"MUMFORD DEGENERATIONS"

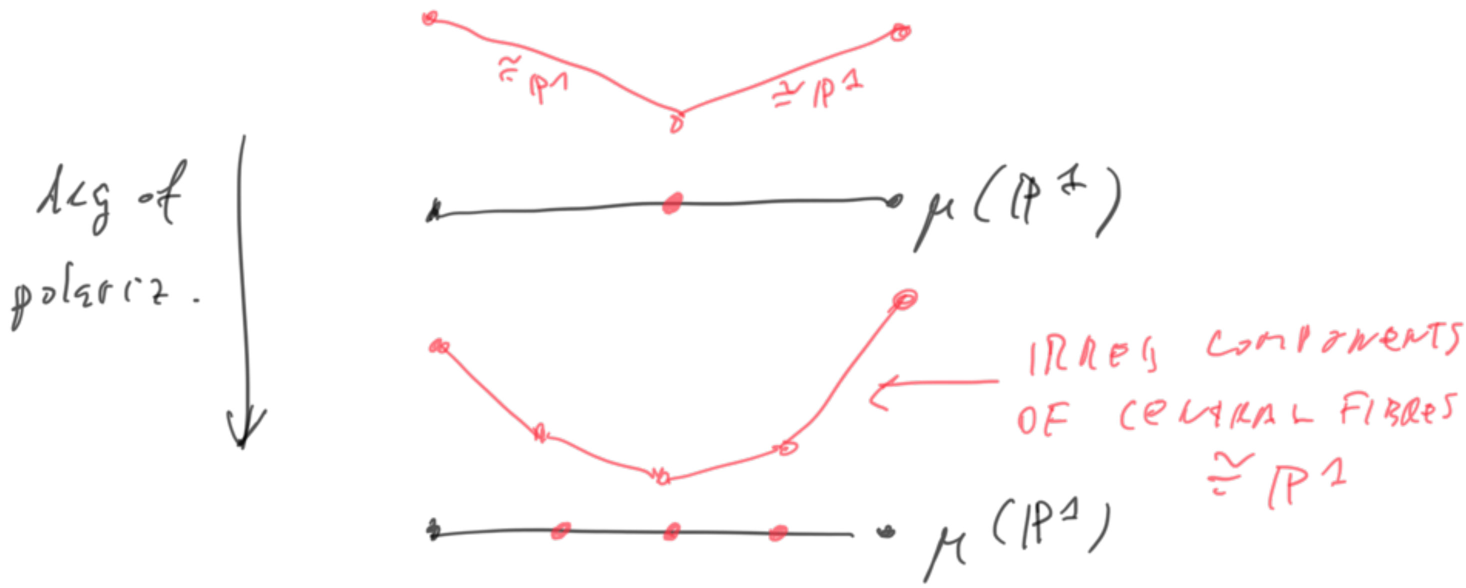
• Toric case

For Y toric, $\exists$ well known corresp.

$$\left\{\begin{array}{l}\text{one-param}\\ \text{deg's of } Y,\\ \text{compatible with}\\ \text{torus action}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{rat'l PL}\\ \text{convex functions}\\ \text{on polytope}\end{array}\right\}$$

widely used in toric cplx diff geom.

Basic idea (for $\mathbb{P}^1$):



Restricting the degenerat. to structure torus gives a deg of $(\mathbb{C}^*)^2$ to a bunch of affine planes $\mathbb{C}^2$ glued along toric strata.

We explain this in a special case, ... FAN instead of polytope.

but consider FAN ...

and allow certain TORIC FAMILIES OF DEGENERATIONS.

$P :=$ fixed e.g. toric monoid

i.e. $P = \sigma_P \cap P^{gp}$

THE LATTICE GENERATED BY MONOID P.

SOME RATIONAL CONVEX POLY CONE

Note: $\operatorname{Spec} \mathbb{C}[P]$ will parametrize (some) toric degenerations of $(\mathbb{C}^*)^2$.

$\Sigma \subset M_{\mathbb{R}}$ a FAN

$|\Sigma| =$ support (Σ)

Suppose: $|\Sigma|$ is convex.

convex, not complete

complete

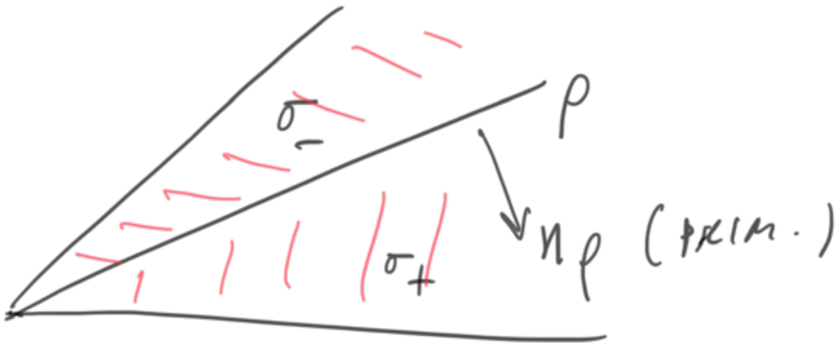
Consider $\varphi : |\Sigma| \longrightarrow P_{\mathbb{R}}^{\cdots}$

a <u>Σ-PL function</u>:

φ is C^0, and $\forall \sigma \in \Sigma_{\max}$ $\varphi|_\sigma$ is given by $\varphi_\sigma \in \mathrm{Hom}_{\mathbb{Z}}(M, P^{gp})$
$\cong N \otimes P^{gp}$

$\Rightarrow$ get <u>bending parameters</u>

$$\varphi_{\sigma_+} - \varphi_{\sigma_-} = n_\rho \otimes p_{\rho,\varphi}$$



Convexity: all $p_{\rho,\varphi} \in P \subset P^{gp}$

strict convexity: all $p_{\rho,\varphi} \in P \setminus P^{\times}$.
(e.g. nonzero if σ_P is strictly convex)

<u>Note</u>: given a hom. $P \longrightarrow \mathbb{R}$,
if φ is convex, we get a rat'l PL convex function on FAN / polytope

as discussed above.

