LOG CALABI YAU SURFACES AND JEFFREY-KIRWAN RESIDUES

(IN PROGRESS WITH R. ONTANI)

Motivation from physics: vast literature on computing partition / correlation functions of susy gauge theories on 2d using susy localization of the path integral. Results expressed as JK residues of explicit meromorphic differential forms on affine complex tori.

Benini, Eager, Hori, Tachikawa:

"$\mathcal{N}=(2,2)$; $\mathcal{N}=(0,2)$ on $T^2 = \mathbb{C}/\langle 1,\tau\rangle$"

Closset, Cremonesi, Park / Benini, Zaffaroni:

"$\mathcal{N}=(2,2)$ on S^2_Ω"

Cordova, Shao / Hwang, Kim, Kim, Park: dimensional reduction to $0+1$ dim. i.e. "$\mathcal{N}=4$ quantum mechanics".

(Also Beaujard, Mondal, Pioline.)

F... $0+1$ for now.

Focus on ...

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Q = finite quiver, no loops, no oriented cycles.

N = dim. vector.

G = gauge group $= \left(\prod_{i \in Q_0} U(N_i) \right) / U(1)$

$$\mathbb{T}(N) = \left(\prod_{i \in Q_0} (\mathbb{C}^*)^{N_i} \right) / \mathbb{C}^*$$

$$Z_Q(N) = \prod_{i \in Q_0} \prod_{s \neq s' = 1}^{N_i} \frac{u_{i,s'} - u_{i,s}}{u_{i,s} - u_{i,s'} - 1} \quad \langle j, i \rangle$$

$$\times \prod_{\{i \to j\} \in Q_1} \prod_{s=1}^{N_i} \prod_{s'=1}^{N_j} \left(\frac{u_{j,s'} - u_{i,s} + 1 - R_{ij}/2}{u_{i,s} - u_{j,s'} + R_{ij}/2} \right)$$

$$\times \prod_{i \in Q_0} \prod_{s=1}^{N_i} du_{i,s} .$$

$Z : \mathbb{Z}Q_0 \to \mathbb{C}$ "central charge";

... stability vector.

$\zeta =$ (real) stability

Relevant quantity for physics:

$$JK(Z_Q(N), \zeta) = \int_C Z_Q(N) \; ;$$

$C \subset \mathbb{T}(N)$ a canonical compact torus, depending on ζ, Q, N, R.

Claims / expectations from physics:

1) $JK(Z_Q(N), \zeta)$ is "Witten index" of $\mathcal{N} = 4$ susy quantum mechanics (obtained as dim. reduction of $\mathcal{N} = (2,2)$ gauge theory on $T^2 = \mathbb{C}^2 / \langle 1, \tau \rangle$);

2) $JK(Z_Q(N), \zeta)$ is also a limit of partition function on $T^2 = \mathbb{C} / \langle 1, \tau \rangle$, as $\begin{cases} \hbar \to 0 \\ \tau \to i\infty \end{cases}$ (LCS limit / cusp).

3) Mathematically:

$$\overline{\chi}_Q(N,s) = \frac{1}{\prod_{i\in Q_0} N_i!}\, JK(Z_Q(N),s)$$

4) Promote Z trivially to $\hat{Z}\in \operatorname{Lie} \mathbb{T}(N)$,

$$\hat{Z}_{i,s} = Z_i.$$

Then, by analogy with correlation functions for S^2_Ω, one would expect

$$\sum_N \frac{1}{N!} JK(Z_Q(N),s) \prod_{i\in Q_0} \prod_{s=1}^{N_i} e^{\hat{Z}_{i,s}}$$

to be related to genus 0 B-model partition function of a CY mfld.
In particular $\hat{Z}_{i,s}$ should be complex structure parameters.

Equivalently, after mirror map, we should get something like a $(g=0)$ GW partition function.

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(with R Ontani) is

Our general aim (with ...) :
to derive / study JK residue formulae
in the context of the Gross-Siebert
approach to mirror symmetry.

For now: mirror symmetry for log
Calabi-Yau surfaces developed by
Gross-Hacking-Keel.

