DEFORMATIONS OF X°_J: INSTANTON CORRECTIONS

Key idea:
start with a modified Mumford family for (B°, Σ), construct mirror family to $(U := Y \setminus D, \Omega)$ by

- deforming the complex structure for $X^\circ_J \longrightarrow \operatorname{Spec} \mathbb{C}[P]/J$
- compactifying the deformed family.

Motivation: symplectic heuristic (SYZ).

Approaches $\nearrow$ GHK (monodromy of "special coordinates")
$\searrow$ Fukaya / CCM (Maurer–Cartan equation).

In the monodromy approach, defos of X°_J are described by "SCATTERING DIAGRAMS".

(AKA "GLOBAL SCATTERING DIAGRAMS")

- (Y, D) Loo pair

Tropicalization $\left(B = B_{(4,0)}, \Sigma \right)$ with integral affine str. on B°

Notation as before:

- P f.g. toric monoid;
- $\varphi = \{\varphi_i\}$ multivalued convex section $\left(\sim \quad \mathbb{P}^{\circ} \longrightarrow B^{\circ} \right)$;
- $J \subset P$ radical monoid ideal, $J = \sqrt{J}$;
- $\forall \tau \in \Sigma$, monoid ring $\mathbb{C}[P_{\varphi_\tau}] \supset J_{\tau,\tau}$;
- $\widehat{\mathbb{C}[P_{\varphi_\tau}]}$ the $J_{\tau,\tau}$-adic completion

A SCATTERING DIAGRAM for

$$\{ (B, \Sigma), P, \varphi, J \}$$

is a set $\mathfrak{D} := \{ (\mathfrak{d}, f_{\mathfrak{d}}) \}$ with

(1) $\mathfrak{d} \subset B$ ray through $0 \in B$ with rational slope (possibly a ray of Σ);

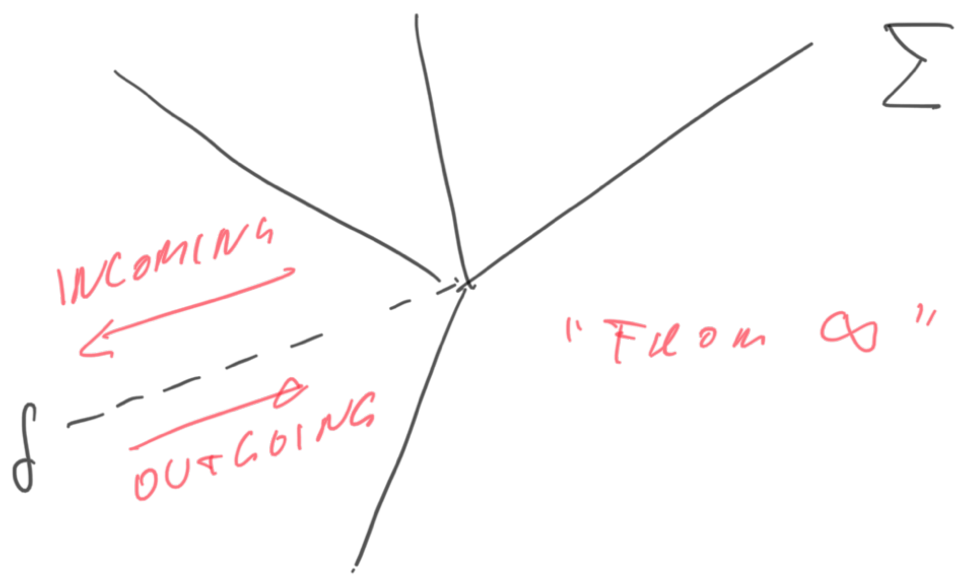
(2) $\mathfrak{d} \subset \tau_{\mathfrak{d}} \in \Sigma$ minimal cone $\Rightarrow$ $\widehat{}$

(2) $\delta \ldots$

$$f_\delta = 1 + \sum_p c_p z^p \in \mathbb{C}[P_{\varphi_{\tau_\delta}}],$$

where $c_p \in \mathbb{C}$, $p \in P_{\varphi_{\tau_\delta}}$ such that $r(p) \neq 0$, $r(p)$ tangent to δ in the sequence

$$0 \to \underline{P}^{gp} \to \Lambda \xrightarrow{r = d\pi} \Lambda_B \to 0$$



;