MONOID OF A P-CONVEX FUNCTION : $P_\varphi := \left\{ (m, \varphi(m) + p) \;\middle|\; \begin{array}{l} m \in M \cap |\Sigma|, \\ p \in P \end{array} \right\}$

$\subset M \times P^{gp}$

So $P_\varphi :=$ " POINTS LYING ABOVE THE GRAPH OF φ ".

Inclusion of monoids

$$P \hookrightarrow P_\varphi$$
$$p \longmapsto (0, p)$$

$$\Rightarrow \quad \mathbb{C}[P] \hookrightarrow \mathbb{C}[P_\varphi],$$

$$\boxed{\operatorname{Spec} \mathbb{C}[P_\varphi] \longrightarrow \operatorname{Spec} \mathbb{C}[P] \quad \text{flat morphism}}$$

General fibre : recall $P \subset P^{gp}$ gives open embedding $\underbrace{\operatorname{Spec} \mathbb{C}[P^{gp}]}_{\text{locus where } z^p\text{'s are invertible}} \hookrightarrow \operatorname{Spec} \mathbb{C}[P]$

family over $\operatorname{Spec} \mathbb{C}[P^{gp}]$ is given by

$$\operatorname{Spec} \underbrace{\mathbb{C}[P_\varphi]}_{\substack{(m,\, \varphi(m)+p) \\ p \in P}} \otimes_{\mathbb{C}[P]} \underbrace{\mathbb{C}[P^{gp}]}_{\substack{(0,\, q) \\ q \in P^{gp}}}$$

(but P

$(m', \varphi(m)) + (0, p)$ cancel

$$\cong \operatorname{Spec} \mathbb{C}[M] \times \operatorname{Spec} \mathbb{C}[P^{gp}]$$

$\Rightarrow$ trivial family with fibre $(\mathbb{C}^*)^2$.

Special fibres: fibre over unique torus fixed pt $0 \in \operatorname{Spec} \mathbb{C}[P]$:

$$\operatorname{Spec} \quad \mathbb{C}[P_\varphi] \otimes_{\mathbb{C}[P]} (\mathbb{C} \cong \operatorname{Spec} \mathbb{C}[P]/m_0)$$

where $\operatorname{Spec} \mathbb{C} \longrightarrow \operatorname{Spec} \mathbb{C}[P]$ is the closed pt;

Multiplication in the tensor product:

$$z^{(m, \varphi(m))} z^{(m', \varphi(m'))} =$$

$$= z^{(m+m', \varphi(m+m') + p)} \text{ for some } p \in P \text{ by convexity}$$

$$= z^{(m+m', \varphi(m+m'))} \underbrace{z^{(0, p)}}_{}$$

(recall m_0 corresponds to monoid ideal $P \setminus P^\times$!) $\longrightarrow$ 0 unless $p \in P^\times$.

$\Rightarrow$ if Σ := fan of max domains of linearity of

$$\bar{\varphi} : |\Sigma| \xrightarrow{\varphi} P^{gp} \longrightarrow P^{gp}/P^\times ,$$

then fibre over fixed pt is

$$C \dots \bigoplus \mathbb{C} z^m$$

$$\mathrm{Spec}\ \mathbb{C} \quad m\in M\cap|\Sigma|$$

where $z^m \cdot z^{m'} = \begin{cases} z^{m+m'} & m, m' \text{ lie in cone of } \Sigma; \\ 0 & \text{otherwise}. \end{cases}$

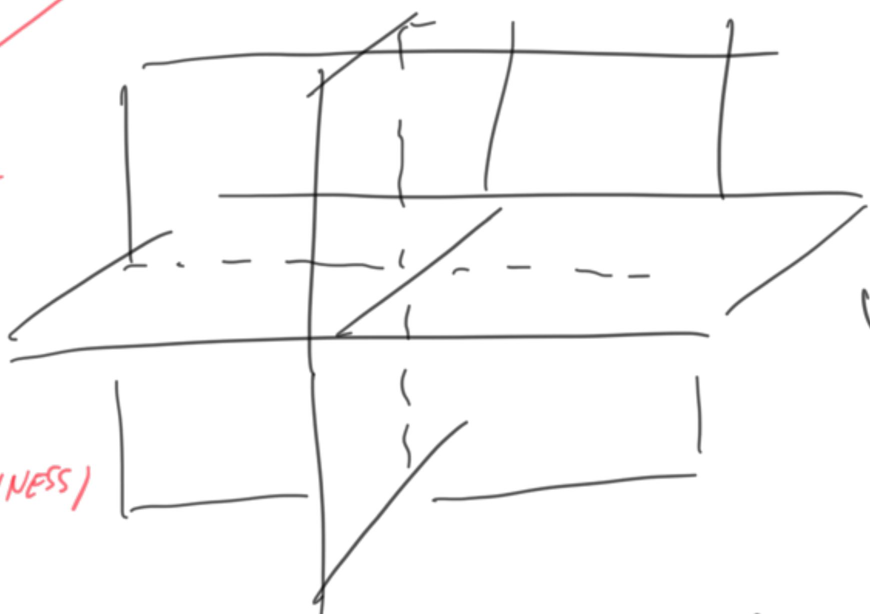
$\Rightarrow$ the irreducible components are

