LCY SURFACE $(Y, D = D_1 + \dots + D_n)$

$D \in |-K_Y|$; D_i RATIONAL.

$D = \{$  ;  $\}$

enough to assume D is connected singular nodal in $-K_Y$ since then $p_a(D) = 1$ by adjunction.

AKA "LOOIJENGA PAIRS"

$$H^0(Y, K_Y(D)) = H^0(Y, K_Y - K_Y) \cong \mathbb{C}$$

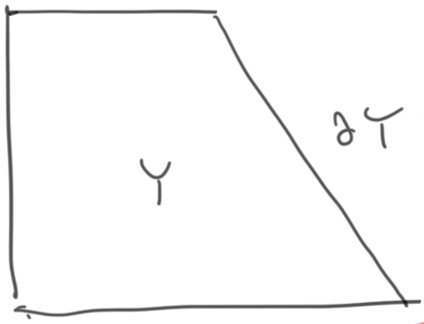
$\Rightarrow \exists!\ \Omega$ holo symp on $Y \setminus D$ up to $\mathbb{C}$

WITH SIMPLE POLES AT ∞.

$\Rightarrow (U = Y \setminus D, \Omega)$ is CY

EG: $(Y, D) = $ (toric surf, toric bdry)

$\boxed{D \sim -K_Y}$



∂Y = torus inv. divisors

toric pair!

EG: $(Y, D) \xrightarrow{p} (\bar{Y}, \bar{D})$ "toric model"

p = blowup at $\{x_{ij}\} \subset \bar{D}_i$

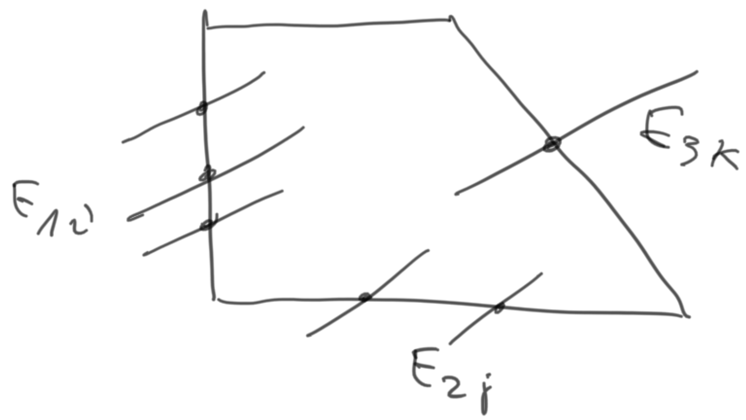
infinitesimally near points allowed

D = prop. transform of $\bar{D}$

$$K_Y = p^* K_{\bar{Y}} + \sum_{i,j} E_{ij}$$

$$D = p^* \bar{D} - \sum_{i,j} E_{ij} \sim -p^* K_{\bar{Y}} - \sum_{i,j} E_{ij}$$

$$= -K_Y.$$



TORIC BLUP: $\pi : \tilde{Y} \xrightarrow[\text{BIR}]{} Y$ such that $(\pi^{-1}(D))^{\text{red}}$ is anti-canonical cycle of rational curves.

RMK: by canonical bdle formula, $\pi : \tilde{Y} \to Y$ is blup along Z with $Z^{\text{red}} \subset \operatorname{Sing} D$.

THM: $\forall$ (Y, D) $\exists$ toric blup $(\tilde{Y}, \tilde{D})$ which has toric model
$$(\tilde{Y}, \tilde{D}) \to (\overline{Y}, \overline{D}).$$

Pf. 2 operations:

(i) $p : Y \to Y'$ bl down -1 curve $E \not\subset D$,
$(Y', D') := (Y', p_*(D))$.
Then, can replace (Y, D) with (Y', D').

(ii) $p : Y'' \to Y$ blup a node of D
$(Y'', D'') := (Y'', (p^{-1}(D))^{\text{red}})$.
Then, can replace (Y, D) with (Y'', D'').