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Geometry:

$(Y, D) =$ "Looijenga pair"

$Y =$ smooth cplx proj (nec. rational)
surface;

$D =$ anticanonical cycle of rational curves
$= D_1 + \dots + D_n$

(i)

(ii)

($D \in |-K_Y|$ nodal $\Rightarrow D =$ (i) or (ii))

Then,
$$U = Y \setminus D$$
is LCY: $\exists !$ (up to scale) holo 2-form Ω, nowhere vanishing, simple poles along D.

Basic invariant: $(D_i \cdot D_j)$.

Suppose $(D_i \cdot D_j)$ <u>not</u> ≤ 0 ("positive case")

Then,

(i) $NE(Y) = NE(Y)_{\mathbb{R}_{\geq 0}} \cap A_1(Y, \mathbb{Z})$ is finitely generated;

(ii) $U = Y \setminus D$ is affine;

(iii) GHK construct a family
$$\mathcal{X} \longrightarrow S = \operatorname{Spec} \mathbb{C}[NE(Y)]$$
of affine surfaces, such that
$$\mathcal{X} \longrightarrow T_Y = \operatorname{Pic}(Y) \otimes \mathbb{C}^* = \operatorname{Spec} \mathbb{C}[A_1(Y)] \subset S$$
(structure torus)

is a versal family of defos of (U, Ω) as a LCY surface.

(iv) Conjecturally, $\mathfrak{X} \to T_Y$ is mirror family to (U, Ω).

Should have:

$$D^\pi \mathcal{F}_{wr}(U, B+i\omega) \cong D^b(\mathrm{Coh}(X_s))$$

for $s = \exp(2\pi i(B+i\omega)) \in S$;

$$SH^0(U, B+i\omega) \cong H^0(X_s, \mathcal{O}_{X_s}).$$

Do have: "periods conjecture" (GHK)

(i) Let $S^\circ \subset S$ locus of smooth fibres.

$\gamma_s \subset H_2(\mathfrak{X}_s, \mathbb{Z})$ monodromy invariant class of a real 2-torus (fibre of a Lagrangian fibration).

then, the local system

$$S^\circ \ni s \mapsto H_2(\mathfrak{X}_s, \mathbb{Z}) / \langle \gamma_s \rangle$$

is trivial. "

(ii) Each fibre is identified canonically with

$$Q = H_2(U,\mathbb{Z})/\langle\gamma\rangle = \langle D_1,\dots,D_n\rangle^{\perp} \subset H_2(Y,\mathbb{Z}),$$

and we have, $\forall\, \beta \in Q \subset H_2(Y,\mathbb{Z})$,

$$z^{\beta} = \exp\Big(2\pi i \int_{\tilde{\beta}} \Omega\Big)$$

(with normalization $\displaystyle\int_{\gamma_s} \Omega = 1$).

Ie: $\{z^{\beta},\ \beta \in Q \subset H_2(Y,\mathbb{Z})\}$ are the "canonical coords" on the cplx moduli space, nearby LCS limit $s \to 0 \in S$.

(Rmk: local / global Torelli theorems hold, $\mathrm{Hom}(D^{\perp}, \mathbb{C}^*)$ is the period domain, and there is an exact sequence

$$T_Y \longrightarrow \mathrm{Hom}(D^{\perp}, \mathbb{C}^*) \longrightarrow 1$$

with kernel T^D = torus spanned by $e_{D_1},\dots,e_{D_n}$.)

Now clearly, by construction, we also have

$$z^{\beta} = \exp\Big(2\pi i \int [B + i\omega]\Big).$$

$\hat{Z} = \exp(- \ldots_\beta \ldots)$

This is compatible with a (more general) result of Ruddat – Siebert, stating that in our case the mirror map is trivial.

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In this context, we are able to give a geometric interpretation/proof for JK residue formulae in gauge theories, in very special cases.

Theorem (Ontani, S.)

Q = quiver as above, dim vect. N

Suppose Q is complete bipartite.

