TROPICALIZATION OF LOOIJ PAIRS

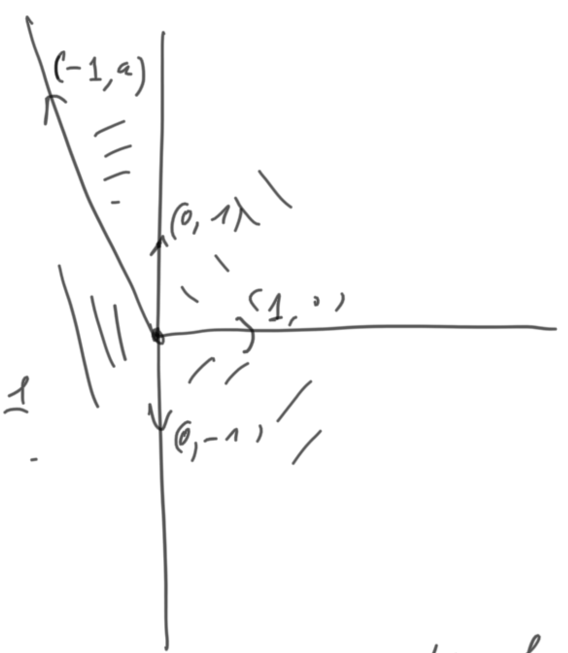
• Toric preliminaries

Recall a toric surface Y is determined by a fan

$$\Sigma = \{\underbrace{\sigma_{i,i+1}, i}_{2d}\} \cup \{\underbrace{\rho_i}_{1d}, i\} \cup \{\underbrace{0}_{0d}\}$$

of strongly convex rat'l polyhedral cones in
$\mathbb{R}^2 \cong M \otimes_{\mathbb{Z}} \mathbb{R}$, $M \cong \mathbb{Z}^2$.

Eg.



Hirzebruch surface
$\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(a)) \longrightarrow \mathbb{P}^1$.

Take $v_1, \dots, v_n$ prim gen's of rays in Σ.

Y smooth $\Rightarrow$ v_i, v_{i+1} generate M $\forall i$

$\Rightarrow \forall i, \exists a_i \in \mathbb{Z}$ s.t.

$$a_i v_i = v_{i-1} + v_{i+1} \quad (*)$$

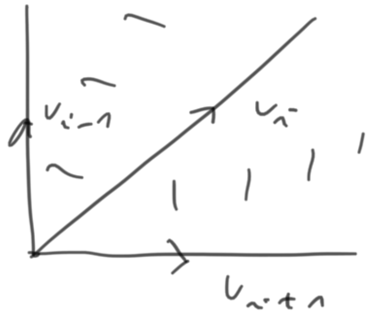
Recall ρ_i corresponds to a divisor (toric) $D_i \cong \mathbb{P}^1 \subset Y$, with

$$D_i^2 = -a_i \quad \forall i \quad (**)$$

Thm For $n \geq 5$, $\exists\, i$ such that

$$v_i = v_{i-1} + v_{i+1}$$ and v_{i-1}, v_{i+1} span a strongly convex cone

so Y is the blp of some $\tilde{Y}$ at a torus fixed pt.



Cor. Y is the blup of $\mathbb{P}^2$ or $\mathbb{F}_a = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(a))$ (smooth) at successive points.

Rmk $(*)$, $(**)$ $\Longrightarrow$ Σ is determined up to $SL(2,\mathbb{Z})$ by $\{D_i^2\}$.

Properties $(*)$, $(**)$ lead to a notion of TROPICALIZATION (B, Σ) for a general Looij pair $(Y, D = D_1 + \dots + D_n)$

- lattices $M_{i,i+1} := \operatorname{span}(v_i, v_{i+1})$ (abstract generators);
- cones $\sigma_{i,i+1} \subset M_{i,i+1} \otimes \mathbb{R}$ gen by v_i, v_{i+1};
- rays $\rho_i := \mathbb{R}_{\geq 0} v_i$;
- glue $\sigma_{i-1,i}$ to $\sigma_{i,i+1}$ along ρ_i

$S, B \simeq \mathbb{R}^2$ $\quad \Sigma := \{\sigma_{i,i+1}\} \cup \{\rho_i\} \cup \{0\}$

$\Rightarrow \quad (B \underset{PL}{\cong} \mathbb{R}^2, \ldots$

This is just PL so far, but now DEFINE INTEGRAL AFFINE STRUCTURE ON $B \setminus \{0\}$ (I.E. CHARTS WITH $SL(2,\mathbb{Z})$ TRANS FUNCTIONS) BY

$$U_i := (\sigma_{i-1,i} \cup \sigma_{i,i+1})^\circ;$$

$$\psi_i : U_i \to \mathbb{R}^2$$

$$v_{i-1} \mapsto (1,0)$$
$$v_i \mapsto (0,1)$$
$$v_{i+1} \mapsto (-1, -D_i^2)$$

extended piecewise linearly.

