The canonical scattering diagram

... is a scattering diagram on B for a pair (Y, D), defined canonically using relative Gromov–Witten theory (which is yet another way to model holomorphic discs, just as broken lines).

It is defined in terms of certain "GHK" cohomology classes on toric blups of (Y, D).

In general, given a pair $(\tilde{Y}, \tilde{D})$, we fix an irred. component $C \subset \tilde{D}$ and consider $\beta \in H_2(\tilde{Y}, \mathbb{Z})$ such that

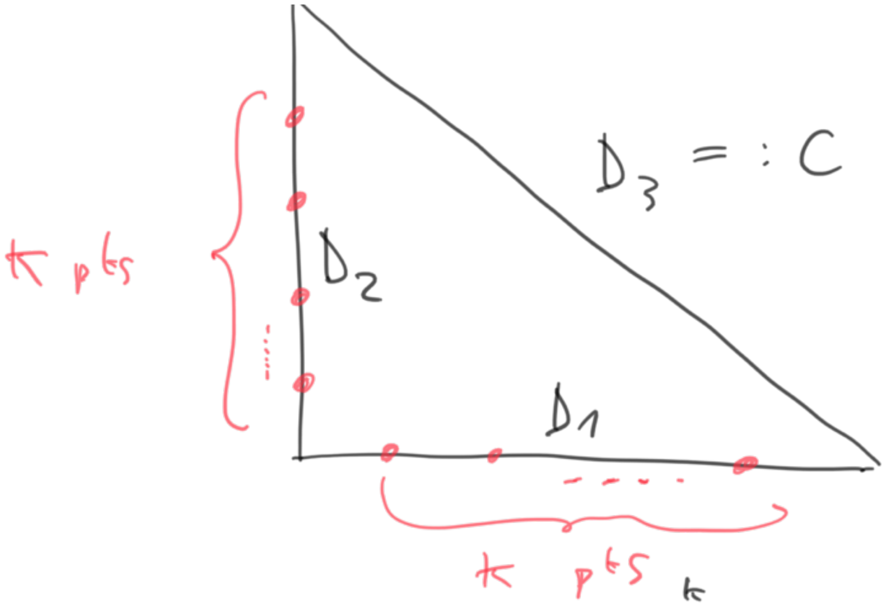
$$\beta \cdot \tilde{D}_i = \begin{cases} k_\beta & \tilde{D}_i = C \\ 0 & \tilde{D}_i \neq C \end{cases}$$

for some $k_\beta > 0$.

<u>Example.</u> Consider a toric model

$$(Y, D) \to (\mathbb{P}^2, \quad)$$

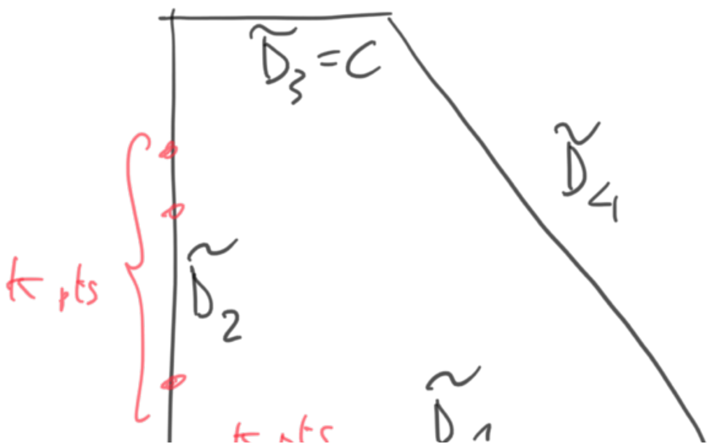
given by blowing up points on 2 toric divisors,



the class $\beta = kH - \sum_{i=1}^{k} E_{1i} - \sum_{j=1}^{k} E_{2j}$

satisfies
$$\begin{cases} \beta \cdot D_1 = \beta \cdot \left(H - \sum_{i=1}^{k} E_{1i}\right) \\ \qquad\quad = k - \sum_{i=1}^{k} (E_{1i})^2 = 0 \\ \beta \cdot D_2 = 0 \quad (\text{same argument}) \\ \beta \cdot D_3 = \beta \cdot H = k . \end{cases}$$

