A Higgs term for the constant scalar curvature equation in Kähler geometry

Carlo Scarpa and Jacopo Stoppa

SISSA

40!

- Many geometers are still obsessed with the classical equations of motion of gauge theory and gravity.
- Mostly with the wrong (Riemannian) signature and the wrong (compact) kind of space.
- Here we make exactly these mistakes, with even worse spaces (compact complex manifolds). But at least we can now use “complex variables" to understand things much better (Atiyah, Hitchin, Donaldson...).
- Plus a few misleading but pretty pictures.

Complex manifolds

M compact complex manifold: local *holomorphic* coordinates $z_i = x_i + \sqrt{-1}y_i$.

Get tensor $J \in \text{End}(TM)$ with $J(\partial_{x_i}) = \partial_{y_i}, J(\partial_{y_i}) = -\partial_{x_i}$. Note $J^2 = -I$.

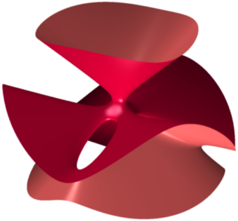


Figure: Clebsch cubic

Hermitian and Kähler metrics

g Riemannian metric on M.

g Hermitian if J is g-isometry.

Then $\omega_g = g(J-, -)$ is skew, i.e. a 2-form.

Strongest compatibility: $\nabla^g(J) = 0$.

It holds iff ω_g is symplectic, $d\omega_g = 0$. This is the Kähler condition.

Hermitian Yang-Mills

(M, g) compact Kähler.

$E \to M$ a *holomorphic* vector bundle, with Hermitian metric h.

$\Rightarrow$ we get a canonical “gauge field", the *Chern connection* $A(h)$.

“Field strength" = curvature $F(h) = F(A(h))$, matrix-valued 2-form.

Hermitian Yang-Mills equation:

$$F(h) = \lambda \, \mathrm{Id} \, \omega_g.$$

Theorem (Donaldson, Uhlenbeck-Yau)

Solvable iff E is “stable".

Geometric interpretation

(Unitary) gauge transformations $\mathcal{G}$: fibrewise isometries of (E, h).

$\mathscr{A}$ = complex structures on E = $\bar{\partial}$-operators $\bar{\partial}_A$ ("half-connections").

$\mathcal{G}$ acts on $\mathscr{A}$, $g \cdot \bar{\partial}_A = g^{-1} \circ \bar{\partial}_A \circ g$.

Geometric interpretation

Theorem (Atiyah-Bott)

The space $\mathscr{A}$ is (∞-dim'l) symplectic. For each infinitesimal gauge transformation ξ, set

$$m_{AB}(\bar{\partial}_A)(\xi) = \int_M [F(\bar{\partial}_A) - \lambda \mathrm{I}\,\omega_g](\xi) \wedge \omega_g^{n-1}.$$

Then $m_{AB}(-)(\xi)$ is a Hamiltonian function for ξ.

That is: $dm_{AB}(-)(\xi)$ is dual under $\omega_{\mathscr{A}}$ to the vector field on $\mathscr{A}$ generated by ξ.

One says m_{AB} **is a "moment" or "momentum" map** (compare to $SO(3)$ acting on phase space!).

Corollary

The HYM equation on metrics h

$$F(h) = \lambda \, \mathrm{Id} \, \omega_g$$

is precisely the **symplectic reduction equation**

$$m_{AB}(\bar{\partial}_A) = 0$$

along **orbits of the complexification $\mathcal{G}^{\mathbb{C}}$.**

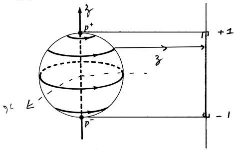


Figure: Moment map

Adding a Higgs field

Hitchin does HYM with a *Higgs field* ϕ, matrix-valued *1-form*.
$\Rightarrow$ harmonic bundle equations:

$$F(h) + [\phi, \phi^{*_h}] = \lambda \,\mathrm{Id}\, \omega_g$$
$$\bar{\partial}\phi = 0.$$

Theorem (Hitchin, Donaldson, Corlette, Simpson)

Solvable iff E is "stable relative to ϕ".