Then, $\exists$ affine LCY surface

$$U = Y \setminus D$$

such that:

(i) the quiver variables $e^{\hat{z}_{i,s}}$ correspond naturally to a set of canonical coords on T_Y, i.e. given by periods of Ω on GHK mirror $\mathcal{X} \to T_Y$ to U;

(ii) given (i) the limit

(ii) given (i),

$$\mathrm{Re}\,\hat{z}_{i,s} \longrightarrow -\infty$$

corresponds to LCS limit

$$s \longrightarrow 0 \in S;$$

(iii) given (i), (ii), the quantity

$$\frac{1}{N!} JK(Z_Q(N), \varsigma) \prod_{i \in Q_0} \prod_{s=1}^{N_i} e^{-\hat{z}_{i,s}/t}$$

appears naturally when considering infinitesimal, finite order defos of $\mathcal{X}_0 \cong \mathbb{V}_n = \mathbb{A}^2_{x_1 x_2} \cup \cdots \cup \mathbb{A}^2_{x_{n-1} x_1}$.

Here, $r = \mathrm{rk}\, G = \sum_{i \in Q_0} N_i - 1$ corresponds to defos over $\mathrm{Spec}\, \mathbb{C}[NE(Y)]/m_0^{r+1}$

And, for fixed $\hat{z}$, the LCS limit is reached along

$$t \in \mathbb{R}_{>0}, \quad t \longrightarrow +\infty;$$

(iv) by (iii), we have both

$$\frac{1}{N!} JK(Z_Q(N), \varsigma) = \bar{\chi}_Q(N, \varsigma)$$

and

$$= \mathrm{DT}(Y, D),$$

N_β

a relative $g=0$ GW invariant of degree $\beta = \beta(N)$.

This is compatible with triviality of the mirror map!

Caveat: this result is conditional on "abelianization" and "R-charges independence" for JK residues. Both are claimed in physics literature.

Ie: result holds for "abelianized, large R" JK residues.

Rmk. The crucial point is showing (iii), ie that JK residues appear in defo theory of $\mathfrak{X}_0 \subset \mathfrak{X} \to S$.

Using GHK theory, we need to show $\{ JK(Z_Q(N), S) \}$ appear in the "saturation" of the relevant "scattering diagram".

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Reduction to the Gross–Siebert locus

$(Y, D) =$ "positive" Looijenga pair

$P = NE(Y)$, fg toric monoid

$\mathfrak{X} \longrightarrow S = \operatorname{Spec} \mathbb{C}[P]$ constructed

initially around $0 \in S$, ie

$$\mathcal{X} \longrightarrow \operatorname{Spec} \mathbb{C}[P]/m^k,$$

$$m = P \setminus \{0\}.$$

D nodal ("max degenerate") $\Longrightarrow$

(up to "toric blps") $\exists$ toric model

$$\boxed{p : (Y, D) \longrightarrow (\overline{Y}, \overline{D})},$$

exc. divisors E_{ij}, $i = 1, \dots, n$
$j = 0, \dots, \ell_i$.

Fix H ample on $\overline{Y}$.

$G = P \setminus (p^* H)^{\perp}$ prime monoid ideal $\subset m$.

$T \subset \operatorname{Spec} \mathbb{C}[P]/G \subset S$ the max. torus orbit = the GS locus.

Reduction to GS locus:
it is enough to construct $\mathfrak{X} \to S$ to all orders around $T \subset S$.

Replace $\quad P \mapsto (P)_{P \setminus G}$

$\mathfrak{m} \mapsto$ max monoid ideal in $(P)_{P \setminus G}$.

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Reduction to scattering diagrams in $\mathbb{R}^2$

In general, $\mathfrak{X} \to S$ is constructed from $\mathbb{R}^2 \underset{\text{homeo}}{\cong} (B, \Sigma)$ with nontrivial $\mathbb{Z}$-affine structure on $B^0 = B \setminus \{0\}$, endowed with a specific consistent scattering diagram $\mathfrak{D}^{can}$.