Similarly, we can consider a nontrivial toric blup $(\tilde{Y}, \tilde{D}) \longrightarrow (Y, D)$,



Write
$$\beta = aH - \sum_{i=1}^{k} a_{1i} E_{1i} - \sum_{j=1}^{k} a_{2j} E_{2j} - bE .$$

GHK classes (for C) determined by
$$\begin{cases} \beta \cdot \tilde{D}_1 = \beta \cdot (H - \sum E_{1i}) = 0 \\ \beta \cdot \tilde{D}_2 = \beta \cdot (H - E - \sum E_{2j}) = 0 \\ \beta \cdot \tilde{D}_3 = \beta \cdot E > 0 \\ \beta \cdot \tilde{D}_4 = \beta \cdot (H - E) = 0 \end{cases}$$

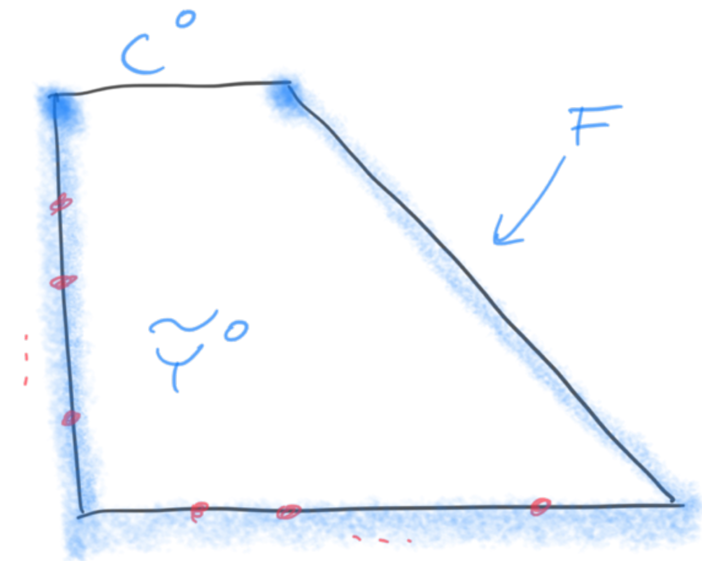
$$\Rightarrow \quad \begin{cases} a - \sum a_{1i} = 0 \\ a - b - \sum a_{2j} = 0 \\ b > 0 \\ a - b = 0 \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} b > 0 \\ \sum a_{1i} = b \\ \sum a_{2j} = 0 \end{cases} .$$

E.g. $\beta = kH - \sum E_{1i} - kE$.

$$\boxed{\text{Relative GW invariants}}$$

Set
$$F = \overline{\tilde{D} \setminus C}$$
$$\tilde{Y}^{\circ} = \tilde{Y} \setminus F$$
$$C^{\circ} = C \setminus F$$



Remove blue part, work relative to C°.

$\overline{\mathcal{M}}(\tilde{Y}^\circ/C^\circ, \beta)$:= moduli space of stable relative maps of genus 0 curves with degree $\beta \in H_2(\tilde{Y}, \mathbb{Z})$ and tangent to C° with order $k_\beta > 0$.

The condition that β is a SHK class is clearly necessary for $\overline{\mathcal{M}} \neq \emptyset$.

Expected dimension of $\overline{\mathcal{M}}$:

$$\underbrace{-K_{\tilde{Y}} \cdot \beta + (\dim \tilde{Y} - 3)}_{\text{standard dim formula for stable maps}} \underbrace{-(k_\beta - 1)}_{\text{imposing tangency condition.}}$$

$$= (\tilde{D}_1 + \dots + C + \dots + D_n) \cdot \beta$$
$$+ (2-3) - (k_\beta - 1)$$

$$= k_\beta - 1 - k_\beta + 1 = 0 .$$

So $\mathrm{vd}\,\overline{M} = 0$, and a bit surprisingly we have:

<u>Theorem</u> (Gross-Pandharipande-Siebert)

$\overline{M}$ is a <u>proper</u> stack, with a virtual fundamental class $[\overline{M}]$ of $\mathrm{vd} = 0$.