$$\mathrm{Spec}\ \mathbb{C}[\sigma\cap M],\ \sigma\in\Sigma_{max}.$$

Eg: If $\mathcal{C}$ is strictly convex and Σ has $\underbrace{n \text{ rays}}_{n\geq 3}$, then fibre is the VERTEX

$$\mathbb{V}_n := \mathbb{A}^2_{x_1x_2} \cup \mathbb{A}^2_{x_2x_3} \cup \cdots \cup \mathbb{A}^2_{x_nx_1} \subset \mathbb{A}^n_{x_1\cdots x_n}.$$

COPIES OF $\mathbb{A}^2$ BY SECANT PROPERTY (SMOOTHNESS)



$\mathbb{V}_3$.

Rmk: fibres over other torus fixed pts can be described in the same way as unions of $\mathbb{A}^2$'s, by localization

$p \longmapsto p - Q$ is a face.

CANONICAL MUMFORD DEGENERATION

Let $NE(Y)_{\mathbb{R}_{\geq 0}}$:= cone in $A_1(Y, \mathbb{R})$ generated by curve classes

aka "cone of num. eff. curves"

$$NE(Y) := NE(Y)_{\mathbb{R}_{\geq 0}} \cap A_1(Y, \mathbb{Z})$$

(monoid)

Rmk. For Y underlying a Loo pair, $NE(Y)$ is not a f.g. monoid, in general.

Toric case: $NE(Y) \subset A_1(Y, \mathbb{Z}) \cong H_2(Y, \mathbb{Z})$ is f.g.

More generally, in "positive case" $(D_i \cdot D_j)$ has a > 0 eigenvalue

$\Rightarrow$ $NE(Y)_{\mathbb{R}_{\geq 0}}$ IS RAT'L POLYH.

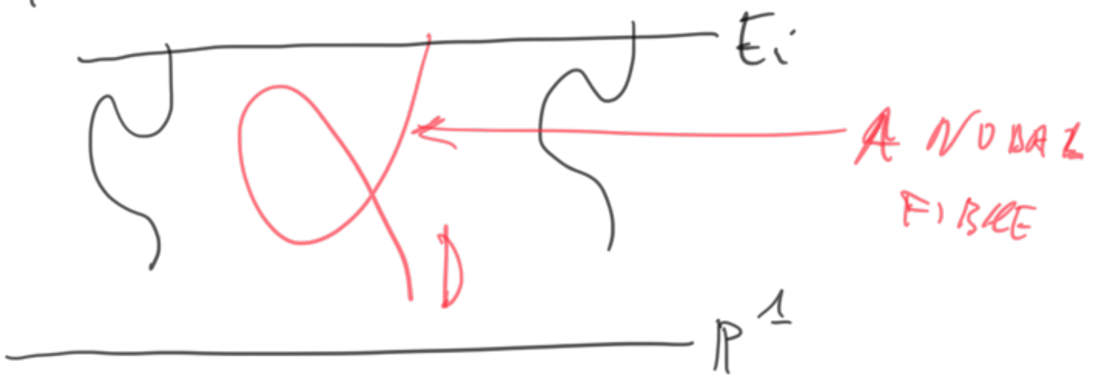
$\Rightarrow$ $NE(Y)$ IS FG MONOID.

This fails for $(D_i \cdot D_j) \leq 0$

E.g. Take rat'l elliptic surface

E.g.

$$Y \longrightarrow \mathbb{P}^1$$



Recall the construction of Y: fix a pencil of irreducible cubics $\subset \mathbb{P}^2$, blup the g base pts, so get map to $\mathbb{P}^1$ with g distinguished sections, the exc. divisors $E_1, \dots, E_g$.

(Y, D) is a Loo pair:

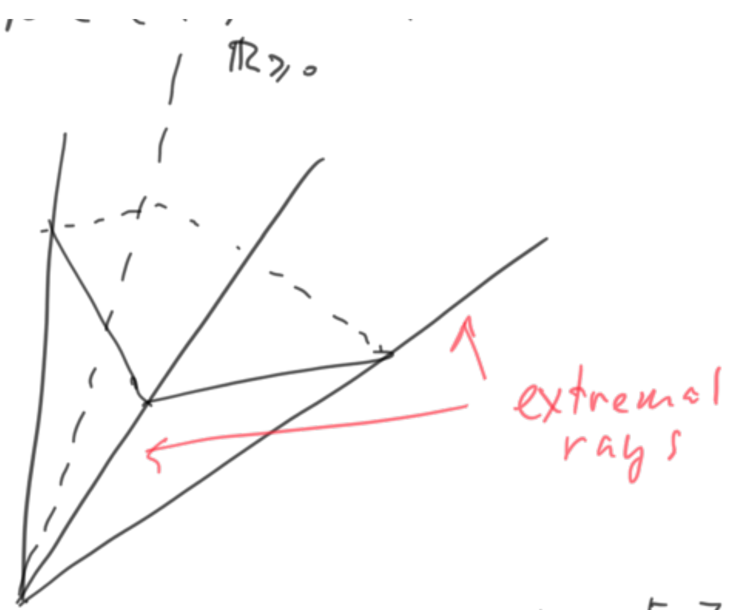
$$-K_Y \sim \pi^*(\text{a nodal cubic in } \mathbb{P}^2) - E_1 \cdots - E_g$$

in the pencil

$$\sim D.$$

Claim: for generic pencil, $NE(Y)$ is NOT f.g. $\Leftarrow$ $NE(Y)_{K_Y \geq 0}$ NOT POLYHED.

