Stability conditions and Frobenius manifolds

Jacopo Stoppa (SISSA)

ICTP, Trieste
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Frobenius manifolds

Definition

A Frobenius manifold is a complex manifold M such that the fibres of the holomorphic tangent bundle TM are endowed with a commutative, associative product $\circ$. Moreover we assume that there are a (unit) field e, a (Euler) field E, and a nondegenerate holomorphic quadratic form g_M on the fibres of TM (the metric) satisfying further constraints.

Frobenius manifolds

- The metric g_M is flat. Denote its Levi-Civita connection by ∇^{g_M}.
- Introducing a Higgs field C^M on M by $C^M_X(Y) = -X \circ Y$, we have $\nabla^{g_M}(C^M) = 0$.
- The unit field e is flat, i.e. $\nabla^{g_M}(e) = 0$.
- Taking Lie derivatives along the Euler field we have $\mathcal{L}_E(\circ) = \circ$ and $\mathcal{L}_E(g_M) = (2 - d)g_M$ for some $d \in \mathbb{C}$ (the conformal dimension of M).
- We have $g_M(C^M_X Y, Z) = g_M(Y, C^M_X Z)$, that is the metric is compatible with the multiplication $\circ$.

Frobenius manifolds: sources

Natural sources of Frobenius manifolds are

- quantum cohomology of projective varieties;
- deformation theory of singularities.

Here we show that in some cases certain spaces of stability conditions in the sense of Bridgeland also give rise to (families of semisimple) Frobenius manifolds. This is based on ideas of Joyce and Bridgeland-Toledano Laredo.

Sketch of correspondence

$\mathcal{D} = \mathcal{D}(Q, W)$ 3CY triangulated category attached to quiver with potential, heart $\mathcal{A}(Q, W)$. Mutation equivalent $(Q', W') \sim (Q, W)$ give $\mathcal{D}(Q', W') \cong \mathcal{D}(Q, W)$.

(1) **Local aspects:** by using (very little) DT theory we can induce a family of semisimple Frobenius manifold structures on Stab($\mathcal{A}(Q, W)$), so a family of Frob structures on an open set in $\mathbb{C}^n$ with canonical coords u_i;

(2) **Global aspects:** the choice of a different heart $\mathcal{A}(Q', W')$ (i.e. mutation) gives another semisimple Frobenius manifold structure on $(\mathbb{C}^n)_{u_1,\dots,u_n}$. We are especially interested in **complete** examples where the two are always related by analytic continuation, i.e. braid group action on corresponding monodromy data.

Local aspects: Frobenius type structures

In deformation theory the construction of Frobenius manifolds often goes via constructing **Frobenius type structures** on auxiliary bundle, then a suitable **section**. Our case is somewhat similar.

Definition (Hertling)

Let $K \to M$ be a holomorphic vector bundle on a complex manifold. A Frobenius type structure on K is given by a collection $(\nabla^r, C, \mathcal{U}, \mathcal{V}, g)$ of *holomorphic* objects with values in K, where

- ∇^r is a flat connection,
- C is a Higgs field, i.e. a 1-form with values in endomorphisms such that $C \wedge C = 0$,
- $\mathcal{U}, \mathcal{V}$ are endomorphisms,
- g is a quadratic form,

satisfying further constraints.

Constraints:

$$\begin{aligned} \nabla^r(C) &= 0, \\ [C, \mathcal{U}] &= 0, \\ \nabla^r(\mathcal{V}) &= 0, \\ \nabla^r(\mathcal{U}) - [C, \mathcal{V}] + C &= 0 \end{aligned}$$

Moreover we require that g is covariant constant with respect to ∇^r, and that $C, \mathcal{U}$ are symmetric and $\mathcal{V}$ is skew-symmetric with respect to g.

Joyce functions and Frob type structures

We claim that there is a canonical Frobenius type structure on Stab($\mathcal{D}$). This is equivalent to work of Joyce, Bridgeland-Toledano Laredo, Kontsevich-Soibelman. Unfortunately this structure is **infinite-dimensional** and **ill-defined**!