Rmk. Suppose (Y, D) toric. Smooth $\Rightarrow \exists$ linear map $\varphi_i : M \to \mathbb{Z}^2$ with $\varphi(v_{i-1}) = (1, 0)$, $\varphi(v_i) = (0, 1)$

Now (*) says

$$v_{i-1} + (D_i)^2 v_i + v_{i+1} = 0$$

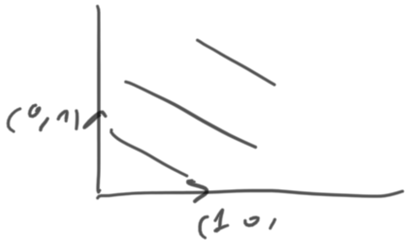
$$\Rightarrow \varphi(v_{i+1}) = (-1, -D_i^2).$$

So in this case, the PL maps ψ_i are linear (no bending!), so $SL(2,\mathbb{Z})$ structure extends to all $B \cong \mathbb{R}^2$.

Prop. The $SL(2,\mathbb{Z})$ str extends across $\{0\} \in B$ iff Y is toric and $D = \partial Y$.

Pf. Let (Y, D) be a Looij. pair with trop (B, Σ). Locally analytically near $D_i \cap D_{i+1}$,

(Y, D) is $\cong$ toric pair $(\mathbb{C}, \partial \mathbb{C})$.



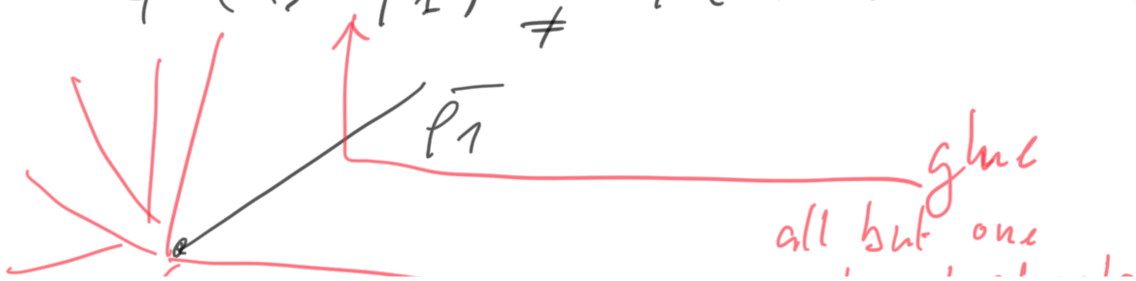
"Toric blups" of (Y, D) at $D_i \cap D_{i+1}$ $\longleftrightarrow$ toric blps of $\mathbb{C}^2$, i.e. subdivisions

Clearly, these do NOT change the $SL(2,\mathbb{Z})$ structure on $B \setminus \{0\}$ (i.e. they give iso structure)

So we are free to do toric blups of (Y, D).
So we have a toric model
$\pi : (Y, D) \to (\bar{Y}, \bar{D})$.

Note $\bar{D}_i^2 \geq D_i^2$ $\forall i$ because we're blowing up points on $\bar{D}$.
Clearly, $(Y; D) \cong (\bar{Y}, \bar{D}) \iff \bar{D}_i^2 = D_i^2$ $\forall i$.
Suppose now (Y, D) NOT toric, so $\bar{D}_i^2 > D_i^2$ for some i, suppose 1.

Then, $\psi(B \setminus \rho_1) \subsetneq \phi(\bar{B} \setminus \bar{\rho}_1) \cong \mathbb{R}^2 \setminus \bar{\rho}_1$



$\varphi(B \setminus p_1)$

local charts

So φ cannot extend to is with $\mathbb{R}^2$, so $SL(2,\mathbb{Z})$ str can't extent across 0 $\square$

IE if we cut along a ray, we can develop. Angle is controlled by the D_i^2 by def of local charts

EG Degree 5 Del Pezzo (simplest nontoric Del Pezzo)

$$Y = Bl_{p_1 p_2 p_3 p_4} \mathbb{P}^2 \quad \left(\text{general position}\right)$$