This is a very classical example. You need to know that if $C \subset Y$ is a irreducible curve with $C^2 < 0$, then $[C] \in A_1(Y)$ generates an extremal ray of $\overline{NE}(Y) \subset A_1(Y, \mathbb{R})$.



Proof: $\forall C$ IRRED., $\overline{NE}(Y)_{\mathbb{R}\geq 0}$ is spanned by $\mathbb{R}_{\geq 0}[C]$ and

$\overline{NE}(Y)_{\mathbb{R}\geq 0, C\geq 0} :=$ classes $[D]$ with $D\cdot C \geq 0$.

Now if $C^2 < 0$, then $[C] \notin \overline{NE}(Y)_{\mathbb{R}\geq 0, C\geq 0}$

$\Rightarrow [C]$ generates extremal ray. $\square$

So to see $NE(Y)_{\geq 0}$ not polyhed it's enough to produce ∞ many classes $[C]$ with $C^2 < 0$.

For example: the classes

$$3k(k+1)\pi^* H - k(k+2)E_1 - k^2 E_2 - k(k+1)(E_3 + \dots + E_9),\ k > 0$$

not imposing too many conditions (exercise).

so they contain IRRED curves C;

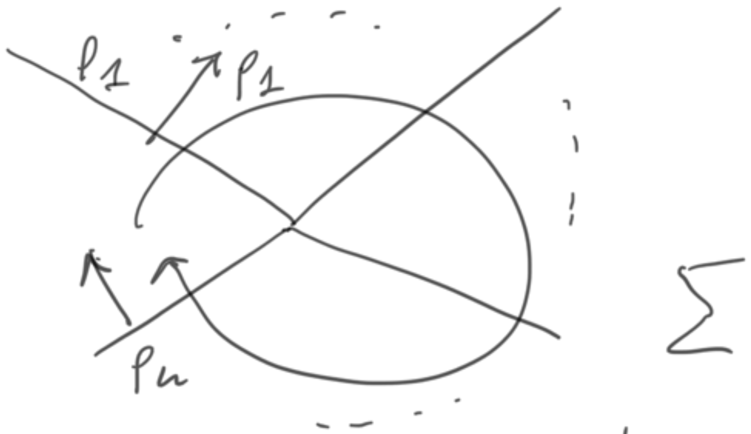
and we have $C^2 = -2K \ldots$

Exercise: show that in fact $\exists$ ∞ many classes of -1 curves!

Lem. Y toric $\Rightarrow$ $\exists$ Σ-PL convex $\varphi : |\Sigma| \to NE(Y)$, unique up to adding a linear function, with prescribed bending parameters

$$p_{\rho,\varphi} = [D_\rho].$$

Pf. This is a monodromy problem: we need to show the function φ is single-valued as we go around $0 \in M_{\mathbb{R}}$. So e.g. if $|\Sigma| \subsetneq M_{\mathbb{R}}$ (i.e. Y not complete) then claim is obvious).



This monodromy is given by

$$\sum_{i=1}^{n} n_{\rho_i} \otimes [D_{\rho_i}] \in N \otimes NE(Y).$$

So we need to show this vanishes.

This is equivalent to
$$\sum_{i=1}^{n} m_i \otimes [D_{\rho_i}] = 0,$$
where $m_i \in M^{prim}$ spans ρ_i.
Fix any $n \in N$, $n \neq 0$. Then, z^n is a function on $(\mathbb{C}^*)^2$ and a rat'l function on Y.
Its divisor is
$$\sum_{i=1}^{n} \langle n, m_i \rangle [D_{\rho_i}].$$
So this is principal, hence 0 in $NE(Y)$.
So $\sum_{i=1}^{n} m_i \otimes [D_i] = 0 \in M \otimes NE(Y)$ $\square$

MODIFIED MUMFORD DEGENERATIONS

We want to extend the construction of "Mumford families" to the integral affine structures on $B \setminus \{0\}$ coming from Loo pairs (Y, D).
These will give defo's of $\mathbb{V}_n^{o} := \mathbb{V}_n \setminus \{0\}$ rather $\mathbb{V}_n$ (since the affine str is only defined on $B \setminus \{0\}$ in general).
This is again a monodromy problem, ... case corresponding to the

with the toric case concept of
"trivial bundle".