Joyce functions and Frob type structures

Choose $K \to \mathrm{Stab}(\mathcal{D})$ the trivial bundle with fibre $\mathbb{C}[K(\mathcal{D})]$ (regarded as Poisson algebra via Euler pairing).
Let Z be the central charge.
Set

$$f^{\alpha}(Z) = \sum_{\alpha_1+\cdots+\alpha_k=\alpha,\, Z(\alpha_i)\neq 0} c(\alpha_1,\ldots,\alpha_k) J_k(Z(\alpha_1),\ldots,Z(\alpha_k)) \prod_{i=1}^{k} \mathrm{DT}_{\mathcal{A}}(\alpha_i, Z),$$

$$\Phi(Z) = \sum_{\alpha\neq 0} f^{\alpha}(Z) x_{\alpha}.$$

Joyce functions and Frob type structures

Choose

- a connection

$$\nabla^r = d + \sum_{\alpha \neq 0} \operatorname{ad} f^\alpha(Z) x_\alpha \frac{dZ(\alpha)}{Z(\alpha)},$$

- a 1-form with values in endomorphisms

$$C = -dZ,$$

- endomorphisms

$$\mathcal{U} = Z,$$
$$\mathcal{V} = \operatorname{ad} \Phi(Z) = \operatorname{ad} \sum_{\alpha \neq 0} f^\alpha(Z) x_\alpha,$$

- a quadratic form

$$g(x_\alpha, x_\beta) = \delta_{\alpha\beta}.$$

Proposition

Work formally (ignoring convergence problems). The choices above give a Frobenius type structure on $K \to \mathrm{Stab}(\mathcal{D})$.

We can turn this into a rigorous result if we fix a heart $\mathcal{A}(Q, W)$ with simples $[S_i]$ and grade everything by length:

$$\alpha = \sum_{i=1}^{n} a_i[S_i], \quad [\alpha]_{\pm} = \sum_{i=1}^{n} [a_i]_{\pm}[S_i],$$

$$f^{\alpha}_{\mathbf{s}}(Z) = \sum_{\alpha_1+\cdots+\alpha_k=\alpha,\, Z(\alpha_i)\neq 0} c(\alpha_1, \ldots, \alpha_k) J_k(Z(\alpha_1), \ldots, Z(\alpha_k))$$

$$\prod_{i=1}^{k} \mathbf{s}^{[\alpha_i]_+ - [\alpha_i]_-} \, \mathrm{DT}_{\mathcal{A}}(\alpha_i, Z),$$

$$\Phi_{\mathbf{s}}(Z) = \sum_{\alpha\neq 0} f^{\alpha}_{\mathbf{s}}(Z) x_{\alpha} \in \mathbb{C}[K(\mathcal{A})][[\mathbf{s}]].$$

Hertling's construction

Proposition

The choices above give a well-defined $\mathbb{C}[[\mathbf{s}]]$*-linear Frob type structure on* $K \to \mathrm{Stab}(\mathcal{A}(Q, W))$*, trivial bundle with fibre* $K(\mathcal{A})[[\mathbf{s}]]$*.*

Remark

Note there is a special point $\mathbf{s}^* = \{s_i = 1\}$*.*

Now we consider applying a general construction due to Hertling for a Frob type $K \to M$, depending of the choice of a good section $\zeta \colon M \to K$.
Note every section ζ gives a map $-C_\bullet(\zeta) \colon TM \to K$, i.e. minus derivative of the section ζ along the Higgs field.

Hertling's construction

Theorem (Hertling)

Suppose ζ is a global section of $K \to M$ such that

- *it is a flat section with respect to the flat connection of the Frobenius type structure, $\nabla^r(\zeta) = 0$,*
- *it is homogeneous with respect to the endomorphism $\mathcal{V}$, i.e. we have $\mathcal{V}(\zeta) = \frac{d}{2}\zeta$ for some $d \in \mathbb{C}$,*
- *the map $-C_\bullet(\zeta)$ is an isomorphism.*

Then the pullback of $(\nabla^r, C, \mathcal{U}, \mathcal{V}, g)$ along the map $-C_\bullet(\zeta)$ gives a Frobenius manifold structure on M with unit field given by the pullback of the section ζ and with conformal dimension $2 - d$.