$\Rightarrow$ get well-defined, defo. invariant GW

$$N_\beta := \int_{[\overline{M}(\tilde{Y}^\circ / C^\circ, \beta)]^{vir}} 1 \quad \in \mathbb{Q} .$$

<u>From now on:</u> assume $NE(Y)$ strictly convex f.g.
(Just to simplify notation!)

<u>Define</u> a collection of rays and weight functions

$$\mathfrak{D}^{can} := \{ (\mathfrak{d}, f_\mathfrak{d}) : \mathfrak{d} \subset B \text{ rational slope} \}$$

by

$$f_\partial := \exp\left[\sum_\beta k_\beta N_\beta z^{\eta(\pi_*\beta) - \underbrace{\varphi_{\tau_\partial}}_{\text{the ``tautological'' (Givental) map}}(k_\beta m_\partial)}\right]$$

where N_β is computed on
$(\tilde{Y}, \tilde{D}) :=$ toric blp of (Y,D) corresp to ∂,

relative to
$C :=$ the anti-canonical component of $\tilde{D}$ corresp. to ∂.

__Lemma.__ Fix $J \subset P$ radical. Suppose:

(1) $\left.\begin{array}{l} \dim \tau_\partial = 2 \\ \dim \tau_\partial = 1,\ p_{\tau_\partial, d} \notin J \end{array}\right\} \Rightarrow \eta(\pi_*\beta) \in J$

(2) $\forall I \subset P$, $\sqrt{I} = J$, $\exists$ finite # of ∂ with $N_\beta \neq 0$, $\eta\pi_*\beta \notin I$.

$\Rightarrow \mathfrak{D}^{can}$ is a scattering diagram for J.

$\square$

__Example.__ $\eta : NE(Y) \xrightarrow{id} NE(Y)$.

$J = m = P \setminus P^* = P \setminus \{0\}$ the max ideal.

This choice works:

if $\sqrt{I} = J = m$, then $P \setminus I$ is finite,

and $\beta \in H_2(\tilde{Y})$ is uniquely determined

by its image $\pi_* \beta \in H_2(Y)$.

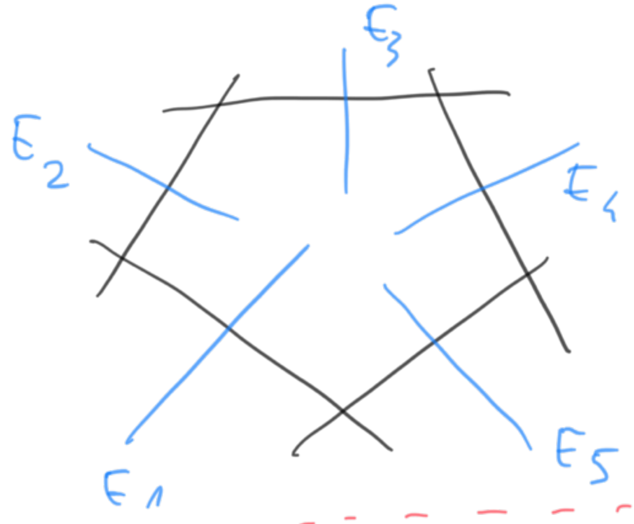
Example. We will show that the diagram

$$\left\{ \left(\rho_i, \ 1 + z^{[E_i] - \varphi_{\rho_i}(v_i)} \right), i = 1, \dots, 5 \right\}$$

$(E_i^2 = -1, \quad E_i \not\subset D, \quad E_i \cdot D_i = 1)$

is precisely the canonical scatt. diag. $\mathfrak{D}^{can}$

for $(Y, D) = ($ deg 5 del Pezzo, cycle of 5 -1 curves $)$



GHK MAIN THEOREM Fix $P_{\rho,d} = \eta([D_\rho])$ as usual.