H = pullback of hyp class

E_1, E_2, E_3, E_4 ex. divisors

-1 curves:

$\{E_i\}$, $i = 1..4$

L_{ij} = proper transform of line through p_i, p_j $i \neq j$ (6 pairs)

ANTICANONICAL CYCLE:

$$D := \underbrace{L_{12}}_{\sim H - E_1 - E_2} + E_2 + \underbrace{L_{23}}_{\sim H - E_2 - E_3} + E_3 + \underbrace{L_{34}}_{\sim H - E_3 - E_4}$$

$$\Rightarrow D \sim 3H - E_1 - E_2 - E_3 - E_4$$

$\sim -K_Y$.

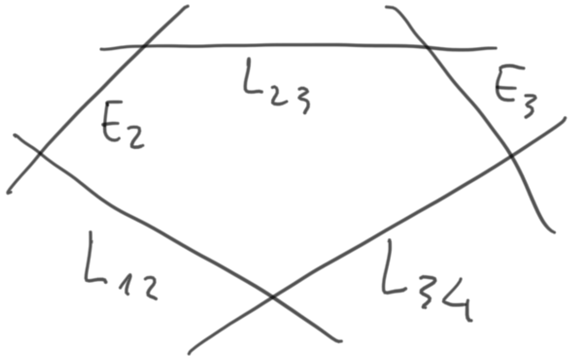
MATRIX:

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\sim \left(\begin{array}{c|cccc} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & & & & \\ 0 & & < 0 & & \\ 0 & & & & \\ 0 & & & & \end{array}\right)$$

$\Rightarrow U$ AFFINE!