(3) If $\dim \tau_\delta = 2$ or $\dim \tau_\delta = 1$ and bending $P_{\tau_\delta, \sigma} \notin J$

$$\Longrightarrow f_\delta \equiv 1 \mod J_{\tau_\delta, \delta};$$

(4) $\forall\ I \subset P$ monoid ideal with $\sqrt{I} = J$, $\exists$ at most finitely many $(\delta, f_\delta) \in \mathfrak{D}$ with $f_\delta \not\equiv 1 \mod I_{\tau_\delta, \tau_\delta}$.

$$\boxed{\text{Construction: def}^{\text{n}}\text{ of } X^\circ_J \to \operatorname{Spec} \mathbb{C}[P]/J \text{ by scattering diagram } \mathfrak{D}}$$

Recall X°_I is obtained by gluing Spec's of $\mathbb{C}[P]/I$ algebras:

$$R_{\rho,I}$$

along open subsets given by

$$\begin{tikzcd} \psi_{\rho,\sigma}: R_{\rho,I} \arrow[r] & R_{\sigma,\sigma,I}. \\ \psi_{\rho',\sigma}: R_{\rho',I} \arrow[ur] & \end{tikzcd}$$

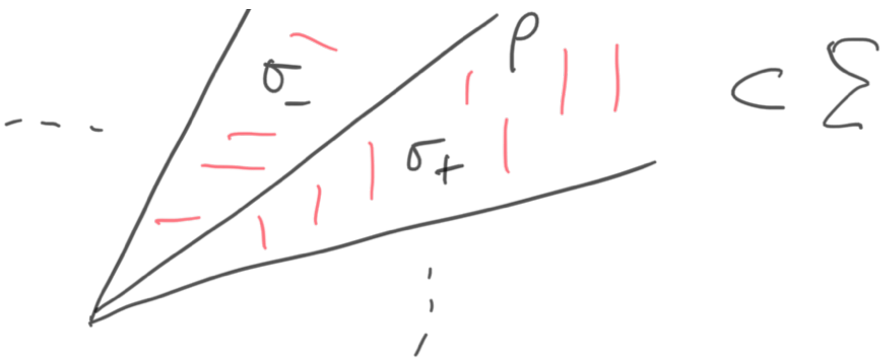
Scattering diagram deforms $\to$ local patches $\operatorname{Spec} R_{\rho,I}$;

Scattering diagram deforms $\to$ gluing maps $\psi_{\rho,\sigma}$.

• Deformation of local pieces $\operatorname{Spec}(R_{\rho,I})$.

We need to recall the fibre product description:

$$R_{\rho,I} \cong R_{\rho,\sigma_-,I} \times_{R_{\rho,\rho}} R_{\rho,\sigma_+,I}$$



We <u>deform</u> this to

$$R_{\rho,I} := R_{\rho,\sigma_-,I} \times_{(R_{\rho,\rho})_{f_\rho}} R_{\rho,\sigma_+,I}$$

where
$$f_\rho := \prod_{\substack{(\gamma, f_\gamma)\in \mathcal{D} \\ \gamma = \rho}} f_\gamma \mod I_{\rho,\rho}$$

(well def. by finiteness mod $I_{\rho,\rho}$);

maps in the fibre product are

$$R_{\rho,\sigma_-,I} \longrightarrow (R_{\rho,\rho,I})_{f_\rho}$$

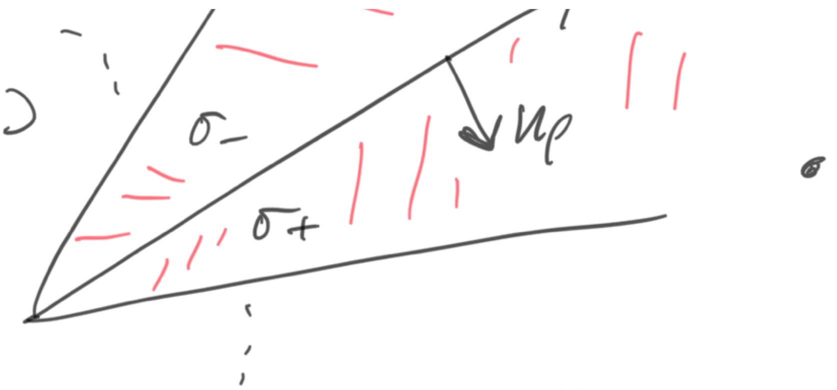
surjection + localization;