Suppose

(i) $\forall \beta$ with $N_\beta \neq 0$, $\eta(\pi_* \beta) \in J$;

(ii) ... $\exists$ finite # of β with

(ii) $\forall\ \sqrt{I} = J$, J finite ...
$N_\beta \neq 0$, $\eta(\pi_* \beta) \notin I$;

(iii) $\eta([D_\rho]) \in J$ for ≥ 2 ρ's.

$\Rightarrow$ $\mathfrak{D}^{can}$ is a consistent scatter. diagram.

E.g. This works when $NE(Y)$ is rat'l strictly convex polyhedral, $\eta : NE(Y) \xrightarrow{id} NE(Y)$, $J = m = NE(Y) \setminus \{0\}$.

Proof of Main Thm consists of several reductions.
Some work for any $\mathfrak{D}$.

Reduction 1. It is enough to prove Main Thm for a toric model $p : (Y, D) \to (\bar{Y}, \bar{D})$.

This works by

Prop. 1. Let $p : (\tilde{Y}, \tilde{D}) \to (Y, D)$ be a toric blup. Taking $\tilde{\eta} := \eta \circ p_*$ gives a scattering diagram $\mathfrak{D}$ for $(\tilde{Y}, \tilde{D})$ (same name!).
$\mathfrak{D}$ consistent for $(\tilde{Y}, \tilde{D}) \Rightarrow$ also for (Y, D).

Reduction 2. We can assume that toric model $p : (Y, D) \to (\bar{Y}, \bar{D})$ is blup at distinct pts ... $\bar{D}$ ... with exc. div. E_{ij}.

say x_{ij} along D_i, ...

Based on:

Prop. 2. Let $(Y, D) \to S$ be a flat family with trivialization $D \cong D \times S$.
Then up to identifying $H_2(Y_s, \mathbb{Z})$ using Gauss-Manin, all fibres (Y_s, D_s) have the same canonical scatter. diagram.

Reduction 3. We may assume

- $\eta : NE(Y) \hookrightarrow P$, $P^\times = \{0\}$; (eg $\eta = id$)
- $J = m = P \setminus \{0\}$;
- a face of $P \cap NE(Y)$ is given by $NE(Y) \cap (p^* H)^\perp$, where H is a fixed ample divisor on $\bar{Y}$.

Define $G :=$ the complement in P of this face (eg $NE(Y) \setminus (p^* H)^\perp$), a radical monoid ideal.

Reduction 4. (i) $\mathfrak{D}^{can}$ is a scatter. diag. for G.
(ii) It is enough to prove that $\mathfrak{D}^{can}$ is consistent as a scatt diag for G.

So we reduced to considering deformations

$$X_I = \operatorname{Spec} \Gamma(X^\circ_{I,\mathfrak{D}^{can}}, \mathcal{O}_{X^\circ_{I,\mathfrak{D}^{can}}}), \sqrt{I} = G$$

over $\operatorname{Spec} \mathbb{C}[P]/I \supset \operatorname{Spec} \mathbb{C}[P]/G$

of the trivial family

$$X_G = \mathbb{V}_n \times \operatorname{Spec} \mathbb{C}[P]/G.$$

Rmk. GHK call the open torus orbit

$$T \subset \operatorname{Spec} \mathbb{C}[P]/G$$

the "Gross-Siebert locus".

Define $E := (P \setminus G)^{gp} \subset P^{gp}$;

$$T := \operatorname{Spec} \mathbb{C}[E] \subset \operatorname{Spec} \mathbb{C}[P]/G;$$

$$T_I := \operatorname{Spec} \mathbb{C}[P+E]/(I+E);$$

an infinitesimal thickening of T;

$$m_{P+E} := (P+E) \setminus E.$$

Reduction 5. It is enough to show that $\mathfrak{D}^{can}$ is consistent as a scatter. diag. for

$P+E$, m_E.

Cle it is enough to consider deformations of the trivial family

$\mathbb{V}_n \times T \subset \mathbb{V}_n \times \mathrm{Spec}\,\mathbb{C}[\ldots]$

So in all the following

$$\text{Notation} \begin{cases} P \longmapsto P + E \\ J \longmapsto \mathfrak{m}_P + E . \end{cases}$$

$\boxed{\text{REDUCTION TO SCATTERING IN } \mathbb{R}^2}$

Recall we have a toric model

$$p : (Y, D) \longrightarrow (\overline{Y}, \overline{D}).$$

Decorate all notation by $\overline{(\ldots)}$ on toric base.