$D =$

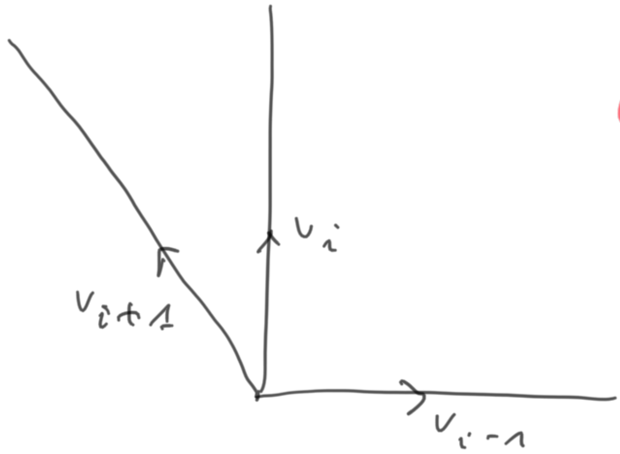


$\Rightarrow (Y, D)$ Log pair

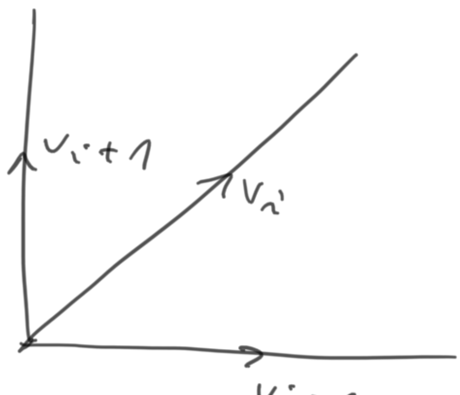
The charts ψ_i all look the same

$\psi_i(v_{i-1}) = (1, 0)$
$\psi_i(v_i) = (0, 1)$
$\psi_i(v_{i+1}) = (-1, 1)$

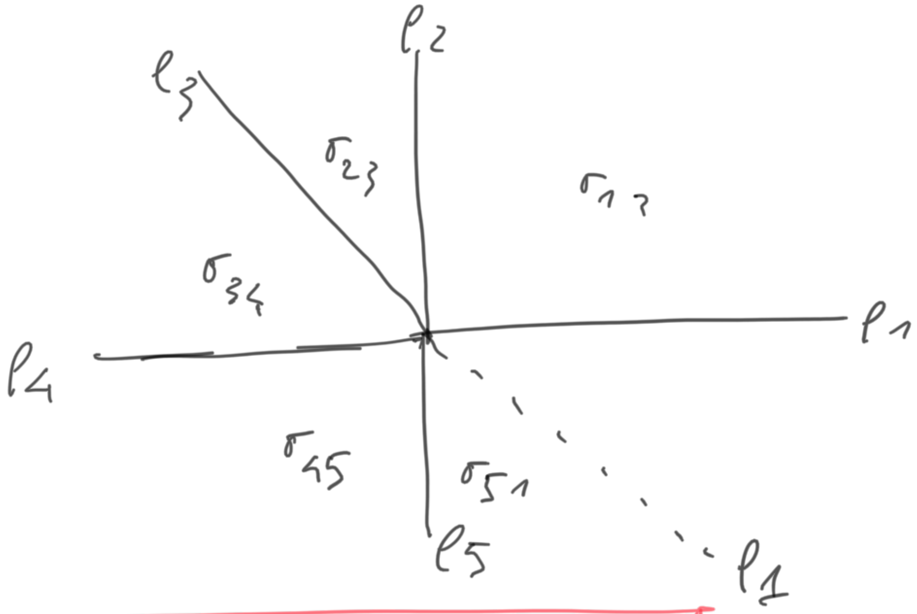
ALSO CLEAR SINCE Y IS DEL PEZZO, $-K_Y > 0$! (I.E. D IS AMPLE)



Composing with $SL(2, \mathbb{Z})$, it can be changed to



So, by changing charts with $SL(2,\mathbb{Z})$, we have $\psi(B \setminus P_1) \subset \mathbb{R}^2$ given by



$\boxed{\text{SYMPLECTIC HEURISTIC (SYZ conjecture)}}$

1) Affine structure on B_0.

Fix ω symp on Y.

SLags in $U := Y \setminus D$: $L \subset U$ such that $\omega|_L = \mathrm{Im}(\Omega)|_L = 0$.

SYZ: there should exist SLag fibration with some singular fibres

$$f : U \longrightarrow B$$

Then B_0 = locus of smooth fibres.

McLean - Hitchin: defos of a SLag L are unobstructed, given by $H^1(L, \mathbb{R})$.

In our case: $T_b B_0 \cong \underbrace{H^1(L_b, \mathbb{R})}$

intersection pairing $H^1 \times H^1 \xrightarrow[D]{} \mathbb{R}$

gives metric on B_0.

Fibration locally trivial $\Rightarrow$ metric is flat

$\Rightarrow$ affine str on B_0

2) Fan Σ on B_0.

Claim (Auroux): up to defo of cplx structure of (Y, D),

$\mathrm{Im}(\Omega)$ is exact.

$= d\gamma$

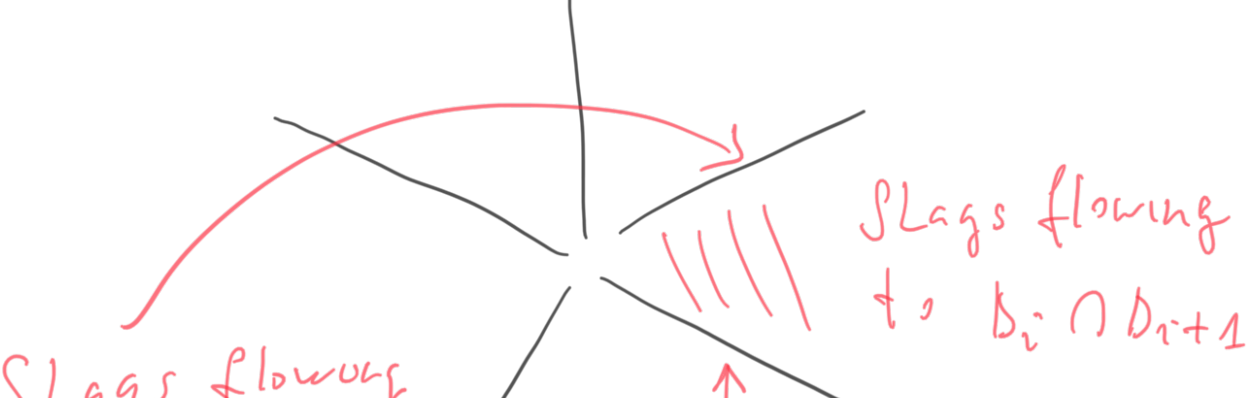
$\Rightarrow \forall\, b,\ L = L_b,\quad d\gamma|_L = \mathrm{Im}(\Omega)|_L = 0$

$\Rightarrow$ get $[\gamma|_L] \in H^1(L, \mathbb{R})$

$\Rightarrow$ get a canonical vector field v on B_0

Claim: v is flat (wrt McLean–Hitchin)
(Auroux)

Conjecture: under the (linear) flow of v, a generic $L = L_b$ flows to a node of D.



to D_{i+1}

Slags flowing to D_i