^{\otimes k})$, $A_k =$ the infinitesimal generator of induced action on $H^0(\mathcal{X}_0, \mathcal{L}_0^{\otimes k})$.
Consider the quantities $w(k) = \mathrm{tr}(A_k)$, $d(k) = \mathrm{tr}(A_k^2)$. Expand

$$h(k) = h^0(X, L^{\otimes k}) = a_0 k^n + a_1 k^{n-1} + \cdots$$
$$w(k) = b_0 k^{n+1} + b_1 k^n + \cdots$$
$$d(k) = c_0 k^{n+2} + c_1 k^{n+1} + \cdots$$

One defines the Donaldson-Futaki invariant (or weight) and the L^2 norm as

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) = \frac{a_1 b_0 - a_0 b_1}{a_0^2}, \quad ||(\mathcal{X}, \mathcal{L})||_{L^2}^2 = c_0 - \frac{b_0^2}{a_0}.$$

K-(semi, poly)stability

Definition

(X, L) is

- K-semistable if $\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq 0$ for all $(\mathcal{X}, \mathcal{L})$;
- K-stable if $\mathrm{DF}(\mathcal{X}, \mathcal{L}) > 0$ for all $(\mathcal{X}, \mathcal{L})$ with normal total space;
- K-polystable if for all $(\mathcal{X}, \mathcal{L})$ with normal total space we have $\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq 0$, with equality if and only if $(\mathcal{X}, \mathcal{L})$ is a product.

Some dislike the word “polystable" and just say “stable (with automorphisms)".

Beyond KE

KE constrains $[\omega_g]$ (except for CYs). Moving beyond this, one looks at cscK metrics

$$s(g) = \hat{s}$$

and more generally at extremal metrics

$$\bar{\partial}\nabla^{1,0} s(g) = 0.$$

They are critical points for all the functionals

$$\int (s(g))^2, \int || \mathrm{Ric}(g) ||_g^2, \int || \mathrm{Riem}(g) ||_g^2.$$

Continuity method

There is no reduction to CMA!
So fix $\alpha > 0$, $[\alpha] = [\omega_g]$ and look at twisted cscK equation (Fine, S., ...)

$$s(g) - \Lambda_{\omega_g}\alpha = c$$

and continuity path for $\omega_{g_t} = \omega_g + \sqrt{-1}\partial\bar{\partial}\varphi_t$

$$ts(g_t) - (1-t)\Lambda_{\omega_{g_t}}\alpha = c_t,$$
$$\alpha \in [\omega_g],\ t \in [0,1].$$

Theorem (Chen-Cheng 2018)

The set of times for which there is a solution is open and contains $t = 0$. It is closed iff a C^0 bound on φ_t holds along the continuity path.

K-stability and canonical metrics

The C^0 bound can certainly fail!
E.g. $X = \mathrm{Bl}_p \mathbb{P}^2$ with any Kähler class.
For Hodge classes $[\omega_g] = c_1(L)$ is the C^0 bound still algebro-geometric?

Theorem (Donaldson)

If there is a Kähler metric g in the class $c_1(L)$ with constant scalar curvature $s(g) = -g^{i\bar{j}}\partial_i\partial_{\bar{j}} \log\det g_{k\bar{l}}$ then (X, L) is K-semistable.

Theorem (S.)

If there is a Kähler metric in the class $c_1(L)$ with constant scalar curvature and $\mathrm{Aut}(X, L)/\mathbb{C}^*$ is discrete then (X, L) is K-stable.

K-stability and canonical metrics

The metric g is called extremal if $\nabla^{1,0}s(g)$ is holomorphic (Euler-Lagrange for $\int s^2(g)$).
There is a formal (Futaki-Mabuchi) inner product on linearised $\mathbb{C}^*$-actions.
Let $T \subset \mathrm{Aut}(X, L)$ be a maximal algebraic torus.