"MMP FOR SURFACES": SINCE $-K_Y$ is EFFECTIVE after a finite # of (i), (ii), we have
$Y \to$ { ruled surface $q : Y \to \mathbb{P}^1$

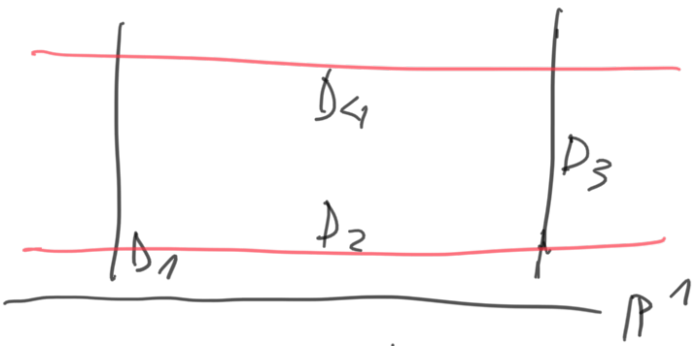
$Y \neq \mathbb{P}^2$: blup node of D, reduce to ruled surface by (ii).

CONSIDER # COMPONENTS $D_i \subset$ FIBRES OF q:

≥ 3

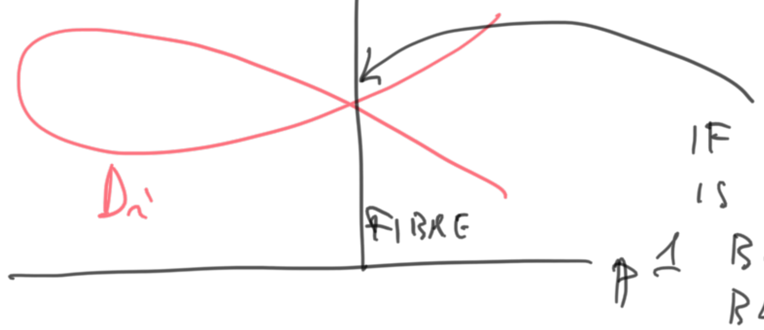
IMPOSSIBLE! ($D_1 + \dots + D_n$ is a cycle)

$= 2$



POSSIBLE: $D_1 + D_2 + D_3 + D_4$ IS TORIC $\equiv$ BDRY OF Y.

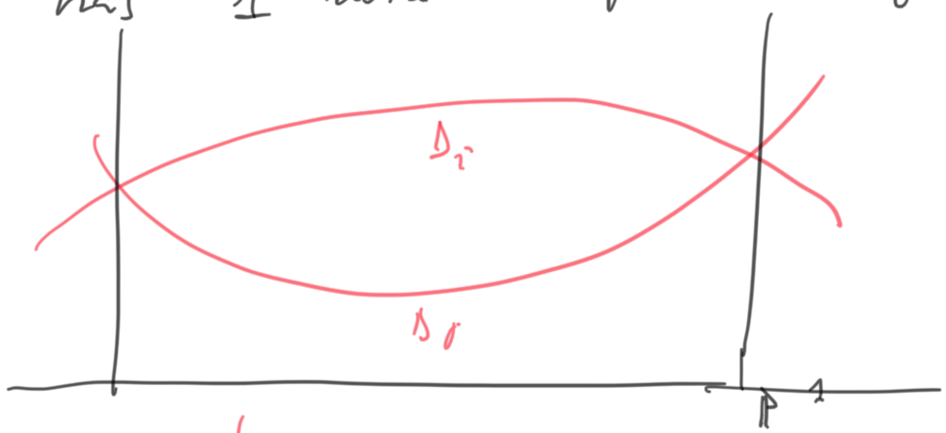
≤ 1



IF $D_i \not\subset$ FIBRE IS NODAL, BLUP NODE, BLDOWN PROPER TRANS OF FIBRE

$\Rightarrow$ resulting divisor is anticanonical and has 1 more component in fibres.