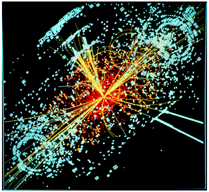


Figure: A Higgs doing something higgsy, unrelated and misleading.
By Lucas Taylor / CERN, CC BY-SA 3.0

Geometric interpretation

Cotangent space $T^*\mathscr{A}$ is both Kähler and *holomorphic symplectic*.

$\Rightarrow$ real and complex symplectic forms $\omega_{T^*\mathscr{A}}, \Omega_{T^*\mathscr{A}}$.

The two structures interact in the nicest possible way (**hyperkähler** condition). Quite easy because $\mathscr{A}$ is affine.

Promote gauge action to $\mathcal{G} \curvearrowright T^*\mathscr{A}$, preserving $\omega_{T^*\mathscr{A}}, \Omega_{T^*\mathscr{A}}$.

Geometric interpretation

Theorem (Hitchin)

The action $\mathcal{G} \curvearrowright T^*\mathscr{A}$ admits moment maps $m_{\mathbb{R}}$, $m_{\mathbb{C}}$ with respect to both $\omega_{T^*\mathscr{A}}$, $\Omega_{T^*\mathscr{A}}$. The harmonic bundle equations

$$F(h) + [\phi, \phi^{*_h}] = \lambda \, \mathrm{Id} \, \omega_g$$
$$\bar{\partial}\phi = 0.$$

are precisely the equations of hyper-symplectic reduction

$$m_{\mathbb{R}}(h, \phi) = 0$$
$$m_{\mathbb{C}}(h, \phi) = 0$$

along orbits of $\mathcal{G}^{\mathbb{C}}$.

Starting point of **nonabelian Hodge theory** (Hitchin, Simpson...)

Curvature of Kähler metrics

Kähler g is Riemannian, so get curvature tensor $\mathrm{Riem}(g)$.

Ricci curvature is equivalent to Ricci form

$$\begin{aligned}\mathrm{Ric}(\omega_g) &= -\sqrt{-1}\partial\bar{\partial}\log\det(g)\\ &= -\sqrt{-1}\partial_i\partial_{\bar{j}}\log\det(g_{k\bar{l}})dz_i\wedge d\bar{z}_j.\end{aligned}$$

Scalar curvature is given by

$$s(g) = -\sqrt{-1}g^{i\bar{j}}\partial_i\partial_{\bar{j}}\log\det(g).$$

Kähler-Einstein metrics

g is Riemannian Einstein iff

$$\mathrm{Ric}(\omega_g) = \lambda \omega_g.$$

Can assume $\lambda \in \{-1, 0, 1\}$.

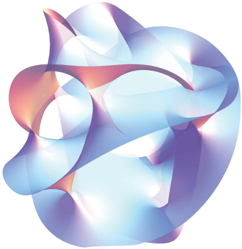


Figure: A solution for $\lambda = 0$

Taking cohomology

$$H^2(M, \mathbb{Z}) \ni c_1(X) = (2\pi)^{-1}[\mathrm{Ric}(\omega_g)] = (2\pi)^{-1}\lambda[\omega_g].$$

So M must be general type ($\lambda = -1$), Calabi-Yau ($\lambda = 0$) or Fano ($\lambda = 1$).

But most manifolds do not fit into one of these categories!
E.g. not preserved by “holomorphic surgery".

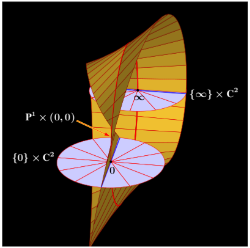


Figure: Holomorphic surgery (Blowup)

Beyond KE

KE constrains $[\omega_g]$ (except for CYs). Moving beyond this, one looks at **cscK metrics**

$$s(g) = \hat{s}$$

and more generally at extremal metrics

$$\bar{\partial}\nabla^{1,0}s(g) = 0.$$

They are critical points for all the functionals

$$\int_M (s(g))^2,\ \int_M ||\,\mathrm{Ric(g)}\,||^2_g,\ \int_M ||\,\mathrm{Riem(g)}\,||^2_g.$$

Geometric interpretation

Gauge group $\mathcal{G} = \mathrm{Ham}(M, \omega_0)$.