So eg $\overline{B} = \mathbb{R}^2$ with linear str.

$\overline{\Sigma}$ = toric fan etc.

Define $\overline{\varphi}$ by $p_{\overline{\rho}, \overline{\varphi}} = p^*[D_{\overline{\rho}}] \in P$.

Get (trivial) idea

$$\overline{\pi} : \overline{\mathbb{P}}_0 \longrightarrow \overline{B}_0,$$

with convex section $\overline{\varphi}$,

$$\overline{\mathbb{P}}_0 \cong \overline{B}_0 \times P^{gp}_{\mathbb{R}},$$

$$\overline{\mathcal{P}} = \overline{\pi}_* \Lambda_{\overline{\mathbb{P}}_0} \cong \overline{B}_0 \times (P^{gp} \oplus M).$$

Let
$$\nu : B \longrightarrow \bar{B}$$
denote canonical PL identification induced by toric model. Note
$$\nu|_\sigma : \sigma \xrightarrow{\cong} \bar{\sigma} \quad \forall \sigma \in \Sigma_{max}$$
as affine linear mflds, so we have
$$B(\mathbb{Z}) = \bar{B}(\mathbb{Z}).$$

Let
$$\tilde{\nu} : \mathbb{P}_0 \longrightarrow \bar{\mathbb{P}}_0$$
denote unique equivariant lift of ν,
$$\tilde{\nu}_\sigma : P_{\varphi_\sigma} \longrightarrow P_{\bar{\varphi}_{\bar{\sigma}}}$$
the induced iso of Mumford monoids,
$$\tilde{\nu}_\rho : \underset{\displaystyle P_{\varphi_\rho}}{\underset{\cap}{\{\varphi_\rho(m)+p : m /\!/ \rho,\ p \in P\}}} \xrightarrow[\text{bijection}]{} \underset{\displaystyle P_{\bar{\varphi}_{\bar{\rho}}}}{\underset{\cap}{\{\bar{\varphi}_{\bar{\rho}}(m)+p \mid m /\!/ \bar{\rho}\}}}.$$

Recall we have a standard notion of ("local") scattering diagram in $\mathbb{R}^2$, as described by Veronica in her lecture.
We can decorate that standard notion with a map of monoids
$$Q \longrightarrow M.$$

$r: Q$

In Veronica's case

$$r: M \xrightarrow{\text{id}} M.$$

We have in mind the case

$$r: \underbrace{P_{\overline{\varphi}}}_{} \longrightarrow M.$$

genuine, toric Mumford monoid of the convex function $\overline{\varphi}$ with $P_{\overline{\varphi},\overline{\varphi}} := p^{*}[D_{\overline{\varphi}}]$.

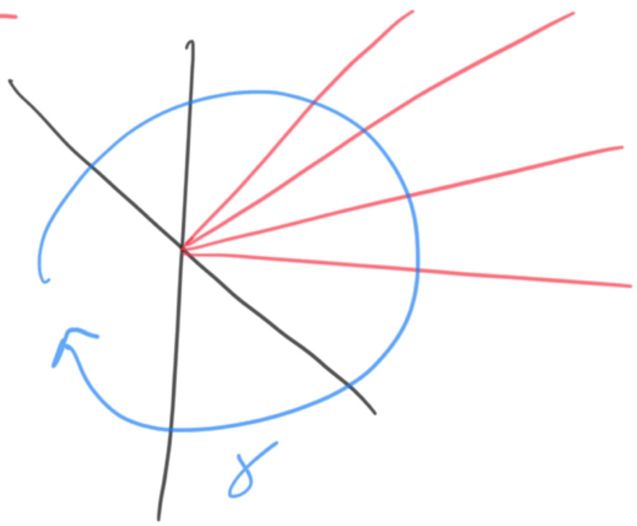
The definition is precisely the same, with scattering automorphisms given by

$$\theta_{\gamma,\mathfrak{d}}(z^{q}) := z^{q} f_{\mathfrak{d}}^{\langle n_{\mathfrak{d}}, r(q)\rangle}$$

for $f_{\mathfrak{d}} = 1 + \sum_{p} c_p z^{p}$, $r(p) \neq 0$ a positive multiple of $m_{\mathfrak{d}}$.