Theorem (S., Szekelyhidi)

If there is a Kähler metric in $c_1(L)$ which is extremal then we have

$$\mathrm{DF}((\mathcal{X}, \mathcal{L})^{\perp}_{T}) > 0$$

for all T-equivariant $(\mathcal{X}, \mathcal{L})$ with normal total space.

$(\mathcal{X}, \mathcal{L})^{\perp}_{T}$ denotes the orthogonal complement with respect to the formal Futaki-Mabuchi inner product.
$\mathrm{DF}((\mathcal{X}, \mathcal{L})^{\perp}_{T})$ is also called the relative DF.

Admissible filtrations

Consider the homogeneous coordinate ring

$$R = R(X, L) = \bigoplus_{k \geq 0} R_k = \bigoplus_{k \geq 0} H^0(X, L^{\otimes k}).$$

Definition

We define a *filtration* χ of R to be sequence of vector subspaces

$$H^0(X, \mathcal{O}) = F_0 R \subset F_1 R \subset \cdots$$

which is

- *exhaustive*: for every k there exists a $j = j(k)$ such that $F_j R_k = H^0(X, L^{\otimes k})$,
- *multiplicative*: $(F_i R_l)(F_j R_m) \subset F_{i+j} R_{l+m}$,
- *homogeneous*: $f \in F_i R$ then each homogeneous piece of f is in $F_i R$.

Rees and graded algebra

Definition

Let χ be a filtration. The corresponding *Rees algebra* is

$$\mathrm{Rees}(\chi) = \bigoplus_{i \geq 0} F_i R\, t^i$$

The graded modules are

$$\mathrm{gr}_i(H^0(X, L^{\otimes k})) = F_i(H^0(X, L^{\otimes k}))/F_{i-1}(H^0(X, L^{\otimes k}))$$

The graded algebra is

$$\mathrm{gr}(\chi) = \bigoplus_{k,i \geq 0} \mathrm{gr}(H^0(X, L^k))$$

The Rees algebra is a subalgebra of $R[t]$.

Filtrations and test-configurations

Definition

A filtration is called finitely generated if its Rees algebra is finitely generated.

If χ is finitely generated then

$$(\mathrm{Proj}(\mathrm{Rees}(\chi)), \mathcal{O}(1))$$

is a test-configuration for (X, L), with central fibre $(\mathrm{Proj}(\mathrm{gr}(\chi)), \mathcal{O}(1))$.

Theorem (Witt Nystrom, Szekelyhidi)

K-(semi, poly)stability can be checked on test-configurations of this form.

DF for general filtrations

There is a canonical notion of finitely-generated approximations $\chi^{(i)}$, and one defines

$$\mathrm{DF}(\chi) = \liminf_{i\to\infty} \mathrm{DF}(\chi^{(i)}), \quad ||\chi||_{L^2} = \liminf_{i\to\infty} ||\chi^{(i)}||_{L^2}.$$

Another important notion is the asymptotic Chow weight

$$\mathrm{Chow}_{\infty}(\chi) = \liminf_{i\to\infty} \mathrm{Chow}(\chi^{(i)}),$$

where

$$\mathrm{Chow}(\chi^{(i)}) = \frac{ib_0}{a_0} - \frac{w_{\chi^{(i)}}(i)}{h_{\chi^{(i)}}(i)}.$$

It is an open problem to understand how $\mathrm{DF}(\chi)$, $\mathrm{Chow}_{\infty}(\chi)$ are related in general.

General filtrations and canonical metrics

Theorem (Szekelyhidi)

If there is a Kähler metric in $c_1(L)$ with constant scalar curvature and $\mathrm{Aut}(X, L)/\mathbb{C}^*$ is discrete then we have $\mathrm{DF}(\chi) > 0$ for all χ with $||\chi||_{L^2} > 0$.

It is not known if this is actually a stronger obstruction.
It is also not easy to construct non-finitely generated filtrations which destabilise (at least conjecturally).
If $\mathrm{Aut}(X, L)$ is non-reductive, there is a canonical (Loewy) filtration which is probably not finitely generated in general, and which conjecturally destabilises (it does in many examples): this is due to Codogni-Dervan.