$$R_{\rho,\sigma_+,I} \longrightarrow (R_{\rho,\rho,I})_{f_\rho}$$

surjection + localization +

$$z^p \longmapsto z^p f_\rho^{\langle n_\rho, r(p)\rangle}$$

with $n_\rho \perp \rho$ and > 0 on σ_+

$\Sigma \supset$  .

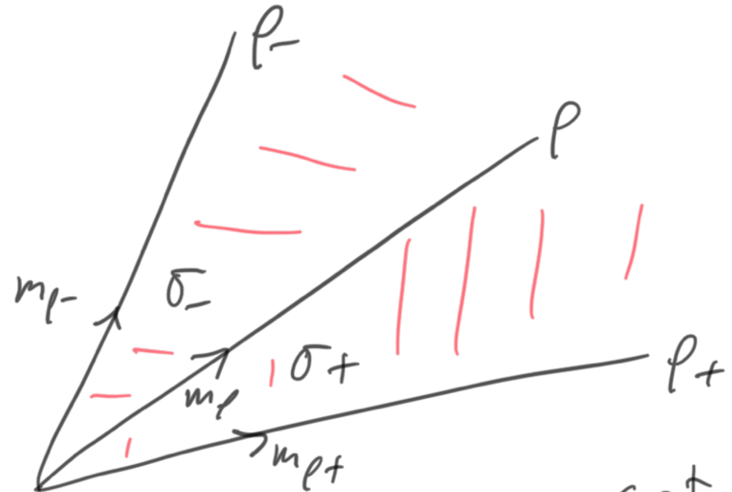
This described the deformed affine pieces of

$X^\circ_{I,\mathbb{D}}$ the $\mathbb{D}$-defo of X°_I.

More explicitly:

Lemma (Coordinate description of $R_{\rho, I}$).

Consider usual setup for Mod. Mumford Degener:



Regard f_ρ as an element of $(\mathbb{C}[P]/I)[X^{\pm}]$

by calling X the element $z^{\varphi_\rho(m_\rho)} \in \mathbb{C}[P_{\varphi_\rho}]$.

(Recall the condition $r(p) \,/\!/\, \rho$).

Define

$R' := (\mathbb{C}[P]/I)[X_+, X_-, X^{\pm}]$.

$$R_{\rho,I} \cdot \overline{(X_+ X_- - z^{p_{\rho,\sigma}} X^{-D_\rho^2} f_\rho)}$$

then the map

$$X \longmapsto (z^{\varphi_\rho(m_\rho)}, z^{\varphi_\rho(m_\rho)})$$
$$X_- \longmapsto (z^{\varphi_\rho(m_{\rho-})}, f_\rho z^{\varphi_\rho(m_{\rho-})})$$
$$X_+ \longmapsto (f_\rho z^{\varphi_\rho(m_{\rho+})}, z^{\varphi_\rho(m_{\rho+})})$$

induces iso $h: R'_{\rho,I} \xrightarrow{\cong} R_{\rho,I}$.

<u>Pf</u> First claim h maps to $R_{\rho,I}$, the fibre product.

Main point is checking
$$X_+ X_- - z^{p_{\rho,\sigma}} X^{-D_\rho^2} f_\rho \longmapsto 0.$$

By definition, it maps to pair

$$\Big(f_\rho z^{\varphi_\rho(m_{\rho-})+\varphi_\rho(m_{\rho+})} - f_\rho z^{p_{\rho,\sigma}} z^{-D_\rho^2 \varphi_\rho(m_\rho)},$$
$$f_\rho z^{\varphi_\rho(m_{\rho-})+\varphi_\rho(m_{\rho+})} - f_\rho z^{p_{\rho,\sigma}} z^{-D_\rho^2 \varphi_\rho(m_\rho)}\Big). \quad (*)$$

By def. of affine str. on B°, we have:

$$(**) \quad m_{\rho-} + D_\rho^2 m_\rho + m_{\rho+} = 0 \in \Lambda_\rho.$$

we know φ_ρ is Σ-PL, with bending parameter $p_{\rho,\sigma}$, so $(**)$ gives

$$(\square)\quad \varphi_\rho(m_{\rho-}) + \varphi_\rho(m_{\rho+}) = p_{\rho,1} - D_\rho^2\,\varphi_\rho(m_\rho).$$

So the image (*) vanishes as required.