(same for



)

$\Rightarrow$ reduce to

Now use $Y \cong \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(e))$ for some e,
$D_2 + f \sim -K_Y$ to compute
$D_2 \sim 2C_0 + (e+1)f$, where
$C_0^2 = -e$, so $C_0 \cdot D_2 = -e+1$
and $e = 0$ or 1.

So $Y \cong \mathbb{P}^1 \times \mathbb{P}^1$ or $Bl_x \mathbb{P}^2 \longrightarrow \mathbb{P}^1$ (projection to line)
and we can reduce easily to previous cases
(EXERCISE) $\square$

NOTIONS OF FAMILIES:

$(\mathcal{Y}, \mathcal{D}) \longrightarrow B$ restricting to Looijenga pairs (Y_b, D_b);

similarly $(\mathcal{U} := \mathcal{Y} \setminus \mathcal{D}, \Omega)$ restricting to holo symp forms Ω_b.

TORELLI THEOREMS (LOCAL AND GLOBAL)

THM (KALEDIN, VERBITSKY) Let (X, Ω) be algebraic holomorphic symplectic with $H^i(\mathcal{O}_X) = 0$ $\forall i > 0$. Then the coarse moduli space of formal deformations of
$\sim$ the completion of $H^2(X, \mathbb{C})$

(X,Ω) is = the
at $[\Omega]\in H^2(X,\mathbb{C})$.

ISOMORPHISM OF LOOP PAIRS: (Y^1,D^1) ; (Y^2,D^2) pairs
$f: Y^1 \to Y^2$ with $f(D^1_i)=f(D^2_i)$
respecting fixed "orientations" i.e.
generators of $H_1(D^1)$, $H_1(D^2)$.

PERIODS OF LOOP PAIR: $D^\perp := \{\alpha\in \mathrm{Pic}(Y) \mid \alpha\cdot[D_i]=0\ \forall i\}$

$$\phi_Y : D^\perp \longrightarrow \mathrm{Pic}^0(D)\cong \mathbb{C}^*,$$
$$L \longmapsto L|_D$$

$\phi_Y \in T_{D^\perp} := \mathrm{Hom}(D^\perp, \mathbb{C}^*)$ is
PERIOD POINT OF (Y,D)

RMK: We have ex. sequence

$$0 \longrightarrow \underbrace{\mathbb{Z}}_{\text{gen by some real 2-torus } \gamma} \longrightarrow H_2(Y\setminus D,\mathbb{Z}) \longrightarrow D^\perp \longrightarrow 0$$

Normalise by $\int_\gamma \Omega = 1$. Then get
$$\int_\cdot \Omega : D^\perp \longrightarrow \mathbb{C}^*.$$
This coincides with period pt ϕ_Y.

A "GLOBAL TORELLI" should say that "periods determine isomorphism type".

THM (GHK) Fix loop pairs
(Y_1, D) (Y_2, D)

identify lattice abstractly.

preserving int. pairing.

Suppose we have
$$\mu : \mathrm{Pic}(Y_1) \xrightarrow{\cong} \mathrm{Pic}(Y_2)$$
(clearly this is necessary to get iso of pairs).

Then $\mu = f^*$ for some $f : (Y_1, D) \xrightarrow{\cong} (Y_2, D)$ iff:

(1) $\mu([D_i]) = [D_i] \ \forall i$;

(2) $\phi_{Y_2} \circ \mu = \phi_{Y_1}$; "same periods"

