Some more details: COMPACTIFICATION PT. 2

(a) Set $\mathcal{O}_0 = 1$.

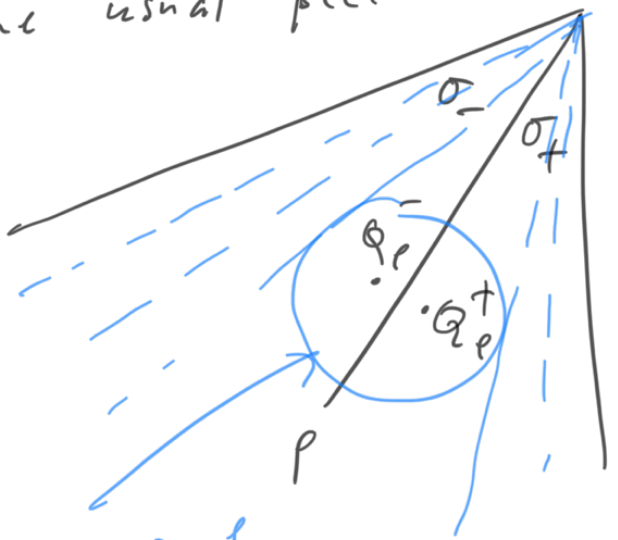
For $q \in B_0(\mathbb{Z})$, define a function $\mathcal{O}_q \in \Gamma(\mathcal{O}_{X^\circ_I})$ as follows: we define an element

$$\mathrm{Lift}_\rho(q) \in R_{\rho, I}$$

as

$$\big(\underset{\substack{\cap \\ R_{\rho,\sigma_-,I}}}{\mathrm{Lift}_{Q^-_\rho}(q)}, \underset{\substack{\cap \\ R_{\rho,\sigma_+,I}}}{\mathrm{Lift}_{Q^+_\rho}(q)}\big)$$

as in the usual picture



no rays of $\mathfrak{D}$ mod I.

Then $\mathrm{Lift}_\rho(q)$ is well def. by consistency, and as ρ varies these elements glue to a regular function $\mathcal{O}_q$ on $X^\circ_{I,\mathfrak{D}}$, again by consistency, so

$$\mathcal{O}_q \in \Gamma(\mathcal{O}_{X^\circ_{I,\mathfrak{D}}}) = \Gamma(\mathcal{O}_{X_{I,\mathfrak{D}}}),$$

since $\quad X_{I,\mathscr{P}} := \operatorname{Spec} \Gamma(\mathcal{O}_{X^\circ_{I,\mathscr{P}}})$.

(b) We claim that the natural map

$$\bigoplus_{q \in B(\mathbb{Z})} (\mathbb{C}[P]/J) \cdot \vartheta_q \longrightarrow \Gamma(\mathcal{O}_{X^\circ_J}) \qquad \Big(\overset{\shortparallel}{\Gamma(\mathcal{O}_{X_J})} \Big)$$

is an iso. of $\mathbb{C}[P]/J$-modules.

<u>Surjectivity</u>: fix $h \in \Gamma(\mathcal{O}_{X^\circ_J})$.

(i) $\forall \sigma \in \Sigma_{max}$, there is a grading

$$h|_{\operatorname{Spec}(R_{\sigma,\sigma,J})} = \sum_{q \in \Lambda_\sigma} \underbrace{h_{\sigma,q}}_{\in r^{-1}(q) \subset \mathscr{P}_\sigma}.$$

This follows at once from our definition of X°_J.

(ii) $\forall \sigma, \sigma' \in \Sigma_{max}$, $h_{\sigma',q}$ is obtained from $h_{\sigma,q}$ by parallel transport in the bundle $\mathscr{P}$.

This follows from the fact that there are no corrections, i.e. $X^\circ_{\mathscr{P},J} = X^\circ_J$.