"$\bar{\partial}$-operators" $\mathscr{J}$ = compatible almost complex structures J.

$\mathcal{G}$ acts on $\mathscr{J}$ by pushforward.

Theorem (Quillen, Donaldson, Fujiki)

The space $\mathscr{J}$ is (∞ dim'l) symplectic. For all $h \in C_0^\infty(M) = \mathfrak{ham}(M, \omega_0)$ set

$$\mu(J)(h) = \int_M (s(g_J) - \hat{s})h\frac{\omega_0^n}{n!}.$$

Then $\mu(-)(h)$ is a Hamiltonian function for h.

In other words scalar curvature is a moment map for the action $\mathcal{G} \curvearrowright (\mathscr{J}, \omega_{\mathscr{J}})$.

Geometric interpretation

Corollary (Donaldson)

The cscK equation on metrics g, with $[\omega_g]$ fixed

$$s(g) = \widehat{s}$$

is precisely the **symplectic reduction equation**

$$\mu(g) = 0$$

along **orbits of the infinitesimal complexification $\mathfrak{ham}^{\mathbb{C}}$.**

So one expects **stability criteria for existence.**

Adding a Higgs field: geometry of $T^* \mathscr{J}$

We extend work of Donaldson for complex curves (Riemann surfaces), with a different approach.

Theorem (Scarpa-S.)

A neighbourhood of the zero section in the holomorphic cotangent bundle $T^* \mathscr{J}$ is endowed with a natural hyperkähler structure. The induced action $\mathcal{G} \curvearrowright T^* \mathscr{J}$ preserves this hyperkähler structure.

This structure is induced by regarding $\mathscr{J}$ as the space of sections of a $\mathrm{Sp}(2n)$-bundle with fibres diffeomorphic to $\mathrm{Sp}(2n)/\mathrm{U}(n)$, and by the Biquard-Gauduchon canonical $\mathrm{Sp}(2n)$-invariant hyperkähler metric on a neighbourhood of the zero section in $T^*(\mathrm{Sp}/\mathrm{U}(n))$.

Adding a Higgs field: moment maps

Theorem (Scarpa-S.)

The action $\mathcal{G} \curvearrowright T^*\mathscr{J}$ is Hamiltonian with respect to the canonical symplectic form Θ; a moment map $\mathfrak{m}_\Theta$ is given by

$$\mathfrak{m}_{\mathbb{C}(J,\alpha)}(h) = -\int_M \frac{1}{2}\mathrm{Tr}(\alpha^{\intercal}\mathcal{L}_{X_h}J)\, d\nu.$$

Moreover the action $\mathcal{G} \curvearrowright (T^*\mathscr{J}, \Omega_I)$ is Hamiltonian; a moment map $\mathfrak{m}_{\mathbb{R}}$ is given by $\mu \circ \pi + \mathfrak{m}$ with

$$\mathfrak{m}_{(J,\alpha)}(h) = \int_M -\mathrm{d}\rho_{(J(x),\alpha(x))}\left(J(\mathcal{L}_{X_h}J), J^*(\mathcal{L}_{X_h}\alpha)\right)\, d\nu(x).$$

Here ρ is the Biquard-Gauduchon function.

Complex moment map is easy

Simple computation shows

$$\mathfrak{m}_{\mathbb{R}}(J, \alpha) = -\mathrm{div}\left(\bar{\partial}^* \bar{\alpha}^{\intercal}\right).$$

Harmonic representatives of infinitesimal deformations give particular solutions.