Now claim h is onto.

Set $\quad R_\pm := R_{\rho,\sigma_\pm,I}$ for convenience.

Recall, by definition,

$$R_\pm = \mathbb{C}[P_{\varphi_\rho}]/I_{\rho,\sigma_\pm}$$

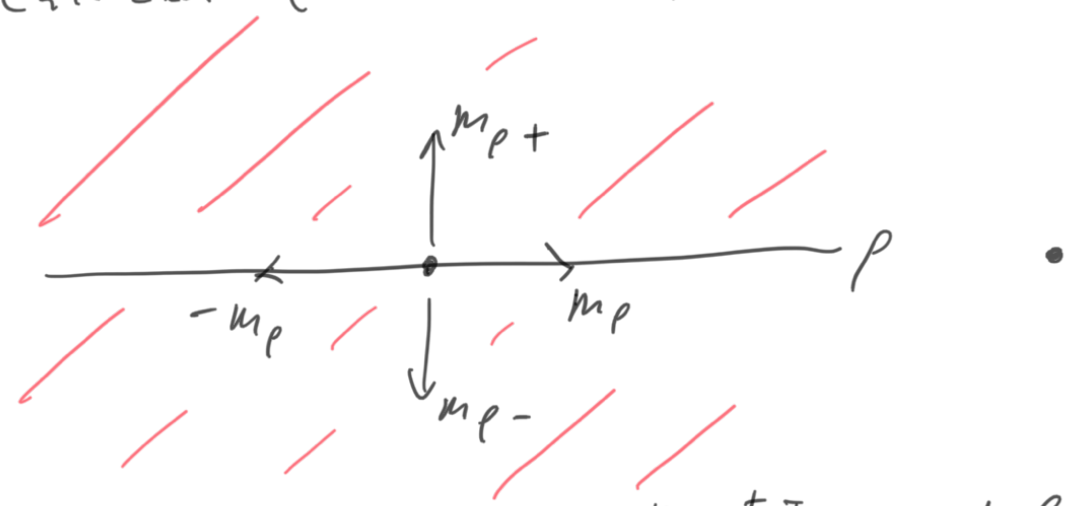
monomial ideal

where

$$I_{\rho,\sigma_\pm} = \{q \in P_{\varphi_\rho} : q - \varphi_{\sigma_\pm}(r(q)) \in I\},$$

(monoid ideal)

and P_{φ_ρ} is the monoid corresponding to the localized fan $\tau^{-1}\Sigma$,



So, R_+ are $\underbrace{(\mathbb{C}[P]/I)}\underbrace{[X^\pm]}$-modules, generated by $\check{S}$

$$1,\ x^j,\ y^k;\ j,k>0,$$

for $x := Z^{\varphi_\ell(m_{\ell-})}$

$y := Z^{\varphi_\ell(m_{\ell+})}$.

Of course, there are relations, eg $(\square)$ mod I.

But, the S-submodules

$$\langle 1, x^j, j>0 \rangle$$

$$\langle 1, y^k, k>0 \rangle$$

are free. So $\forall g_\pm \in R_\pm$ write uniquely

$$g_- = \sum_{j\geq 0} a_j x^j + h_-(y)$$

$$g_+ = \sum_{k\geq 0} b_k y^k + h_+(x)$$

for some $a_j, b_k \in S$, $h_- \in S[y]$, $h_+ \in S[x]$,

with $h_\pm(0) = 0$.