(Working modulo powers of $m_Q = Q \setminus Q^{\times}$).

Then, we have the usual notion of saturation ("local consistency"):



$$\Theta_{\gamma,\mathfrak{D}} = \mathrm{id} \quad \forall \text{ loop } \gamma.$$

Recall saturation exists and is essentially unique:

$$\mathfrak{D} \longmapsto \mathrm{Scatter}(\mathfrak{D}).$$

Problem: reduce scattering diagrams from

$$\left.\begin{array}{l} (B, \Sigma), P, \varphi, m_P \\ \\ \text{to} \quad \underbrace{(\bar{B}, \bar{\Sigma})}_{\text{toric base}}, \bar{P}, \bar{\varphi}, m_{\bar{P}} \end{array}\right\} \text{toric model.}$$

Step I. Description of $f_{\rho_i}^{\mathrm{can}}$ mod G: we have

$$f_{\rho_i} = g_{\rho_i} \prod_{j=1}^{\ell_i} \left(1 + b_{ij} X_i^{-1}\right)$$

for $X_i := z^{\varphi_{\rho_i}(v_i)}$

$b_{ij} := z^{\eta([E'_{ij}])}$

and $g_{\rho_i} \equiv 1 \mod G$.

Reason: first show that a stable map contributing to N_β must map to E_{ij} for some j, ie, it is a k_β-fold cover of E_{ij}. Follows from rather standard

Then the claim follows from multiple cover computation in GW theory, i.e. the contribution is

$$\frac{(-1)^{\kappa_\beta - 1}}{\kappa_\beta^2},$$

writing $f_{\rho_i} = \exp \log(\cdots)$.

Step II. Now define a "local" scatter diag

$$\nu(\mathfrak{D})$$

on $\overline{B}$ with rays

$$\begin{cases} (\nu(\mathfrak{d}), \tilde{\nu}_{\tau_\mathfrak{d}}(f_\mathfrak{d})), \ \mathfrak{d} \neq \rho_i \quad i = 1, \dots, n; \\ (\overline{\rho_i}, \underbrace{\tilde{\nu}_{\tau_{\rho_i}}(g_{\rho_i})}), \ i = 1, \dots, n; \\ \qquad \text{well defined because weight functions are // to rays!} \\ (\overline{\rho_i}, \prod_{j=1}^{l_i} (1 + \underbrace{b_{ij}^{-1} \overline{X}_i}_{\text{note change of signs!}})). \end{cases}$$

Warning: we want this to be a scatter diagram on $\mathbb{R}^2$, with monoid

$$P_{\bar{\varphi}} = \{(m, \bar{\varphi}(m) + p) : m \in M, p \in P\} \subset M \times P^{gp}$$

$$\downarrow$$
$$M,$$

the genuine toric Mumford monoid.

So $\forall\, \mathfrak{d}$ we should have

$$f_{\mathfrak{d}} \in \widehat{\mathbb{C}[P_{\bar{\varphi}}]} \quad (m_{P_{\bar{\varphi}}}\text{-completion}).$$

<u>But</u> a priori, by construction, we only have

$$f_{\mathfrak{d}} \in \widehat{\mathbb{C}[P_{\bar{\varphi}_{\tau_{\mathfrak{d}}}}]}, \text{ where}$$

$P_{\bar{\varphi}_{\tau_{\mathfrak{d}}}}$ is a localization of $P_{\bar{\varphi}}$.

