Main reference:

GHK "Mirror symmetry for log CY surfaces I"
(unabridged version!)

Basically GHK prove an interesting <u>existence result</u> in algebraic geom.

Rough idea, special case:

U smooth affine surface with good compactification (Y, D)

Ω holomorphic 2 form on U, simple poles at ∞ (along D)

$\boxed{\text{GHK:}}$ the universal* family of deformations of (U, Ω) (as alg holo sympl. surface) can be constructed explicitly, as Spec of algebra A with "canonical generators" ϑ_q, with explicit multiplication rule.

*UP TO AN EXPLICIT TORUS ACTION

The construction depends only ... theory of on the Gromov-Witten ... of the compactification (Y, D).

<u>Conjecture:</u> $\mathcal{X} := \operatorname{Spec} A \to T_Y$ DEF'N SPACE

is MIRROR to (U, Ω) in the sense of HMS. "Classical MS":

$A \cong$ symplectic cohomology $SH^0(U)$.

(Aside: in joint wk with R. Ontani, we'd like to use GHK construction to give a geometric interpretation/proof of certain "Jeffrey-Kirwan residue formulae" in physics, in some very special cases.

Basic idea: they are equivalent to the behaviour of canonical generators ϑ_q, nearby the "large cplx structure limit"

$$0 \in \overline{T_Y},$$

along the "Gross-Siebert locus"

$$GS \subset \overline{T_Y}.$$)

Example (of GHK thm): Suppose the "compactification" of (affine) (U, Ω) is

$Y :=$ del Pezzo of deg. 5 (blup $\mathbb{P}^2$ at $p_1 \dots p_4$) ; $D = D_1 + \dots + D_5$ = anti-canonical cycle of -1 curves

$\Rightarrow$ the GHK family is given by

$$\mathcal{X} \longrightarrow \operatorname{Spec} \mathbb{C}[NE(Y)]$$

with $\mathcal{X} = \operatorname{Spec} A$, and A generated over $\mathbb{C}[NE(Y)]$ by $\vartheta_1, \dots, \vartheta_5$ satisfying

$$\left\{ \vartheta_{i-1} \vartheta_{i+1} = z^{[D_i]} (\vartheta_i + z^{[E_i]}) \right\} \quad i = 1, \dots, 5.$$

exc. divisors of $Y \to \mathbb{P}^2$.

The restriction

$$\mathcal{X} \longrightarrow T_Y := \operatorname{Spec} \mathbb{C}[A_1(Y)] \subset \operatorname{Spec} \mathbb{C}[NE(Y)]$$

is a versal family of defo's of (U, Ω).

But there's a effective "gauge group" action on the family by torus

T^D with character group generated by e_{D_i}, $i = 1 \dots 5$

This is a general fact: on the base

$$T_Y = \operatorname{Spec} \mathbb{C}[A_1(Y)]$$

the action is dual to

$$C \longmapsto \sum (C \cdot D_i) e_{D_i}.$$

$\Rightarrow$ # moduli: $\dim T_Y - \dim T^D = 5 - 5 = 0$

$\Rightarrow (U, \Omega)$ is INF. RIGID.

But the GHK family is still nontrivial and interesting!

Nonrigid example: Y = smooth cubic surface
$\cong$ blup $\mathbb{P}^2$ at $p_1 \dots p_6$

$D = L_1 + L_2 + L_3$
triangle of lines $\subset Y$

$\Rightarrow U := Y \setminus D$ is log CY

(U, Ω)

AFFINE $\longrightarrow U$; $\Omega \longleftarrow \exists!$ up to scale

GHK family is

$$\begin{array}{c} \mathcal{X} \subset \operatorname{Spec}(\mathbb{C}[NE(Y)]) \times \mathbb{A}^3 \\ \downarrow \\ \operatorname{Spec} \mathbb{C}[NE(Y)] \end{array}$$

Spec ...

cut out by single equation

$$\vartheta_1\vartheta_2\vartheta_3 = \sum_i z^{L_i}\vartheta_i^2 + \sum_i\sum_j z^{E_{ij}} z^{D_i}\vartheta_i + \sum_\pi z^{\pi^*H} + 4z^{L_1+L_2+L_3}$$

for some functions ϑ_i on $\mathrm{Spec}(NE(Y))\times\mathbb{A}^3$,
E_{ij} := exc. divisors meeting L_i,
π := any contraction $(Y,D)\longrightarrow(\mathbb{P}^2, \triangle)$.