Good sections

Let $K(\zeta) = \mathrm{im}(-C_\bullet(\zeta)) = \mathrm{im}(dZ(\zeta))$.

Proposition

The following are equivalent:

- *the map* $-C_\bullet(\zeta)\colon T\,\mathrm{Stab}(\mathcal{A}) \to K(\zeta)$ *is an isomorphism,* ζ *is a section of* $K(\zeta)$*, and* $K(\zeta)$ *is preserved by* $C = -dZ$ *and* $\mathcal{U} = Z$*;*
- *the section has the form*

$$\zeta(Z) = \sum_i c_i(Z, \mathbf{s}) x_{\alpha_i}$$

where α_i *are a basis for* $K(\mathcal{A}) \otimes \mathbb{R}$*.*

From now on work with this finite dim $K(\zeta)$ with section ζ. For now it has some of the ingredients of a Frob type, but not all!

Good sections

Note $K(\zeta)$ as above comes with the projected structure

$$\pi^{\zeta} \circ \nabla^{r}_{\mathbf{s}}|_{K(\zeta)}, C|_{K(\zeta)}, \mathcal{U}|_{K(\zeta)}, \pi^{\zeta} \circ \mathcal{V}_{\mathbf{s}}|_{K(\zeta)}, g|_{K(\zeta)}$$

where $\pi^{\zeta} \colon K \to K(\zeta)$ is orthogonal projection onto $K(\zeta)$ with respect to the quadratic form g. Denote this by

$$(K(\zeta), \nabla^{r,\zeta}_{\mathbf{s}}, C, \mathcal{U}, \mathcal{V}^{\zeta}_{\mathbf{s}}, g).$$

In general, it is not a Frobenius type structure! The problem is that $\nabla^{r,\zeta}_{\mathbf{s}}$ is not flat and $\mathcal{V}^{\zeta}_{\mathbf{s}}$ is not covariant constant with respect to it.

Good sections

Consider the following conditions on the basis underlying the section ζ.
Fix $i, j = 1, \ldots, n$. Then either $\alpha_j - \alpha_i$ has length at least 3, or

- for all $k \neq i, j$ we have

$$\langle \alpha_j, \alpha_i \rangle \langle \alpha_j - \alpha_k, \alpha_k - \alpha_i \rangle = \langle \alpha_j, \alpha_k \rangle \langle \alpha_k, \alpha_i \rangle,$$

  and

- for all nontrivial decompositions $\alpha_j - \alpha_i = \beta + \gamma$ with β, γ not equal to $\alpha_j - \alpha_k$, $\alpha_k - \alpha_i$ the product

$$\langle \beta, \gamma \rangle f^{\beta}_{\mathbf{s}}(Z) f^{\gamma}_{\mathbf{s}}(Z)$$

  is at least cubic in **s**. This holds automatically if $\alpha_j - \alpha_i = \pm[S_a] \pm [S_b]$ implies $[S_a] = \pm(\alpha_j - \alpha_k)$, $[S_b] = \pm(\alpha_k - \alpha_i)$ for some k.

Good sections

We call a section ζ satisfying these conditions for all i, j a good section, and the underlying basis α_i of $K(\mathcal{A}) \otimes \mathbb{R}$ a good basis.

Proposition

Let ζ be a good section. Then the curvature $F(\nabla^{r,\zeta}_{\mathbf{s}})$ and the covariant derivative $\nabla^{r,\zeta}_{\mathbf{s}}(\mathcal{V}^{\zeta}_{\mathbf{s}})$ vanish modulo terms which are at least cubic in $\mathbf{s}$.

Corollary

If ζ is a good section, the structure

$$(K(\zeta), \nabla^{r,\zeta}_{\mathbf{s}}, C, \mathcal{U}, \mathcal{V}^{\zeta}_{\mathbf{s}}, g).$$

is a Frobenius type structure modulo $(\mathbf{s})^3$, i.e. a quadratic jet of Frob type structures.