$\Rightarrow (g_-, g_+)$ lies in fibre product

$$R_\ell = R_+ \times_{(R_{\ell,\ell}, I)_{f_\ell}} R_-$$

iff

$$a_0 = b_0;$$

$$h_-(y) = \sum_{k>0} b_k f_\ell^k y^k;$$

$$h_+(x) = \sum_{j>0} a_j f_\ell^j x^j$$

(by def of maps to $(R_{\ell,\ell,I})_{f_\ell}$).

But then we have

$$(g_-, g_+) = h\left(\sum_{j\geq 0} a_j X_-^j + \sum_{k\geq 0} b_k X_+^k\right).$$

$\Rightarrow$ h surjective.

Injectivity is similar (exercise!) $\square$

$$\boxed{\text{Deformation of glueing maps } \varphi_{\rho,\sigma}}$$

We have natural maps

$$\varphi_{\rho,\pm} : R_{\rho,I} \longrightarrow R_{\sigma_\pm,\sigma_\pm,I}$$

$$\cong$$

$$\mathbb{C}[P_{d\sigma_\pm}]/I\mathbb{C}[P_{\varphi_{\sigma_\pm}}]$$

$\sigma_\pm^{-1}\Sigma$

generated by $z^{\pm d_\rho(m_\rho \pm)}$

IN COORDINATES:

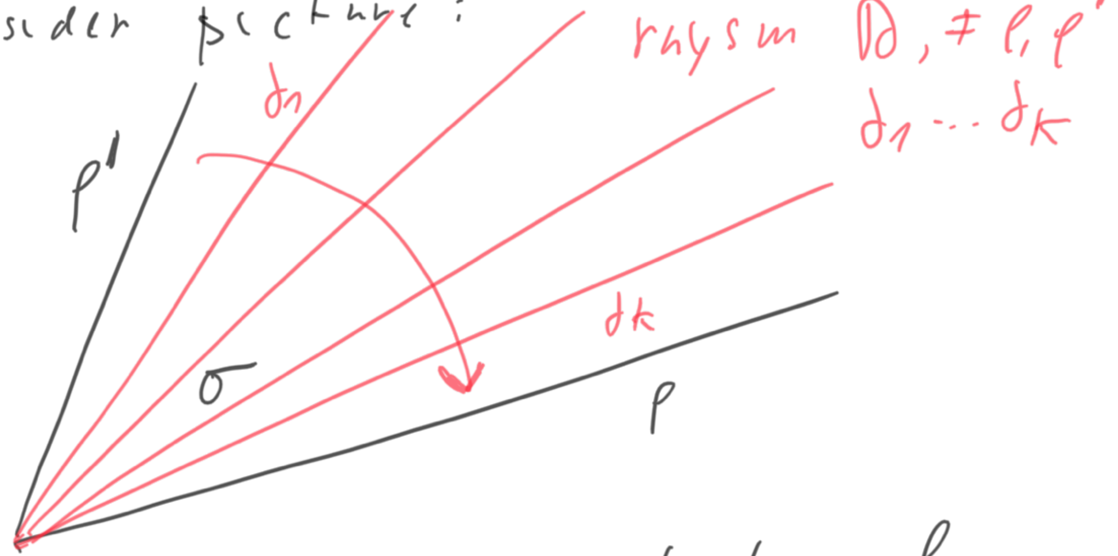
$$\varphi_{\rho,\pm} : R'_{\rho,I} \longrightarrow (R'_{\rho,I})_{X_\pm}$$

are the localization maps at $X_\pm$.

$\Rightarrow$ get open subsets

$$\operatorname{Spec} R_{\sigma_\pm,\sigma_\pm,I} =: U_{\rho,\sigma_\pm,I} \hookrightarrow U_{\rho,I} := \operatorname{Spec} R_{\rho,I}$$

Now consider picture:



We have isomorphic open subsets of

$$\operatorname{Spec} R_{\rho,I} \; ; \; \operatorname{Spec} R_{\rho',I}$$

given by

$$\operatorname{Spec} R_{\sigma,\sigma,I}.$$

We glue them with nontrivial iso:

$$\theta_{\gamma,\mathfrak{D}} : R_{\sigma,\sigma,I} \xrightarrow{\cong} R_{\sigma,\sigma,I}$$

with

$$\theta_{\gamma,\mathfrak{D}} := \theta_{\gamma,d_k} \circ \dots \circ \theta_{\gamma,d_1},$$

$$\theta_{\gamma,d_i}(z^p) := z^p f_{d_i}^{\langle n_{d_i}, r(p)\rangle},$$

$n_{\gamma_i} \in \Lambda^*_{\sigma, \mathrm{prim}}$,

$\langle n_{\gamma_i}, \partial_i \rangle = 0$

$\langle n_{\gamma_i}, \gamma'(t_*) \rangle < 0$.