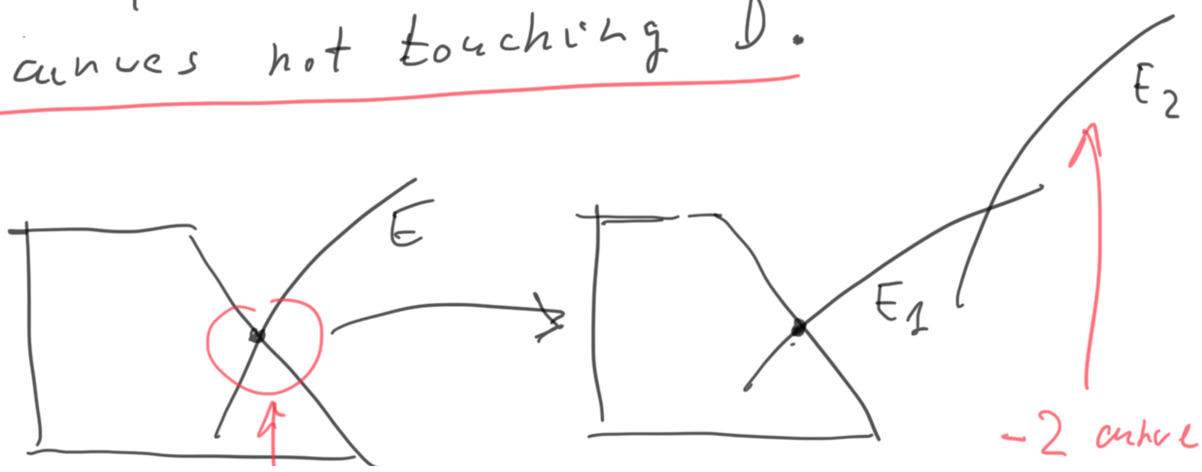
extra conditions:

(3) certain special classes $\Delta_Y \subset \mathrm{Pic}(Y)$, appearing at special pts in moduli, are preserved:
$$\mu(\Delta_{Y_1}) = \Delta_{Y_2};$$

(4) a suitable subcone $C^{++} \subset C^+$ of ample cone is preserved:
$$\mu(C_1^{++}) = C_2^{++}.$$

Here $\Delta_Y \subset \mathrm{Pic}(Y)$ are the classes of -2 curves not touching D.

E.g.



TORIC MODEL

BLOW UP AGAIN

THE INTERSECTION MATRIX

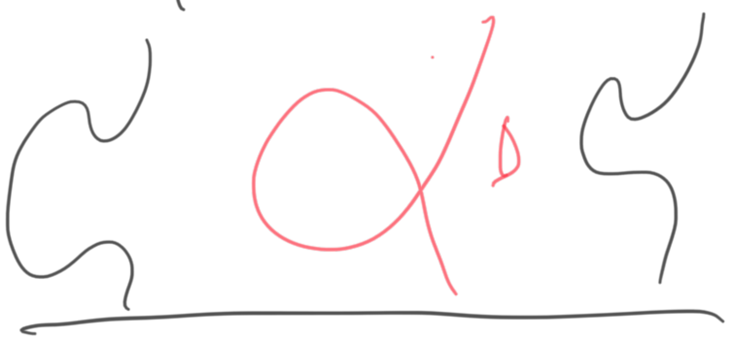
This is the matrix $(D_i \cdot D_j)$.

"Positive case": $(D_i \cdot D_j)$ is not negative semidef.

$\Rightarrow U = Y \setminus D$ is AFFINE.

"Strictly semidef case": up to contracting -1 curves in D and defo equivalence, D is fibre of a rat'l elliptic fibration $f : Y \to \mathbb{P}^1$.

Generically:



$\mathbb{P}^1$

Note $U = Y \setminus D$ is not affine (it contains compact holomorphic curves).

"Negative case" $(D_i \cdot D_j) < 0$

Then $\exists\, f : Y \longrightarrow \overline{Y}$ a birational morphism contracting D to a singular pt $p \in \overline{Y}$ (a "cusp").

Note $U = Y \setminus D$ is <u>not</u> affine:

$Y \setminus D \cong \overline{Y} \setminus p$ the complement of a pt in a projective surface.