Good bases: examples

- The triangular basis

$$\alpha_i = \sum_{j=i}^{n} [S_j]$$

is good for the Dynkin quiver A_n with all possible orientations.

- The basis

$$\alpha_1 = S_1 + S_3, \quad \alpha_2 = -[S_2] + [S_3], \quad \alpha_3 = [S_3]$$

is good for the oriented affine Dynkin quiver $\widetilde{A_2}$.

Monodromy

One notion of monodromy attached to a Frobenius type structure is the generalised monodromy (Stokes data) of any connection in the family

$$\nabla^{\zeta}_{\mathbf{s}}(Z) = d + \left(\frac{\mathcal{U}(Z)}{z^2} - \frac{\mathcal{V}^{\zeta}_{\mathbf{s}}(Z)}{z}\right) dz$$

Corollary

In our case the family $\nabla^{\zeta}_{\mathbf{s}}(Z)$ has constant generalised monodromy (product of Stokes factors) modulo $(\mathbf{s})^{\mathbf{3}}$.

Monodromy

- Stokes rays

$$\ell_{ij}(Z) = \mathbb{R}_{>0} Z(\alpha_i - \alpha_j) \subset \mathbb{C}^*$$

for $i \neq j$,

- Stokes factors

$$\mathcal{S}_{\ell_{ij}}(Z) = I - (-1)^{\langle \alpha_i, \alpha_j \rangle} \langle \alpha_i, \alpha_j \rangle \, \mathrm{DT}_{\mathcal{A}}(\alpha_i - \alpha_j, Z) \mathbf{s}^{\alpha_i - \alpha_j} E_{ij}$$

if $\alpha_i - \alpha_j$ is the class of a simple object or the sum of classes of simple objects (or shifts),

- Stokes matrix

$$\mathcal{S} = \prod_{\ell \subset \bar{\mathcal{H}}}^{\rightarrow} \mathcal{S}_{\ell}(Z) \in GL(K(\zeta)[[\mathbf{s}]]/\mathbf{s}^3)$$

Lifting

We wish to lift $(K(\zeta), \nabla_{\mathbf{s}}^{r,\zeta}, C, \mathcal{U}, \mathcal{V}_{\mathbf{s}}^{\zeta}, g)$ to a genuine Frobenius type structure, i.e. lift to formal power series in **s** so that the Frobenius type constraints hold to all orders and the fps are convergent for sufficiently small $||\mathbf{s}||$.

Proposition

*There exist (many) lifts of $(K(\zeta), \nabla_{\mathbf{s}}^{r,\zeta}, C, \mathcal{U}, \mathcal{V}_{\mathbf{s}}^{\zeta}, g)$ to a family of genuine Frobenius type structures, parametrised by **s** for sufficiently small $||\mathbf{s}||$. They are parametrised by lifts $\tilde{\mathcal{S}}$ of $\mathcal{S}$. There is a canonical one given by the trivial lift.*

Pullback

We pull this family of Frob type structures back to the tangent bundle via $-C = dZ$.

If ζ is an eigenvalue of $\mathcal{V}$ we get a Frobenius manifold with flat identity and Euler field. $\mathcal{V}$ acts generically semisimply on flat sections of $K(\zeta)$ with spectrum locally determined by Stokes matrix.

So at least locally $\mathcal{V}(\zeta) = \frac{d}{2}\zeta$ is ODE on the coefficients $c_i(Z, \mathbf{s})$ which we can always solve.

Pullback

- Canonical coordinates are given by

$$u_i = Z(\alpha_i),$$

- the Euler field is

$$E = \sum_i Z(\alpha_i)\partial_{Z(\alpha_i)}$$

- the flat metric is given by

$$g_{\mathbf{s}}(u) = \sum_i c_i^2(Z, \mathbf{s}) du_i^2$$

- the conformal dimension is $2 - d(\mathbf{s})$.
- Stokes rays and matrix are the same.