IMPACT TIME ALONG $\gamma(t)$.

Rmk: $\langle n_{\gamma_i}, r(p) \rangle$ can have any sign, so need to check $f_{\partial_i} \in R_{\sigma,\sigma,I}$ is invertible.

By def of scattering diagram (3),

$$f_{\partial_i} \equiv 1 \mod J_{\sigma,\sigma}$$

$\Rightarrow$ $(f_{\partial_i} - 1)^k \in I_{\sigma,\sigma}$ for some k since $\sqrt{I} = J$

$\Rightarrow$ $f_{\partial_i} - 1$ nilpotent in $R_{\sigma,\sigma,I}$

$\Rightarrow$ f_{∂_i} invertible " " .

Upshot: fixing a scattering diagram $\mathfrak{D}$, $\forall I$ get a family of schemes

$$X^0_{I,\mathfrak{D}} \longrightarrow \operatorname{Spec} \mathbb{C}[P]/I$$

which is a deformation of the modified Mumford family

$$X^o_I \longrightarrow \operatorname{Spec} \mathbb{C}[P]/I.$$

Main problem: for which $\mathcal{D}$ is $X^o_{I,\mathcal{D}}$ well-behaved? E.g.

1) Admits relative flat partial compactif.?
2) Relatively log Calabi-Yau?

At least, $X^o_{I,\mathcal{D}} \longrightarrow \operatorname{Spec} \mathbb{C}[P]/I$ is always "relatively CY", i.e. the relative dualizing sheaf is trivial.

Lemma: The sheaf $\omega_{X^o_{I,\mathcal{D}}/(\operatorname{Spec}\mathbb{C}[P]/I)}$ is generated by the global section given on local patches $U_\rho = \operatorname{Spec} R_{\rho,I}$ by

$$\Omega|_{U_\rho} := d\log X_+ \wedge d\log X$$
$$= d\log X \wedge d\log X_- .$$

Pf. First consider local situation. We can work with "local coordinates", i.e. with ring

$$R'_{\rho,I} = \frac{(\mathbb{C}[P]/I)[X_\pm, X^{\pm 1}]}{(X_+X_- - z^{P_{\rho,\sigma}} X^{-D_\rho^2} f_\rho)} \cong R_{\rho,I}.$$

This is the coord ring of the affine ...

$$U_\rho \hookrightarrow \mathbb{A}^2_{X_+X_-} \times \mathbb{C}^*_X \times \operatorname{Spec} \mathbb{C}[P]/I$$

cut out by the eqn

(*) $\{ X_+ X_- = z^{\varphi_{\rho\rho'}} X^{-b_\rho^2} f_\rho \}$.

It follows from (*) that the 1-forms

$$d\log X_\pm \quad , \quad d\log X$$

are well def and satisfy

$$d\log X_+ \wedge d\log X = d\log X \wedge d\log X_- .$$

One checks that this 2-form generates $\omega_{U_\rho / \operatorname{Spec} \mathbb{C}[P]/I}$ eg by the adjunction formula over $\operatorname{Spec} \mathbb{C}[P]/I$.

The main point is showing that these local 2-forms patch to a global form on $X^\circ_{I,\mathfrak{A}}$.