<u>Main "local" result</u> (Kontsevich-Soibelman; Gross-Pandharipande-Siebert)

Let $\bar{\mathfrak{D}}_0 = \left\{\left(\bar{\rho}_i, \prod_{j=1}^{l_i} (1 + b_{ij}^{-1} \bar{X}_i)\right), i = 1, \dots, n\right\}.$

Then,

$$\nu(\mathfrak{D}^{can}) = \mathrm{Scatter}(\bar{\mathfrak{D}}_0) \; (=: \bar{\mathfrak{D}}).$$

So in particular $\nu(\mathfrak{D}^{can})$ is a scatter. diag for $P_{\bar{\varphi}}$, and it is saturated.

Main global result

(i) The saturated scatter diag $\nu(\mathfrak{D}^{can})$ is consistent: $\forall$ path γ, $\forall \sqrt{I} = m_{P_{\bar{\sigma}}}$,

$$\overline{\mathrm{Lift}_{Q'}(q)} = \theta_{\gamma,\nu(\mathfrak{D})_I}\left(\overline{\mathrm{Lift}_Q(q)}\right)$$

as elements of $\mathbb{C}[P_{\bar{\sigma}}]/I$.

(ii) As a consequence, the canonical scattering diagram $\mathfrak{D}^{can}$ is consistent.

Rmk. Indeed, there is an isomorphism

$$X^{\circ}_{I,\mathfrak{D}} \cong \bar{X}^{\circ}_{I,\nu(\mathfrak{D})}$$

over $\mathrm{Spec}\,\mathbb{C}[P]/I$, although it cannot extend across the singular locus (because the $SL(2,\mathbb{Z})$-structures on B°, $\bar{B}^{\circ}$ are different!).

Very rough idea of proof. I only want to make plausible the fact that on $\bar{B}$, $\nu(\mathfrak{D})$, consistency is implied by saturation (i.e. trivial monodromy).

Step I. Consider deformation theory of broken $\bar{\ }\ \ \ (\mathfrak{D})$ for fixed $q \in \bar{B}(\mathbb{Z})$

lines on B, $\nu(\vartheta)$, [illegible] and varying endpoint $Q \in \overline{B}_0$, modulo I.

Show that γ deforms continuously, and $\mathrm{Mono}(\gamma)$ is constant, as long as Q does not cross $\underline{\mathrm{Supp}}_I(\nu(\vartheta))$ or γ does not cross $O \in \overline{B}$.

Equivalently, as long as Q does not cross a wall in

$$M_{\mathbb{R}} \setminus \mathcal{U}_I$$

where

$$\mathcal{U}_I := \mathrm{Supp}_I(\nu(\vartheta)) \cup \mathbb{R}_{\geq 0}\{-r(p) : p \in P_{\overline{\varphi}} \setminus I\}.$$

Main point:

__Step II.__ Show that as Q crosses a wall, say $\partial \subset M_{\mathbb{R}} \setminus \mathcal{U}_I$, curves γ that do not deform continuously, but s.t. $\Theta_\partial(\mathrm{Mono}(\gamma)) = \mathrm{Mono}(\gamma)$, nevertheless give the same contribution to