Example

$$\begin{cases} \alpha_1 = [S_1] + [S_2] \\ \alpha_2 = \qquad\quad [S_2] \end{cases}$$

$$0 \longrightarrow 1 \, , \quad \tilde{\mathcal{S}} = \begin{pmatrix} 1 & -s_1 \\ & 1 \end{pmatrix}$$

$$0 \longleftarrow 1 \, , \quad \tilde{\mathcal{S}} = \begin{pmatrix} 1 & s_1 \\ & 1 \end{pmatrix}$$

Example

Choose $(Q, W) = (A_n, 0)$ with standard orientation.
Good basis: $\alpha_i = \sum_{r=i}^{n} [S_r]$.

$$\mathcal{S}_{i,i+1} = I - (-1)^{\langle [S_i],[S_{i+1}]\rangle} \langle [S_i], [S_{i+1}]\rangle s_i E_{i,i+1}$$
$$= I - s_i E_{i,i+1}.$$

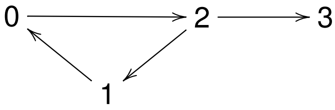
$$\tilde{\mathcal{S}} = \prod_{j=1}^{n-1} \mathcal{S}_{n-j,n-j+1} = I - \sum_{i=1}^{n-1} s_i E_{i,i+1}$$
$$= \begin{pmatrix} 1 & -s_1 & & & \\ & 1 & -s_2 & & \\ & & \ddots & & \\ & & & 1 & -s_n \\ & & & & 1 \end{pmatrix}.$$

Example

$$0 \longleftarrow 1 \longrightarrow 2 \longrightarrow 3$$

$$\alpha_i = \sum_{r=i}^{n} [S_r] \qquad \tilde{\mathcal{S}} = \begin{pmatrix} 1 & s_1 & -s_1 s_2 & 0 \\ 0 & 1 & -s_2 & 0 \\ 0 & 0 & 1 & -s_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

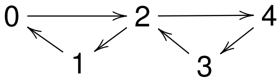
Example



$$\begin{cases} \alpha_1 = [S_1] + \qquad\quad [S_3] + [S_4] \\ \alpha_2 = \qquad\quad -[S_2] + [S_3] + [S_4] \\ \alpha_3 = \qquad\qquad\qquad [S_3] + [S_4] \\ \alpha_4 = \qquad\qquad\qquad\qquad\quad [S_4] \end{cases}$$

$$\tilde{\mathcal{S}} = \begin{pmatrix} 1 & -s_1 s_2 & -s_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & s_2 & 1 & -s_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

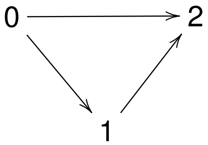
Example



$$\begin{cases} \alpha_1 = [S_1] + [S_3] + [S_5] \\ \alpha_2 = -[S_2] + [S_3] + [S_5] \\ \alpha_3 = [S_3] + [S_5] \\ \alpha_4 = -[S_4] + [S_5] \\ \alpha_5 = [S_5] \end{cases}$$

$$\tilde{S} = \begin{pmatrix} 1 & -s_1 s_2 & -s_1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & s_2 & 1 & -s_3 s_4 & -s_3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & s_4 & 1 \end{pmatrix}$$

Example



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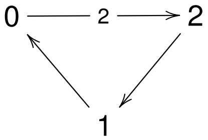
$$\tilde{\mathcal{S}} = \begin{pmatrix} 1 & -s_1 s_2 & s_1 \\ 0 & 1 & 0 \\ 0 & s_2 & 1 \end{pmatrix}$$

Example



$$\begin{cases} \alpha_1 = [S_1] + [S_2] + [S_3] \\ \alpha_2 = \qquad\quad [S_2] + [S_3] \\ \alpha_3 = \qquad\qquad\qquad [S_3] \end{cases}$$

$$\tilde{S} = \begin{pmatrix} 1 & -s_1 & -s_1 s_2 \\ 0 & 1 & s_2 \\ 0 & 0 & 1 \end{pmatrix}$$