For this, recall that in the is

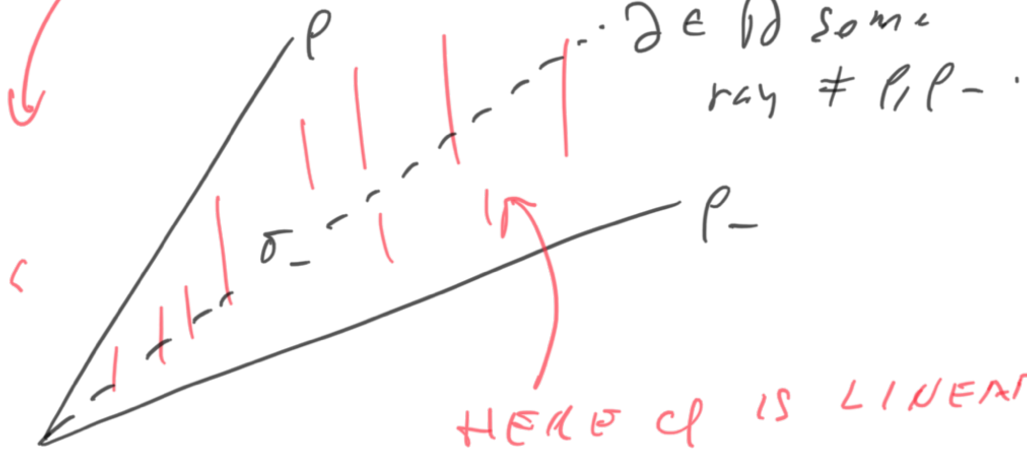
$$R'_{\rho,I} \xrightarrow{\cong} R_{\rho,I}$$

we have in particular

(**)
$$X \mapsto (z^{d_\rho(m_\rho)}, z^{d_{\rho'}(m_{\rho'})})$$
$$X_- \mapsto (z^{d_\rho(m_{\rho-})}, f_\rho z^{d_{\rho'}(m_{\rho'-})})$$

$\in R_{\rho,\sigma_-,I} \times R_{\rho,\sigma_+,I}$.

CONSIDER FIRST PROJ. FOR GLUEING U_ρ, U_{ρ_-}



HERE φ IS LINEAR!

Now,

$$f_\partial = 1 + \sum_p c_p z^p \in \widehat{\mathbb{C}[P_{\varphi_{\sigma_-}}]},$$

$$r(p) \neq 0 \quad , \; r(p) \parallel \partial$$

so, $\quad p = k\,\varphi_{\sigma_-}(m)$ for some prim. m,

$$\Rightarrow \quad p = k\left(\varphi_{\sigma_-}(a m_\rho + b m_{\rho_-})\right)$$

$$= ka\,\varphi_{\sigma_-}(m_\rho) + kb\,\varphi_{\sigma_-}(m_{\rho_-}).$$

Rmk. Here we are ignoring factors in $\mathbb{C}[P]/I$, which are treated as constants and have trivial differential.

Then by (**)

$x^{ka}\, y^{kb}$

$$f_\gamma = 1 + \sum_{k \neq 0} c_k X \quad X_- ,$$

so
$$\log f_\gamma = \sum_{k \neq 0} \hat{c}_k X^{ka} X_-^{kb} ,$$

$$d \log f_\gamma = \frac{\partial}{\partial X} \log f_\gamma \, dX +$$
$$\frac{\partial}{\partial X_-} \log f_\gamma \, dX_-$$
$$= \sum_k \hat{c}_k \, ka \, X^{ka-1} X_-^{kb} \, dX +$$
$$\sum_k \hat{c}_k \, kb \, X^{ka} X_-^{kb-1} \, dX_- .$$

By (**) we also have
$$\Theta_\gamma(X) = X f_\gamma^{\langle n, m_\rho \rangle} ,$$
$$\Theta_\gamma(X_-) = X_- f_\gamma^{\langle n, m_{\rho_-} \rangle}$$

$$\Rightarrow d \log \Theta_\gamma(X) = d \log X + \overbrace{\langle n, m_\rho \rangle}^{= -b} d \log f_\gamma ,$$
$$d \log \Theta_\gamma(X_-) = d \log X_- + \underbrace{\langle n, m_{\rho_-} \rangle}_{= a} d \log f_\gamma$$

$$\Rightarrow \ d\log \Theta_\gamma(X) \wedge d\log \Theta_\gamma(X_-) =$$

$$= \ d\log X \wedge d\log X_-$$

$$- b \, d\log f_\gamma \wedge d\log X_- + a \, d\log X \wedge d\log f_\gamma$$

$\parallel$

0 by our computation of $d\log f_\gamma$ $\square$