(iii) Finally in the special case when $\sigma = \sigma'$ we have

$q \in \sigma$,

$$h_{\sigma,q} = a_q \vartheta_q$$

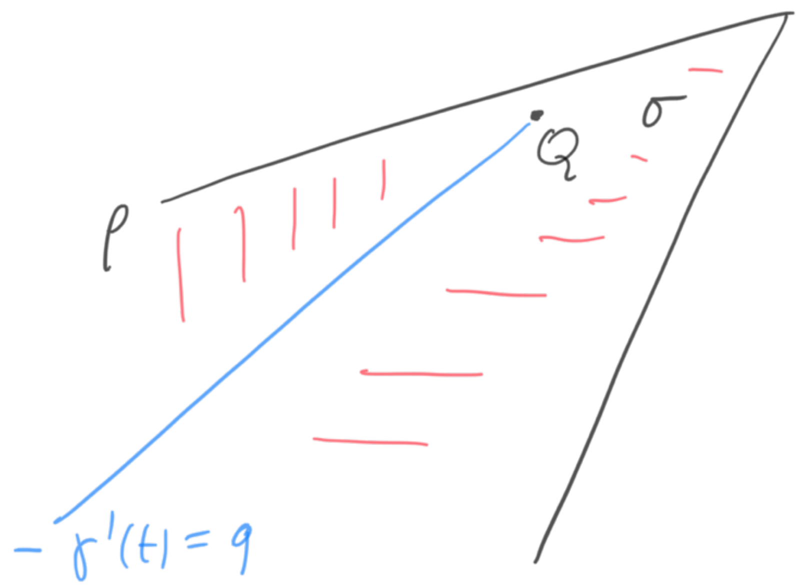
for some $a_q \in \mathbb{C}[P]/J$.

Reason: we know
$$h_{\sigma,q} = a_q z^{\varphi_\sigma(q)}$$
for some $a_q \in \mathbb{C}[P]/J$ because
$$h_{\sigma,q} \in r^{-1}(q).$$

On the other hand by definition,

$$\vartheta_q|_\sigma = \mathrm{Lift}_Q(q) \text{ for } Q \in \sigma$$
close to a ray ρ

$$= \mathrm{Mono}(\gamma),$$
where γ is the unique (un)broken line through Q in direction q,



Mono$(\gamma) = z^{\varphi_\sigma(q)}$, so

By definition, ... ,

$$h_{\sigma,q} = a_q z^{q_\sigma(q)} = a'_q \vartheta_q .$$

Injectivity: check on all $\mathrm{Spec}(R_{\sigma,\sigma,J})$.

(c) Use of deformation theory.

(i) Fix $\sqrt{I} = J$. Show that

$((X_{I,0} =:) X_I) / \mathrm{Spec}(\mathbb{C}[P]/I)$

is a flat deformation of

$((X_{J,0} =:) X_J) / \mathrm{Spec}(\mathbb{C}[P]/J)$,

ie $X_I \longrightarrow \mathrm{Spec}(\mathbb{C}[P]/I)$ is flat

and $X_I \times_{\mathrm{Spec}(\mathbb{C}[P]/I)} \mathrm{Spec}(\mathbb{C}[P]/J)$

$$= X_J .$$

(ii) We know $\{\vartheta_q : q \in B(\mathbb{Z})\}$ form a basis of $\Gamma(X_J, \mathcal{O}_{X_J})$ as $\mathbb{C}[P]/J$-mod.

And $X_I / \mathrm{Spec}(\mathbb{C}[P]/I)$ is a flat infinitesimal defo of $X_J / \mathrm{Spec}(\mathbb{C}[P]/J)$.

$\Rightarrow$ they are also a $\mathbb{C}[P]/I$-mod basis of $\Gamma(\mathcal{O}_{X_I})$.

$\square$

Rmk. The crucial ingredient in this proof is of course consistency.

...

In particular, it says that the complicated functions $$\mathrm{Lift}_Q(q) = \sum_\gamma \mathrm{Mono}(\gamma)$$

(sums over "holomorphic discs")

are actually regular functions on an (affine) scheme $X_I \to \mathrm{Spec}\, \mathbb{C}[P]/I$, and so enjoy a nice deformation theory, which is not at all obvious from their definition.

<u>But</u>, <u>are there consistent scattering diagrams?</u>

<u>Theorem (GHK).</u> Given (Y, D), there is a canonical choice of a consistent scatter. diagram $\mathfrak{D}^{can}$, depending only on the deformation class of (Y, D).