General filtrations and canonical metrics

Open:

1. remove assumption on $\mathrm{Aut}(X, L)$;
2. prove an analogue of this result for extremal metrics with non-constant scalar curvature.

Limits of filtrations

(X, L) polarised variety, $T \subset \mathrm{Aut}(X, L)$ torus, $\lambda \colon \mathbb{C}^* \to T$ 1PS.
$(\mathcal{X}, \mathcal{L})$ test-configuration, $\mathcal{F}$ corresponding (admissible) filtration of $R(X, L)$.
$\mathcal{F}_\lambda = \lim_{\tau \to 0} \lambda(\tau) \cdot \mathcal{F}$ is λ-equivariant filtration.

Theorem

$\mathcal{F}_\lambda$ is a an admissible filtration, and we have

$$\mathrm{Chow}_\infty(\mathcal{F}_\lambda) \leq \mathrm{DF}(\mathcal{F}).$$

Iterating on a basis of 1PS for T get T-equivariant, admissible $\mathcal{F}_T$ with

$$\mathrm{Chow}_\infty(\mathcal{F}_T) \leq \mathrm{DF}(\mathcal{F}).$$

Limits of filtrations

(X, L) normal, $(\mathcal{X}, \mathcal{L})$ normal, nontrivial.

Proposition

Suppose $\mathcal{F}_T$ is finitely generated, corresponding to $(\mathcal{X}', \mathcal{L}')$. Then the normalisation $(\widehat{\mathcal{X}', \mathcal{L}'})$ is a nontrivial, T-equivariant test-configuration with

$$\mathrm{DF}(\widehat{\mathcal{X}', \mathcal{L}'}) \leq \mathrm{DF}(\mathcal{X}, \mathcal{L}).$$

Moreover if $(\widehat{\mathcal{X}', \mathcal{L}'})$ is a product then strict inequality holds.

Limits of filtrations

Lemma

$\mathcal{F}_T$ is not finitely generated in general (even for $X = \mathbb{P}^1$).

Remark

This gives another class of non-finitely generated filtrations occurring naturally in K-stability.

Example

Consider the polynomial algebra $\mathbb{C}[t][x, y]$ over the ring $\mathbb{C}[t]$ and let A denote the $\mathbb{C}[t]$-subalgebra generated by

$$t(x+y), txy, txy^2, t^2y.$$

Then $A \subset R[t]$ is the Rees algebra of a homogeneous, multiplicative, exhaustive finitely generated filtration $\mathcal{F}$ of the homogeneous coordinate ring $R = \mathbb{C}[x, y]$ of the projective line $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$.

Consider the 1-parameter subgroup $\lambda\colon \mathbb{C}^* \to SL(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)))$ acting by

$$\lambda(\tau)\cdot x = \tau^{-1}x, \quad \lambda(\tau)\cdot y = \tau y.$$

The limit $\mathcal{F}_T$ is not finitely generated (adapted from Robbiano-Sweedler).

A result of Datar-Szekelyhidi

(M, K_M^{-1}) Fano manifold, $G \subset \mathrm{Aut}(M)$ compact.

Theorem (Datar-Szekelyhidi)

If (M, K_M^{-1}) is K-polystable with respect to G-equivariant special degenerations then it is Kähler-Einstein.

In particular (M, K_M^{-1}) is K-polystable with respect to *all* (normal) test-configurations (by important results of Berman and Li-Xu). Combining with results of Ilten-Süss gives the first higher-dimensional, non-toric examples where K-polystability can be checked explicitly.

Remark

Datar-Szekelyhidi's theorem follows from their more fundamental construction of Kähler-Einstein metrics along the smooth continuity method.

Equivariance

Question

Can one prove the equivariance property for general polarised varieties? At least for a (maximal) torus?