Infinitesimally around GS locus $T \subset S$,

$(B, \Sigma) \mapsto (\bar{B}, \bar{\Sigma})$

• $(\ldots, \Sigma, \ldots$

$$= (\mathbb{R}^2, \text{toric fan})$$

with trivial $\mathbb{Z}$-affine structure ;

• consistency $\mapsto$ saturation :

$\forall \sqrt{I} = m$, we have

$$\begin{tikzcd}
p : \mathcal{X}^{\circ}_{I, \mathfrak{D}^{can}} \arrow[rr, "\cong"] \arrow[dr] & & \overline{\mathcal{X}}^{\circ}_{I, \nu(\mathfrak{D}^{can})} \arrow[dl] \\
& \operatorname{Spec} \mathbb{C}[P]/I &
\end{tikzcd},$$

where

$$\nu(\mathfrak{D}^{can}) = \mathrm{Scatter}(\overline{\mathfrak{D}}_0),$$

$$\overline{\mathfrak{D}}_0 = \left\{ \mathbb{R}_{\geq 0} m_i, \prod_{j=0}^{l_i} (1 + z^{(m_i, p^*[\overline{D}_i] - [E_{ij}])}) \right\}$$

$$\subset \mathbb{R}^2 = \overline{B},$$

over $P_{\overline{\varphi}}$ = Mumford monoid .

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To apply "periods conjecture", pass to

$$\overline{\mathfrak{D}}' = \left\{ \mathbb{R}_{\geq 0} m_i, \prod^{l_i} (1 + t_i z^{(m_i, p^*[\overline{D}_i] + [E_{i0} - E_{ij}])}) \right\}$$

$\ldots_{j=1}$

(with τ_i ~~...~~ $z^{(0,-[E_{i0}])}$)),

so $[E_{i0}-E_{ij}] \in Q = D^{\perp} \subset H_2(Y,\mathbb{Z})$

(recall period domain is $\mathrm{Hom}(D^{\perp},\mathbb{C}^*)$)).

Then, $\bar{\mathfrak{D}}_0'$ is isomorphic to scattering diagram for the LCY with toric model

$$(Y',D') \longrightarrow (\bar{Y},\bar{D})$$

with exc. divisors E_{ij}, $i=1,\dots,n$, $j=1,\dots,l_i$.

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The canonical scattering diagram

Recall the GHK mirror family to

$$(Y',D') \longrightarrow (\bar{Y},\bar{D})$$

is obtained from the saturation

$$\bar{\mathfrak{D}}' = \{(\mathfrak{d}', f_{\mathfrak{d}'})\} \subset \mathbb{R}^2$$

of

$$\bar{\mathfrak{D}}' = \{ \ldots \prod^{l_i} (1+\tau_i z^{(m_i,\, p^*[\bar{D}_i]+[E_{i0}-E_{ij}])})\}$$

$\mathfrak{D}_0 = \{ \mathbb{R}_{\geq 0} m_i, \ldots$
$j=1$

over $\mathbb{C}[P_{\bar{\sigma}}][\tau_1, \ldots, \tau_n]$, with specialization $\tau_j \mapsto z^{(0, -[E_{i0}])}$.

This can be described by the appropriate canonical scattering diagram:

$$\mathrm{Scatter}(\bar{\mathfrak{D}}'_0) = \nu((\mathfrak{D}')^{\mathrm{can}}).$$

That is, we have

$$\log f_{\mathfrak{d}'} = \nu\left(\sum_\beta k_\beta N_\beta (\tau z)^{\pi_* \beta - \bar{\varphi}_{\tau_{\mathfrak{d}'}}(k_\beta m_{\mathfrak{d}'})} \right),$$

where

$$N_\beta = \int_{[\overline{\mathcal{M}}((\tilde{Y}')^\circ / C^\circ, \beta)]^{\mathrm{vir}}} 1,$$

$\pi : \tilde{Y}' \to Y'$ "toric blp" corresp to $\mathfrak{d}'$,

and $\beta \in H_2(\tilde{Y}', \mathbb{Z})$ satisfies

$$\beta \cdot \tilde{D}'_i = \begin{cases} k_\beta & \tilde{D}'_i = C \\ 0 & \tilde{D}'_i \neq C \end{cases}$$

where $C \subset \tilde{D}'$ is dual to $\mathfrak{d}'$.