$$\overline{\mathrm{Lift}_{Q^{\pm}}(a)},$$

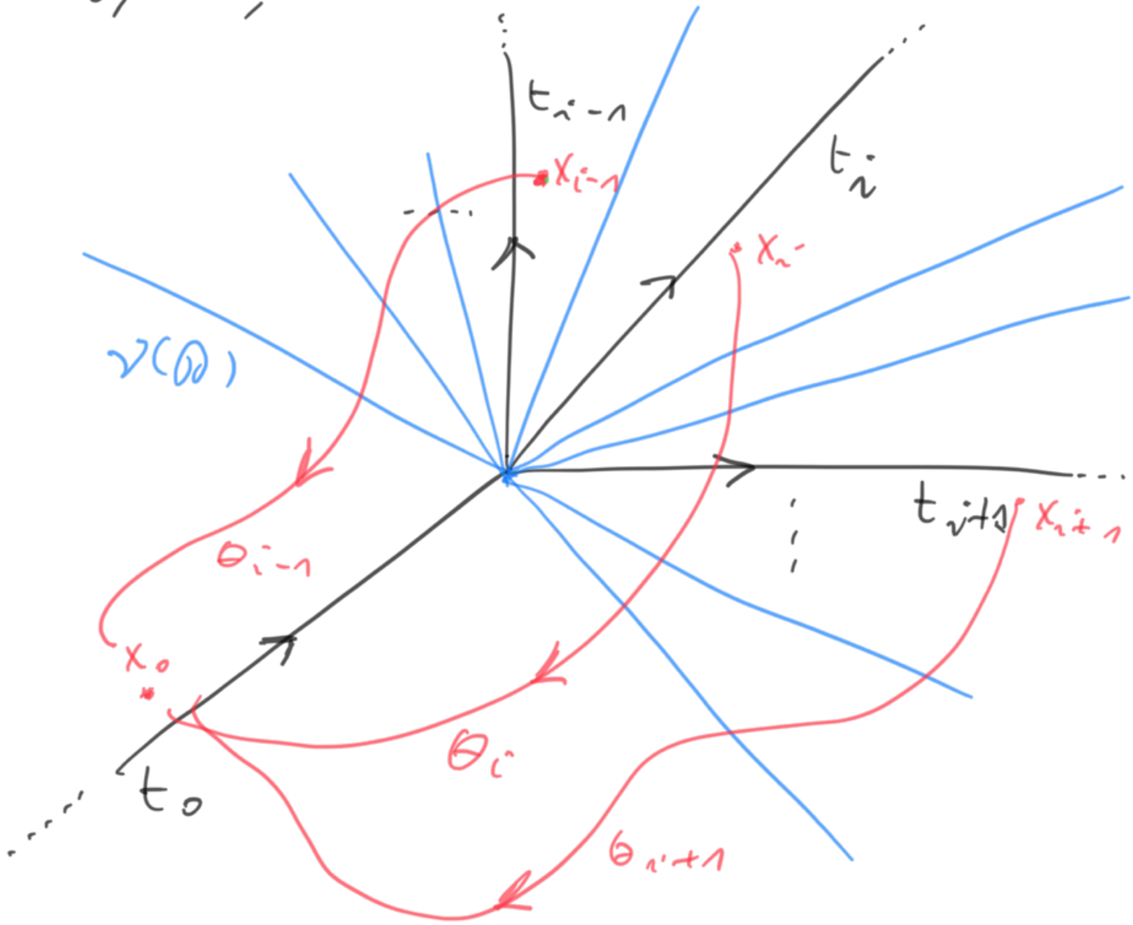
ie

$$\sum_{\gamma^-} \mathrm{Mono}(\gamma^-) = \sum_{\gamma^+} \mathrm{Mono}(\gamma^+).$$

this follows (nontrivially) from the following "conserved quantity":

following

$\forall\, t_0$, $\exists$ (essentially <u>unique</u>) t_i such that
$$c_0 z_0^{q_0} = \sum_i \theta_i (c_i z^{q_i}) \bmod I,$$
where t_0, t_i, θ_i are given by picture



But for this uniqueness, we need θ_i to be <u>independent</u> of the choice of path.
this holds precisely because $\nu(\mathfrak{D})$ is <u>saturated</u>. $\square$

<u>Exercise</u>. (i) Consider the toric fan with rays

$$\mathbb{R}_{\geq 0}\{(1,0),(1,1),(0,1),(-1,0),(0,-1)\}.$$

which defines $(\bar{Y}, \bar{D})$ with toric divisors

$$\{\bar{D}_1, \bar{D}_2, \bar{D}_3, \bar{D}_4, \bar{D}_5\}.$$

Fix $p \in \bar{D}_4$, $q \in \bar{D}_5$.

Show that

$$(Y, D) := Bl_{p,q}(\bar{Y}, \bar{D}) \longrightarrow (\bar{Y}, \bar{D})$$

is a toric model for the deg 5 del Pezzo.

(ii) Consider the scatt diagram.

$$\mathfrak{D} := \{(\rho_i, 1 + z^{[E_i] - \varphi_{\rho_i}(v_i)}), i = 1, \dots, 5\}$$

on B (usual notation).

Show that $\mathcal{S}(\mathfrak{D})$ on $\bar{B}$ is saturated.

$\wedge$ $\wedge$ can