$\#$ moduli: $\dim T_Y - \dim T^D$
$= \mathrm{rk}(\mathrm{Pic}(Y)) - \#$ bdry divisors
$= 7 - 3 = 4.$

This can also be computed as the $\#$ of independent periods of normalized $\gamma\,\Omega$ on cycles of $H_2(U,\mathbb{Z})$, i.e. $\mathrm{rk}\, H_2(U,\mathbb{Z})$:

$$0\to\underset{\substack{\| \\ \mathbb{Z}\langle\gamma\rangle}}{\mathbb{Z}}\to H_2(U,\mathbb{Z})\to\{D^\perp\subset H_2(Y,\mathbb{Z})\}\to 0$$

$\mathbb{Z}\langle\gamma\rangle$; normalize by $\int_\gamma\Omega = 1$.

3 linear conditions out of rk 7.

Kähler vs cplx moduli

Symplectic side

$(U, i\omega + B)$

$[i\omega + B] \in iH^2(Y,\mathbb{R}) \oplus H^2(Y,\mathbb{R})/H^2(Y,\mathbb{Z})$

Cplx side

$(\mathcal{X}_s, \Omega_s)$

$s \in \operatorname{Spec} \mathbb{C}[A_1(Y,\mathbb{Z})] =: T_Y \cong \operatorname{Pic}(Y) \otimes \mathbb{C}^*.$

$$\underset{\hat{T}_Y}{\overset{\cap}{s}} = \exp([i\omega + B])$$

Rmk: In both our examples, we have

$$\mathcal{X} \subset \operatorname{Spec} \mathbb{C}[NE(Y)] \times \mathbb{A}^n$$

($n = \#$ components of D).

So $\mathcal{X}_0 \subset \mathbb{A}^n$.

Deg 5 Del Pezzo: $\mathcal{X}_0 = \{ \vartheta_{i-1} \vartheta_{i+1} = 0 \}$

$$\cong \mathbb{A}^2_{x_1 x_2} \cup \mathbb{A}^2_{x_2 x_3} \cup \cdots \cup \mathbb{A}^2_{x_5 x_1} \subset \mathbb{A}^5_{x_1 \ldots x_5}$$

Cubic surface: $\mathcal{X}_0 = \{ \vartheta_1 \vartheta_2 \vartheta_3 = 0 \}$

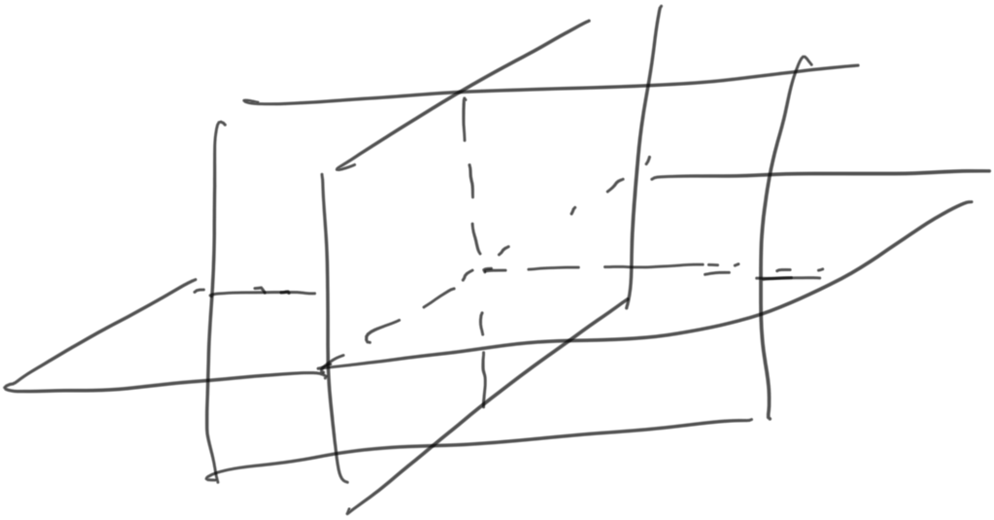
$$\cong \mathbb{A}^2_{x_1 x_2} \cup \mathbb{A}^2_{x_2 x_3} \cup \mathbb{A}^2_{x_3 x_1} \subset \mathbb{A}^3$$

$\subset \mathbb{A}^3_{x_1 x_2 x_3}$

THESE ARE GENERAL FACTS ABOUT GHK!

S_0 fibre over $0 \in \mathrm{Spec}\,\mathbb{C}[NE(Y)]$ is VERTEX

$$\mathbb{V}_n := \mathbb{A}^2_{x_1 x_2} \cup \dots \cup \mathbb{A}^2_{x_n x_1}$$



This is the "Large Cplx Str Limit" of GHK family ("The most singular fibre")

The relation

$$S = \exp([i\omega + B])$$

suggs that "MIRROR MAP NEAR $\mathbb{V}_n$" is TRIVIAL.

This seems a special case of general result of Ruddat - Siebert "Period integrals from wall structures..."