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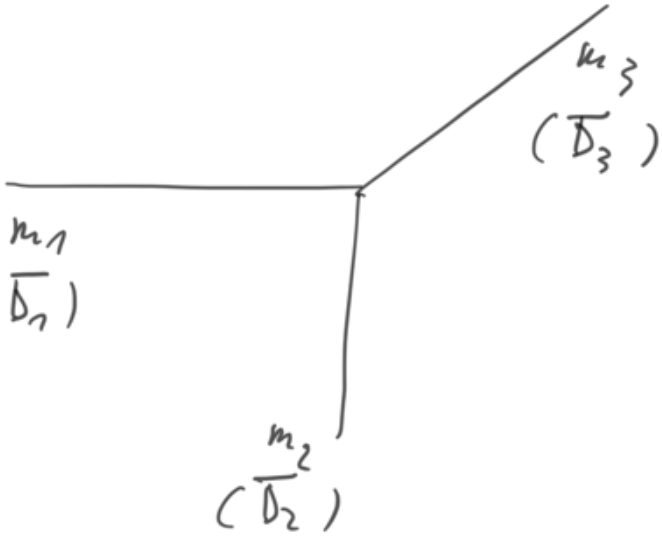
We apply this to special case

$(\overline{Y}, \overline{D}) = (\mathbb{P}^2,$  $),$

and with $\ell_3 = -1$ (no blps).

$\overline{\Sigma} =$



Then,

$$\beta = \beta_{(P_1, P_2)} = \pi^* \beta_k - \sum_{i=1}^{\ell_1} p_{1i} [E_{1i}] - \sum_{j=1}^{\ell_2} p_{2j} [E_{2j}]$$

for $(P_1, P_2) = \left(\sum_{i=1}^{\ell_1} p_{1i}, \sum_{j=1}^{\ell_2} p_{2j} \right)$,

$(|P_1|, |P_2|) = k(a,b)$, $k > 0$,

$(a,b) \in (\mathbb{Z}^2)_{prim}$,

$\mathfrak{d}' = \mathbb{R}_{>0}(a,b)$.

Write

$$N[(P_1, P_2)] = N_{\beta(P_1,P_2)} .$$

Then,

$$f'_{\gamma'} \equiv f'_{(a,b)} \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}][[s_1, \dots, s_{e_1}, t_1, \dots t_{e_2}]]$$

with

$$f'_{(a,b)} = \sum_{k \geq 0} \sum_{\substack{(|P_1|, |P_2|) \\ = k(a,b)}} k N[(P_1,P_2)](s,t)^{(P_1,P_2)} \cdot (\tau x)^{ka} (\tau y)^{kb},$$

$$s_i = e^{2\pi i \int_{\tilde{\beta}_{1i}} \Omega}, \quad t_j = e^{2\pi i \int_{\tilde{\beta}_{2j}} \Omega},$$

$$(\tau x)^{ka} = z^{(k a m_1, -ka[E_1 \cdot J])},$$

$$(\tau y)^{kb} = z^{(k b m_2, -kb[E_2 \cdot J])}.$$

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ABELIANIZATION

Q = any quiver (no potential)

$i \in Q_0$ $\qquad N \in \mathbb{Z}_{\geq 0} Q_0$

Abelianization at i.

$$\hat{Q}^i = \begin{cases} i \mapsto i_{k,\ell}, \quad k,\ell \geq 1 \\ i \to j \mapsto i_{k,\ell} \xrightarrow{(\ell)} j \\ j \to i \mapsto j \xrightarrow{(\ell)} i_{k,\ell} \end{cases}$$

$$\hat{N}_{i_{k,\ell}}(m_*) = \begin{cases} 1 & 1 \leq k \leq m_\ell \\ 0 & k > m_\ell \end{cases}$$

$m_* \vdash N_i$

$\left(N_i = \sum_\ell \ell m_\ell \right)$

$$\hat{S}_{i_{k,\ell}} = \ell S_i, \quad k,\ell \geq 1.$$

Abelianization for JK residues:

$JK(Z_Q(N), S) =$

$$N_i! \sum_{m_* \vdash N_i} \prod_{\ell \geq 1} \frac{1}{m_\ell!} \left(\frac{(-1)^{\ell-1}}{\ell^2} \right)^{m_\ell} JK\left(Z_{\hat{Q}^i}(\hat{N}(m_*)), \hat{S}\right).$$

Physical argument by Beaujard, Mondal, Pioline.