Notation:

integral linear

$\pi : \mathbb{P}_0 \longrightarrow B_0$ loc. trivial principal $P^{gp}_{\mathbb{R}} := P^{gp} \otimes_{\mathbb{Z}} \mathbb{R}$ bdle.

integral affine mfld

$\Lambda_{B_0} =: \Lambda_{\mathbb{R}}$:= sheaf of integral constant vector fields on B_0;

$\Lambda_{\mathbb{P}_0}$:= " " on $\mathbb{P}_0$;

$\mathcal{P} := \pi_* \Lambda_{\mathbb{P}_0}$ the pushfor. sheaf;

$\tau, \sigma, \ldots$ = cones of Σ;

$\forall\, \mathcal{U} \subset B_0$, $\tau \subset \mathcal{U}$, $\mathcal{U} \hookrightarrow \Lambda_{\mathbb{R},\tau}$ canonical linear immersion

simply connected

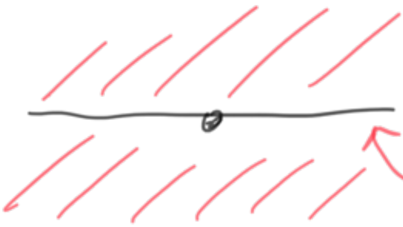
choice of "base pt"

STALK AT ANY PT OF τ

given by // transport

$\tau^{-1}\Sigma :=$ localized fan

$= \begin{cases} \Lambda_{\mathbb{R},\tau} & \text{if } \dim \tau = 2 \\ & \text{if } \dim \tau = 1 \end{cases}$



TANGENT LINE TO τ.

(Convex sections)

Note: a SECTION OF $\pi: \mathbb{P}_0 \longrightarrow \mathcal{B}_0$ gives a collection of Σ-PL functions

$$\{ d \} = \{ d_i : U_i \longrightarrow P^{gp}_{\mathbb{R}} \}$$

$U_i = \mathrm{Int}(\sigma_{i-1,i} \cup \sigma_{i,i+1})$

unique up to LINEAR FUNCTIONS (because transition functions for bdle are affine).

$\Rightarrow$ bending param $p_{\rho_i, d} \in P^{gp}_{\mathbb{R}}$ is well def;

integrality $p_{\rho_i, d} \in P$ and

convexity $p_{\rho_i, d} \in P \setminus P^{\times}$

are well def.

Conversely: (construction of bundles)

Given a collection $\{d\} = \{d_i\}$ as above, we get a bundle

$$\pi: \mathbb{P}_0 \longrightarrow \mathcal{B}_0$$

by gluing

$$(x, p) \longmapsto (x, p + d_{i+1}(x) - d_i(x)).$$

$$\underset{U_i \times P^{gp}_{\mathbb{R}}}{\cap} \qquad \qquad \underset{U_{i+1} \times P^{gp}_{\mathbb{R}}}{\cap}$$

The linear iso type of $\pi : \mathbb{P}_0 \to B_0$ is determined by $p_i, q \in P$.

$\Rightarrow$ equivalence

$$\left\{ \begin{array}{l} \pi : \mathbb{P}_0 \to B_0 \\ \text{up to iso} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{"multi-valued functors"} \\ \{q\} - \{d_i\} \\ \text{up to linear functors} \end{array} \right\}$$

$$\updownarrow$$

$$\left\{ d_i : \Lambda_{\mathbb{R}, p_i} \to P^{gp}_{\mathbb{R}}, \; p_i^{-1}\Sigma\text{-linear} \right\}$$

Canonical exact sequence:

$$0 \to \underline{P^{gp}} \to \Gamma \xrightarrow{r = d\pi} \Lambda_{B_0} \to 0 \qquad (*)$$

Bundle attached to a Loo pair (Y, D):

$$P_{p_i, q} := [D_i] \in NE(Y)$$

or in general

$$:= \eta([D_i]) \in P$$

$\forall \; \eta : NE(Y) \to P$ a monoid homom.

Symplectic heuristic for (*) . local systems as L varies

$$0 \to H_2(Y,\mathbb{Z}) \to \underbrace{H_2(Y,L,\mathbb{Z}) \to H_1(L,\mathbb{Z})}_{\text{local systems as } L \text{ varies}} \to 0$$

for $L \subset U$ a SLag fibre.

Now construct the "Mumfor deg" of a collection $\{\varphi\} = \{\varphi_i\}$:

$\forall i$ $\exists$ canonical toric Mumford deg for φ_i: $\rho_i^{-1}\Sigma \to P_{\mathbb{R}}^{gp}$,

and we glue them!

BUT: this works only "infinitesimally";

i.e. modulo a suitable nontrivial monomial ideal

$$I \subset \mathbb{C}[P].$$

Notation: given $\{\varphi\} = \{\varphi_i\}$ convex multi-val,

under φ_τ : $t^{-1}\Sigma \cong \Lambda_{\mathbb{R},\tau} \to P_{\mathbb{R},\tau}$;

convex

monoid

$$P_{d_\tau} := \{ q \in \mathcal{P}_\tau \mid q = p + q_\tau(m) \text{ for some } p \in \underline{P},\ m \in \Lambda_\tau \} \subset \mathcal{P}_\tau ;$$

satisfying (WITH OBVIOUS CONE NOTATION)

$$P_{d_\rho} \subset P_{d_\sigma} \subset \mathcal{P}_\rho$$

and $$P_{d_{\sigma_+}} \cap P_{d_{\sigma_-}} = P_{d_\rho} ;$$

ideals

$$I_{\tau,\sigma} := \{ q \in P_{d_\tau} \mid q - q_\sigma(r(q)) \in \underline{I} \},$$

$$I_{\tau_1,\tau_2} := \bigcup_{\tau_2 \subset \sigma} I_{\tau_1,\sigma} ;$$

rings

$$R_{\tau_1,\tau_2,I} := \mathbb{C}[P_{d_{\tau_1}}]/I_{\tau_1,\tau_2},$$

satisfying

$$R_{\sigma,\sigma,I} \cong \mathbb{C}[\Lambda_\sigma] \otimes_{\mathbb{C}} \mathbb{C}[P]/I ;$$

a collection of maps

$$R_{\rho,\sigma,I} \xrightarrow{\text{onto}} R_{\rho,\rho,I}$$

$$R_{\rho,\sigma,I} \hookrightarrow R_{\sigma,\sigma,I} ;$$

$R_{\rho,\sigma,I}$ into ...