The case of complex curves

Theorem (Donaldson+ϵ)

M a compact complex curve. Set

$$\psi(\alpha) = \frac{1}{1 + \sqrt{1 - \frac{1}{4}||\alpha||^2}},$$

$$Q(J,\alpha) = \frac{1}{4} g_J(\nabla^a \alpha, \bar{\alpha})\partial_a + \frac{1}{4} g_J(\nabla^{\bar{b}} \bar{\alpha}, \alpha)\partial_{\bar{b}}.$$

Then under the L^2 product

$$\mathfrak{m}_{\mathbb{R}}(J,\alpha) = 2\, s(g_J) - 2\,\hat{s} + \Delta\left(\log\left(1 + \sqrt{1 - \frac{1}{4}||\alpha||^2}\right)\right)$$
$$+ \operatorname{div}\left(\psi(\alpha)\, Q(J,\alpha)\right).$$

The case of complex surfaces

$$A = \mathrm{Re}(\alpha^{\intercal}).$$

$$\delta^{\pm}(A) = \frac{1}{2}\left(\frac{\mathrm{Tr}(A^2)}{2} \pm \sqrt{\left(\frac{\mathrm{Tr}(A^2)}{2}\right)^2 - 4\det(A)}\right).$$

$$\psi(A) = \frac{1}{\sqrt{1-\delta^{+}(A)} + \sqrt{1-\delta^{-}(A)}},$$

$$\tilde{\psi}(A) = \frac{1}{\left(\sqrt{1-\delta^{+}(A)} + \sqrt{1-\delta^{-}(A)}\right)}$$

$$\times \frac{1}{\left(1+\sqrt{1-\delta^{+}(A)}\right)\left(1+\sqrt{1-\delta^{-}(A)}\right)}.$$

$\tilde{A}$ = *adjugate*.

Introduce a complex vector field

$$
\begin{aligned}
&X(J,\alpha)\\
&= -\psi(A)\,\mathrm{grad}\left(\frac{\mathrm{Tr}(A^2)}{2}\right) + 2\,\psi(A)\,\left(g_J(\nabla^a A^{0,1}, A^{1,0})\partial_a + \mathrm{c.c.}\right)\\
&- 2\,\nabla^*(\psi(A)A^2) + \tilde{\psi}(A)\,\mathrm{grad}\,(\det(A))\\
&- 2\,\tilde{\psi}(A)\,\left(g_J(\nabla^a A^{0,1}, \tilde{A}^{1,0})\partial_a - \mathrm{c.c.}\right) - 2\,\mathrm{grad}(\tilde{\psi}(A)\det(A)).
\end{aligned}
$$

Complex surfaces

Theorem (Scarpa-S.)

Suppose J is integrable. Then under L^2 product

$$\mathfrak{m}_{\mathbb{R}}(J, \alpha) = 2(s(g_J) - \hat{s}) + \frac{1}{2}\mathrm{div} X(J, A).$$

Complex surfaces

So harmonic bundle equations become the “**HcscK equations**"

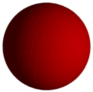
$$2\,s(g) + \frac{1}{2}\mathrm{div}X(g, A) = 2\,\hat{s}$$
$$\mathrm{div}\left(\bar{\partial}_g^* A^{1,0}\right) = 0$$

for fixed J and g varying in $[\omega_g]$.

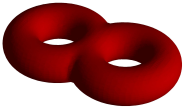
“H" for Higgs or Hitchin accoring to taste.

An existence result

We concentrate on **ruled surfaces**, “twisted products" of Riemann sphere and curves.



$$\tilde{\times}$$



Theorem (Scarpa-S.)

Fix compact complex curve Σ of genus at least 2, endowed with the hyperbolic metric g_Σ. Let M be the ruled surface $M = \mathbb{P}(T\Sigma \oplus \mathcal{O})$, with projection $\pi : M \to \Sigma$ and relative hyperplane bundle $\mathcal{O}(1)$, endowed with the Kähler class

$$\alpha_m = [\pi^* \omega_\Sigma] + m c_1(\mathcal{O}(1)),\ m > 0.$$

Then for all sufficiently small m the HcscK equations can be solved on (M, α_m).

It's is well-known that, for all positive m, (M, α_m) does not admit a cscK metric.