Abelianization for quiver invariants:

$$\bar{\chi}_Q(N,\zeta) = \sum_{m_* \vdash N_i} \sum_{\ell \geq 1} \frac{1}{m_\ell!} \left(\frac{(-1)^{\ell-1}}{\ell^2} \right)^{m_\ell} \chi_{\hat{Q}_i}(\hat{N}(m_*), \hat{\zeta})$$

(Manschot, Pioline, Sen; Reineke, S., Weist).

Special notation for $Q = K(\ell_1, \ell_2)$:

$$N_0 = \{ i_{(w,m)} : (w,m) \in \mathbb{N}^2 \} \cup \{ j_{(w,m)} : (w,m) \in \mathbb{N}^2 \}$$

$$N_1 = \{ \alpha_1 \dots \alpha_{ww'} : i_{(w,m)} \to j_{(w',m')} \};$$

$$\mathbb{Z}_{>0} K(\ell_1, \ell_2) \equiv \{ (P_1, P_2) \};$$

Refinements $(k^1, k^2) \vdash (P_1, P_2)$:

$$p_{1i} = \sum_w w k^1_{wi}; \quad p_{2j} = \sum_w w k^2_{wj};$$

$$m_w(k^i) = \sum_{j=1}^{\ell_i} k^i_{wj}, \quad i = 1,2;$$

$$d(k^1,k^2)(q_{(w,m)}) = \begin{cases} 1 & 1 \leq m \leq m_w(k^p) \\ 0 & m > m_w(k^p) \end{cases}$$

$\mathbb{Z}_{\geq 0}$ N_0

$(p = 1, 2$ accord to $q = i, j)$;

$$\hat{\zeta}(i_{(w,m)}) = w\zeta_1 \quad (= w\zeta(i))$$

$$\hat{\zeta}(j_{(w,m)}) = w\zeta_2 \quad (= w\zeta(j)).$$

Then (conj.)

$$JK(Z_{K(l_1,l_2)}(P_1,P_2), \zeta) =$$

$$\sum_{(k^1,k^2)\vdash(P_1,P_2)} \prod_{i=1}^{2} \prod_{j=1}^{l_i} \prod_{w} \frac{(-1)^{k^i_{w,j}(w-1)}}{k^i_{w,j}!\, w^{2k^i_{w,j}}}$$

$$\times JK(Z_N(d(k^1,k^2)), \hat{\zeta}).$$

And ([MPS], [RSW]),

$$\overline{\chi}_{K(l_1,l_2)}((P_1,P_2),\zeta) =$$

$$\sum \prod^{2} \prod^{l_i} \prod \frac{(-1)^{k^i_{w,j}(w-1)}}{\cdots}$$

$$\sum_{(k^1,k^2)\vdash(p^1,p^2)} \prod_{i=1} \prod_{j=1} \prod_{w} \frac{\cdots}{k^i_{wj}!\, w^{k^i_{wj}}}$$

$$\times \overline{\chi}_N (d(k^1,k^2), \hat{S}).$$

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DEGENERATION IN GW THEORY

Back to $\gamma((\mathbb{D}')^{can})$. We have by

$$N[(P_1,P_2)] =$$

$$\sum_{(k^1,k^2)\vdash(P_1,P_2)} \prod_{i=1}^{2} \prod_{j=1}^{l_i} \prod_{w} \frac{(-1)^{k^i_{wj}(w-1)}}{k^i_{wj}!\, w^{k^i_{wj}}}$$

$$\times N^{rel}_{\mathbb{P}^2}[w(k^1), w(k^2)].$$

Under the correspondence

$$N[(P_1,P_2)] = \overline{\chi}_{K(l_1,l_2)}((P_1,P_2),\underline{S}),$$

[Reineke, Weist]

we have

(GW deg. formula) $\equiv$ (abelianization).