$\mathbb{C}[P]/I$ algebras attached to rays

$$R_{\rho,I} := R_{\rho,\sigma_+,I} \times_{R_{\rho,\rho,I}} R_{\rho,\sigma_-,I}.$$

Lem. (local picture):
$$\mathrm{Spec}(R_{\rho,I}) \to \mathrm{Spec}\,\mathbb{C}[P]/I$$
is the base change of the toric Mumford deg given by
$$\varphi_\rho : \rho^{-1}\Sigma \to P_{\mathbb{R}}^{gp}.$$

Pf. Enough to show
$$R_{\rho,I} \cong \mathbb{C}[P_{\varphi_\rho}] \otimes_{\mathbb{C}[P]} \mathbb{C}[P]/I.$$

the iso is given by
$$\mathbb{C}[P_\varphi]/I\mathbb{C}[P_{\varphi_\rho}] \to R_{\rho,I}$$
$$f \mapsto (f \bmod I_{\rho,\sigma_+},\ f \bmod I_{\rho,\sigma_-}) \quad \square$$

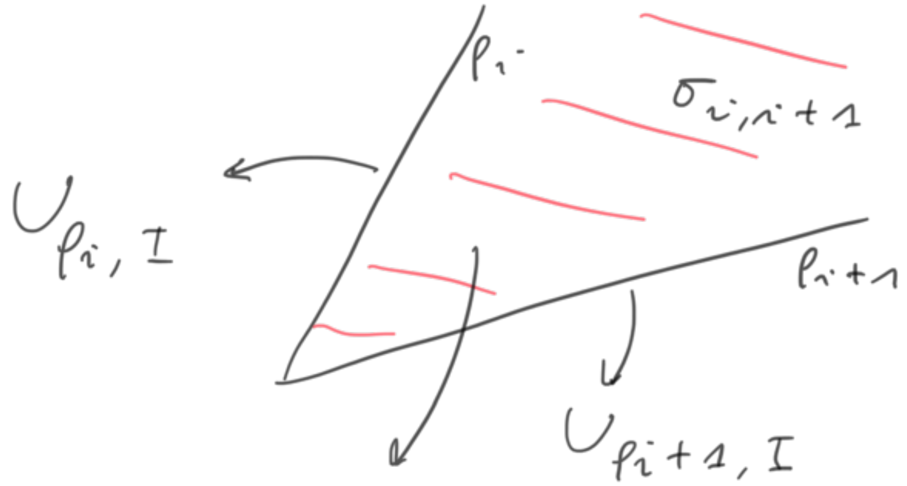
Gluing: we have localization maps
$$\psi_{\rho,\pm} : R_{\rho,I} \to R_{\sigma_\pm,\sigma_\pm I}$$
localizing at $z^{\varphi_\rho(m)}$ for

localizing at

m orthogonal to ρ.

$\Rightarrow$ for $\rho \subset \sigma$ get open subschemes

$$\operatorname{Spec} R_{\rho,\sigma,I} =: U_{\rho,\sigma,I} \underset{\text{open}}{\hookrightarrow} U_{\rho,I} := \operatorname{Spec} R_{\rho,I}$$



contain open $U_{\rho_i,\sigma_{i,i+1},I}$

$$\cong U_{\rho_{i+1},\sigma_{i,i+1},I}$$

So we want to define Mumf deg as

$$X_I^0 := \bigsqcup_i U_{\rho_i,I}$$

gluing along equiv. relation gen by $U_{\rho_i,\sigma_{i,i+1},I} \cong U_{\rho_{i+1},\sigma_{i,i+1},I}$

But there's a problem: if

$$U \cap U_{\rho,\sigma,I} \neq \emptyset$$

$U_{\rho,\sigma_+,I}$ ... $_{\rho,\sigma_-,I}$

the data do not satisfy the axioms for gluing schemes.

The condition
$$U_{\rho,\sigma_+,I} \cap U_{\rho,\sigma_-,I} = \emptyset$$
holds iff $p_{\rho,d} \in \sqrt{I}$, because
$$U_{\rho,\sigma_+,I} \cap U_{\rho,\sigma_-,I} \cong \operatorname{Spec} \mathbb{C}[\Lambda_\rho] \times \operatorname{Spec}(\mathbb{C}[P]/I)_{z^{p_{\rho,d}}}.$$

So it's convenient to set
$$\Sigma_I := \Sigma \setminus \left\{ \begin{array}{l} \text{rays } \rho \text{ for} \\ \text{which } p_{\rho,d} \notin \sqrt{I} \end{array} \right\}$$

Lem. (existence) If Σ_I contains ≥ 2 rays then X_I^o is a scheme $/\operatorname{Spec}\mathbb{C}[P]/I$. $\square$

Lem. (EIRRES)

Suppose all cones in Σ_I are strictly convex.

Let $x \in \operatorname{Spec} \mathbb{C}[P]/I$ be a closed pt with ideal $m_x \subset \mathbb{C}[P]/I$.