[Reineke, S., Weist]

I.e. we have

$$\overline{\chi}_N(d(\kappa^1,\kappa^2),\underline{\hat{\xi}}) = N^{rel}[w(\kappa^1),w(\kappa^2)] \prod_{i=1}^{2}\prod_{j=1}^{l_i}\prod_{w} w^{\kappa^i_{wj}}.$$

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Upshot:

the invariants $N[(P_1,P_2)]$ computing $\gamma((\mathfrak{D}')^{can})$, and so the GHK mirror to $U' = Y' \setminus D'$ around T_{GS},

are also given by

$$\sum_{(\kappa^1,\kappa^2)\vdash(P^1,P^2)} \prod_{i=1}^{2}\prod_{j=1}^{l_i}\prod_{w} \frac{(-1)^{\kappa^i_{wj}(w-1)}}{\kappa^i_{wj}!\, w^{2\kappa_{wj}}} \overline{\chi}_N(d(\kappa^1,\kappa^2),\underline{\hat{\xi}})$$

Then, our main result follows from

claim:

$$\boxed{\overline{\chi}_N(d(\kappa^1,\kappa^2),\underline{\hat{\xi}}) = JK(Z_N(d(\kappa^1,\kappa^2)),\underline{\hat{\xi}}).}$$

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On JK side, we show:

$\forall\, Q$ (no loops or oriented cycles),

$$JK(Z_Q(\mathbb{1}, \varsigma)) = \int_C Z_Q(\mathbb{1})$$

$$\Big(\underline{\mathbb{1}} = \sum_{i \in Q_0} i \in \mathbb{Z} Q_0\Big),$$

for generic $\overline{R} = \overline{R}_{ij}$,

is given by a sum of contributions,

one for each $T \subset \overline{Q}$ a spanning tree of the flattened quiver, with fixed root

$$\varepsilon'(T) \prod_{\{i \to j\} \subset T_1} \frac{1}{2\pi i} \int_{C_i'} \left(\frac{u_j - u_i + 1 - \overline{R}_{ij}/2}{u_i - u_j + \overline{R}_{ij}/2} \right)^{\langle j,i \rangle} du_j.$$

Moreover, $\quad \varepsilon'(T) = \overline{\chi}_T(\mathbb{1}, \varsigma)$.

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On the GHK side, we compute

$\overline{\chi}_N(d(\kappa^1, \kappa^2), \hat{\underline{\varsigma}})$ by the method of generalized monodromy (isomonodromic deformation), following in particular Filippini, Garcia-Fernandez, S.

Here, one constructs a holomorphic family of $\mathrm{Aut}(\mathbb{C}[\mathbb{Z}_{\geq 0} N_0]^{\wedge})$ - connections $\nabla(\hat{\underline{z}})$ on $\mathbb{P}^1 \setminus \{0, \infty\}$, param by period domain

$$\mathrm{Hom}(D^{\perp}, \mathbb{C}^*) \subset \mathrm{Hom}(\mathbb{Z} N_0, \mathbb{C}),$$

such that

(i) for some "initial point" $\hat{\underline{z}}_0$, $\nabla(\hat{\underline{z}}_0)$ and its canonical flat sections $Y(t, \hat{\underline{z}}_0)$ have a rather simple explicit expression in terms of graph (tree) integrals;

(ii) the jumps of $Y(t, \hat{\underline{z}})$ along branch cuts $\mathbb{R}_{>0}\, \hat{\underline{z}}(d)$ compute the invariants $\bar{\chi}_N(d, \hat{\underline{s}})$.

This leads naturally to

$$-\hat{\underline{z}}(d(k^1 k^2))/t$$

$$\overline{\chi}_{\mathcal{N}}(d(k^1,k^2),\hat{\underline{\varsigma}})\, e^{\cdots} =$$

$$\sum_{\substack{\sum_{i\in T_0}\alpha_i = d(k^1,k^2),\\ \alpha_T=\tau}} \; \sum_{\{i\to j\}\subset T_1} \varepsilon(T)\prod \frac{1}{2\pi i}\int_{C_i} \frac{\langle\alpha_i,\alpha_j\rangle w_i}{w_j-w_i}\,\frac{dw_j}{w_j}\, e^{-2\pi i\int_{\beta}\Omega/t},$$

$$\text{for } \beta = \sum_{i=1}^{2}\sum_{j=1}^{l_i} p_{ij}\beta_{ij}.$$

Moreover,

$$\varepsilon(T) = \overline{\chi}_T(d(k^1,k^2),\hat{\underline{\varsigma}}).$$

<u>Rmk.</u> The generalized monodromy perspective is natural from GHK theory, because of the consistency property

$$\overline{\mathrm{Lift}_{Q^+}(q)} = \theta_{\gamma',\overline{\mathfrak{D}}'}\left(\overline{\mathrm{Lift}_{Q^-}(q)}\right).$$

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Finally, our main claim follows from

$$\cdots \ell(m_i - m_j + 1 - \lambda\overline{R}_{ij}/2)\ \ \langle j,i\rangle \cdots$$

$$\lim_{\lambda \to +\infty} \frac{1}{2\pi i} \int_{C_i'} \left(\frac{\cdots}{u_i - u_j + \lambda R_{ij}/2} \right) du_j$$

$$= \frac{1}{2\pi i} \int_{C_i} \frac{\langle i,j \rangle w_i}{w_j - w_i} \frac{dw_j}{w_j} .$$

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$Z_Q(N)$ is $\hbar \to 0$ limit of

$Z_Q(N,\hbar)$ given by

$$\left(\frac{\pi\hbar}{\sin(\pi\hbar)}\right)^{\sum_{i\in Q_0} N_i - 1} \prod_{\substack{i\in Q_0 \\ s,s'=1\dots N_i \\ s\neq s'}} \frac{\sin(\pi\hbar(u_{i,s'}-u_{i,s}))}{\sin(\pi\hbar(u_{i,s}-u_{i,s'}-1))}$$

$$\prod_{\substack{\{i\to j\}\\ \subset Q_1}} \prod_{\substack{s=1\dots N_i \\ s'=1\dots N_j}} \left(\frac{\sin(\pi\hbar(u_{j,s'}-u_{i,s}+1-R_{ij}/2))}{\sin(\pi\hbar(u_{i,s}-u_{j,s'}+R_{ij}/2))} \right)^{\langle j,i \rangle}$$

$$\widetilde{\prod_{\substack{i\in Q_0 \\ s=1\dots N_i}}} du_{i,s} .$$

$JK(Z_Q(N,\hbar))$ should appear when considering appropriate defo quantizations

of (T... Q...) (cf. "quantum mirrors"

of (U, SL) (the [illegible]
of P. Bousseau)

* * *

In turn, $Z_Q(N,\hbar)$ is $\tau \to i\infty$ $\left(\begin{smallmatrix}\text{LCS,}\\ \text{cusp}\end{smallmatrix}\right)$ limit of

$$Z_Q(N,\hbar,\tau) = \left(\frac{2\pi\eta^3\hbar}{\theta_1(\hbar)}\right)^{\sum_{i\in Q_0} N_i - 1}$$

$$\times \prod_{\substack{i\in Q_0\\ s,s'=1..N_i\\ s\neq s'}} \frac{\theta_1(\pi\hbar(u_{j,s'} - u_{i,s'}))}{\theta_1(\pi\hbar(u_{i,s} - u_{j,s'} - 1))}$$

$$\times \prod_{\substack{\{i\to j\}\\ \subset Q_1}} \prod_{\substack{s=1..N_i\\ s'=1..N_j}} \left(\frac{\theta_1(\pi\hbar(u_{j,s'} - u_{i,s} + 1 - R_{ij}/2))}{\theta_1(\pi\hbar(u_{i,s} - u_{j,s'} + R_{ij}/2))}\right)^{\langle j,i\rangle}$$

$$\widetilde{\prod_{\substack{i\in Q_0\\ s=1...N_i}}} du_{i,s} \; ,$$

with

$$\theta_1(v) = 2q^{1/8}\sin(\pi v)\prod_{n=1}^{\infty}(1-q^n)(1-q^n e^{2\pi i v})(1-q^n e^{-2\pi i v}),$$

$$q = e^{2\pi i \tau},$$

the corresponding Benini-Eager-Hori-Tachikawa form.

Is there any interpretation in terms of $(\mho, \Omega)$?