Let $\Sigma_x := \Sigma_I \cup \{\text{all rays with } z^{p_{\rho,d}} \in m_x\}$

Then

a) The fibre of $X_I^\circ \to \operatorname{Spec} \mathbb{C}[P]/I$ over x is $\operatorname{Spec} \mathbb{C}[\Sigma_x] \setminus \{0\}$;

b) If "Delzant property" all cones in Σ_x are $\cong$ 

then the fibre is $\mathbb{V}^{\circ}_{(\#\text{rays in } \Sigma_x)}$.

Pf. We work with Σ_I.
We know that locally $X_I^\circ \to \operatorname{Spec} \mathbb{C}[P]/I$ is basechange of toric Mumf deg, for which we have computed fibres, at least over 0.
In general, if bending param $z^{\rho_i, \varphi} \in \mathfrak{m}_x$, we have locally around ρ_i,

$$z^{(m, \varphi(m))} z^{(m', \varphi(m'))} = z^{(m+m', \varphi(m+m') + p)} = z^{(m+m', \varphi(m+m'))} \underset{\substack{\text{III} \\ z^{p}}}{z^{(0,p)}}$$

for some p a multiple of $\rho_{i,\varphi}$.

$= 0$.

The claims follow easily □

Σ_I for Loo pairs

Let (Y, D) be a Loo pair with trop. (B_0, Σ). Fix

Let $\{\varphi\} = \{\varphi_i\}$ be the "canonical multivalued convex function" with

$$p_{\rho_i, \varphi} = \eta([D_i])$$

for some fixed

$$\eta : NE(Y) \to P.$$

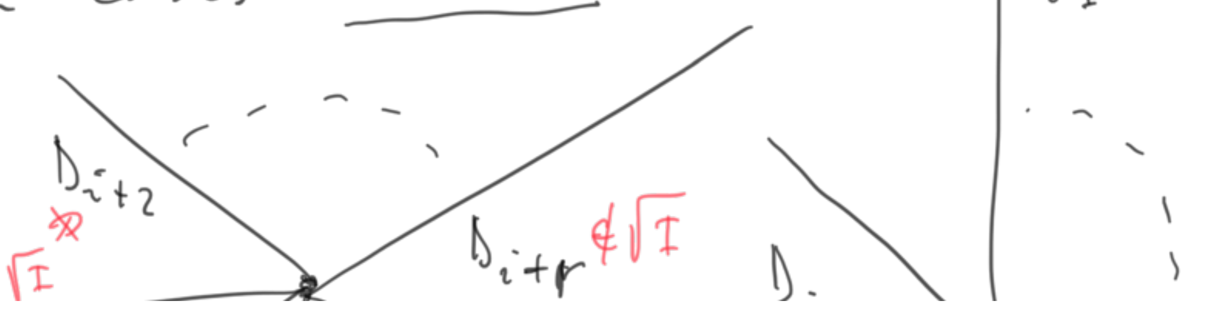
Fix $I \subset P$ monomial ideal with $\sqrt{I}$ PRIME.

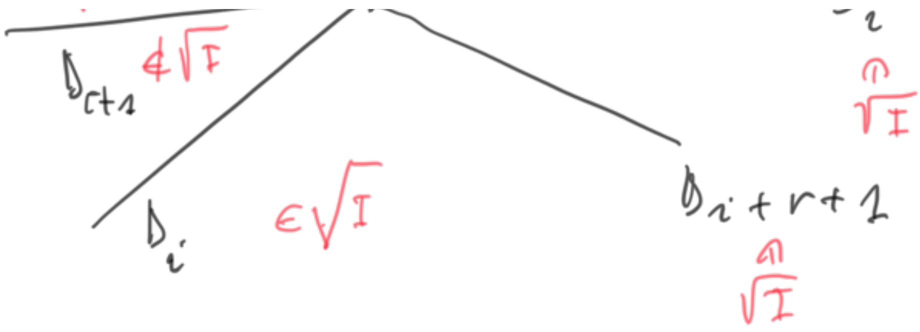
Suppose $p_{\rho, \varphi} \in \sqrt{I}$ for ≥ 2 rays.

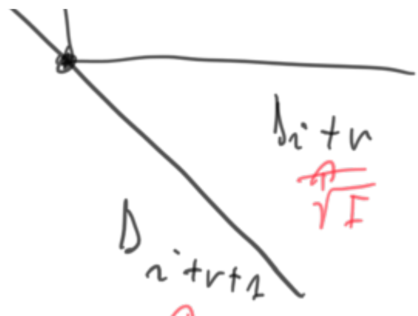
We want to understand when all cones in Σ_I are strictly convex (so that we can describe all fibres).

How can a cone in Σ_I fail to be strictly convex?

2 cases in Σ:







$\Downarrow$ not convex in Σ_I

$\Downarrow$ convex but not strictly convex in Σ_I

• <u>not convex case:</u> <u>claim</u> $\exists\, a_j \in \mathbb{N}$ such that

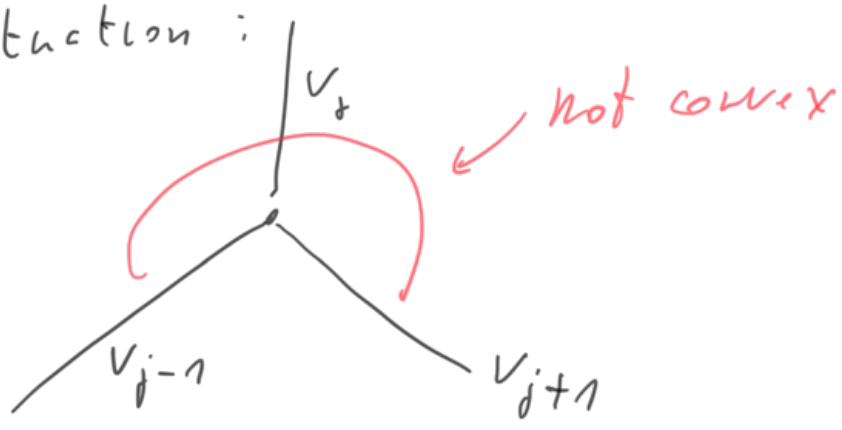
$$(a_1 D_{i+1} + a_2 D_{i+2} + \dots + a_r D_{i+r})^2 > 0.$$

<u>Pf:</u> the affine structure on $B \setminus \{0\}$ was constructed precisely so that $\forall j$ $\exists$ linear embeddings of $B \setminus \{0\} \hookrightarrow \mathbb{R}^2$ with

$$v_{j-1} + (D_j)^2 v_j + v_{j+1} = 0$$

just as in the toric case.

In the situation:



$(D_i)^2 > 0$.

The general case is very similar. □

Now <u>claim</u> the not convex case cannot happen.

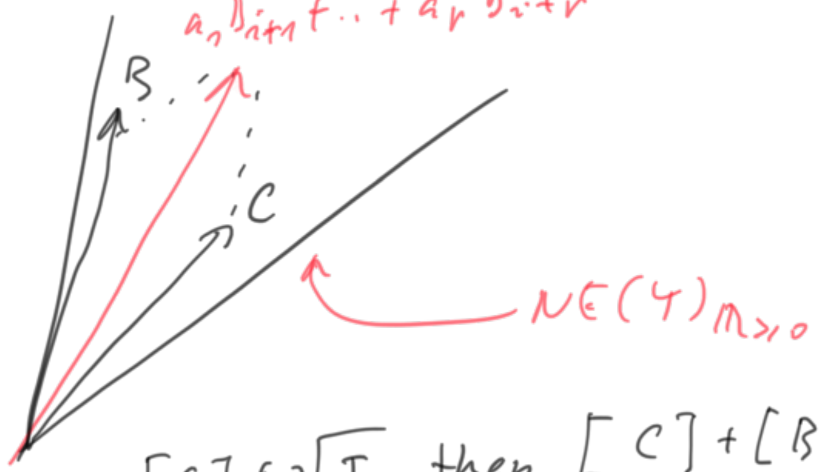
<u>Pf:</u> We need to recall that a class $[C]$ lies on $\partial NE(Y)_{\mathbb{R}\geq 0} \subset A_1(Y,\mathbb{R})$ iff

$$C^2 \leq 0.$$

$\Rightarrow [a_1 D_{i+1} + \dots + a_r D_{i+r}] \in \mathrm{Int}(NE(Y)_{\geq 0})$.

So, $\forall\, [C] \in NE(Y)$, $\exists\, [B] \in NE(Y)$ such that

$$[a_1 D_{i+1} + \dots + a_r D_{i+r}] = [C] + [B].$$



In part. if $[C] \in \sqrt{I}$ then $[C] + [B] \in \sqrt{I}$ by ideal property so

$$[a_1 D_{i+1} + \dots + a_r D_{i+r}] \in \sqrt{I}.$$

But then $D_{i+1}, \dots, D_{i+r} \in \sqrt{I}$ because $\sqrt{I}$ is PRIME $\Rightarrow$ contradiction

$\Rightarrow [C] \notin \sqrt{I}\ \forall\ C \in NE(Y),$

$\Rightarrow [C] \notin \sqrt{I}$

$\Rightarrow [D_i] \notin \sqrt{I}$ $\underline{\underline{\forall i}}$ $\Rightarrow$ contradiction. $\square$
$i = 1 \ldots n$

• <u>not strictly convex case</u>:

similarly, $\exists\, a_j \in \mathbb{N}$ s.t.

$$(a_1 D_{i+1} + \cdots + a_r D_{i+r})^2 = 0.$$

$\Rightarrow$ the linear system $|a_1 D_{i+1} + \ldots + a_r D_{i+r}|$ on Y

gives a map $f : Y \longrightarrow \mathbb{P}^1$.

(The proof is similar to the argument for existence of toric models).

So if $C := f^{-1}(x)$ is a fibre,

$$C \sim a_1 D_{i+1} + \ldots + a_r D_{i+r}$$

$\Rightarrow C \notin \sqrt{I}$ by PRIMARITY

and if C_i is a component of C, then $C_i \notin \sqrt{I}$ by ideal property

$\Rightarrow \forall\, C$ contracted by f, $[C] \notin \sqrt{I}$.

We can assume (Y, D) is <u>not</u> toric,

...

because otherwise this whole argument is not needed.

Then a simple argument using the index $\rho(Y)$ shows $\exists$ a -1 curve $C \not\subset D$ contracted by f (possibly after a toric blup), so with $[C] \notin \sqrt{I}$.

But such $[C]$ is a typical example of an "$\mathbb{A}^1$ class":

a class $\beta \in H_2(\tilde{Y}, \mathbb{Z})$ on some toric blup such that

N_β = a relative GW invariant counting rat'l curves of class β on $\tilde{Y}$

$\neq 0$.

So if we assume the condition

$[C] \in \sqrt{I}$ $\forall$ $\mathbb{A}^1$ classes $[C]$