COMPACTIFICATION AND BROKEN LINES

Recall we need to find scattering diagrams $\mathfrak{D}$ for which the family

$$X^{o}_{I,\mathfrak{D}} \longrightarrow \operatorname{Spec} \mathbb{C}[P]/I$$

is well behaved and in particular admits a partial fibrewise compactification.

Indeed, the situation in the toric case for $\mathfrak{D} = \emptyset$ is the following:

<u>Lemma.</u> For (Y,D) toric and $\mathbb{P} = B \times P^{gp}_{\mathbb{R}}$,

X^{o}_{I} is an open subscheme of the Mumford degeneration $\operatorname{Spec} \mathbb{C}[P_{\alpha}]/I\mathbb{C}[P_{\alpha}]$

and $H^{0}(X^{o}_{I}, \mathcal{O}_{X^{o}_{I}}) = \mathbb{C}[P_{\alpha}]/I\mathbb{C}[P_{\alpha}]$.

Moreover, the fibres of

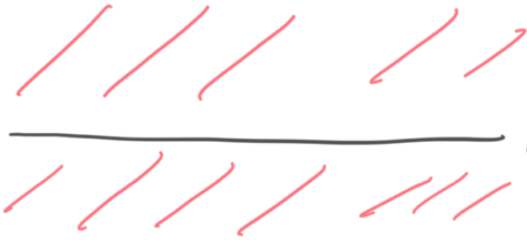
$$X^{o}_{I} \longrightarrow \operatorname{Spec} \mathbb{C}[P]/I$$

are obtained by removing the zero-dim torus orbit from fibres of Mumford degeneration.

<u>Pf.</u> Fix $\tau \in \Sigma$, $\dim \tau = 1$.

Recall P_{φ_τ} is defined ...

attached to the convex function φ_τ on

$\tau^{-1}\Sigma =$  $\mathbb{R}\tau$, where

φ_τ is defined up to linear functions by

bending $P_{\tau,\varphi}$.

Thus, P_{φ_τ} is the localization of the monoid P_φ along the face

$$\{(m, \varphi(m)) \mid m \in \tau \cap M\},$$

ie $\mathbb{C}[P_{\varphi_\tau}]$ is the localiz. of $\mathbb{C}[P_\varphi]$ along the multiplicative subset

$$\{z^{(m,\varphi(m))} \mid m \in \tau \cap M\}.$$

So $\operatorname{Spec} \mathbb{C}[P_{\varphi_\tau}] \subset \operatorname{Spec} \mathbb{C}[P_\varphi]$ is open and

$$\operatorname{Spec} \mathbb{C}[P_{\varphi_\tau}] \otimes_{\mathbb{C}[P]} \mathbb{C}[P]/I \hookrightarrow \operatorname{Spec} \mathbb{C}[P_\varphi]/I\mathbb{C}[P_\varphi]$$

is open too.

These inclusions are preserved by gluing

$\Rightarrow \quad X^0 \subset \operatorname{Spec} \mathbb{C}[P_\varphi]/I\mathbb{C}[P_\varphi]$

$\Rightarrow$ ... I ... Spec ...

is open.

Let us go back to

$$\operatorname{Spec} \mathbb{C}[P_{\varphi_T}] \longrightarrow \operatorname{Spec} \mathbb{C}[P].$$

From a previous computation, we know the fibre over smallest toric stratum is the union of toric affine pieces, corresponding to domains of linearity of φ_T:

$$(\mathbb{A}^1 \times \mathbb{C}^*) \cup (\mathbb{A}^1 \setminus \{0\}) \cup (\mathbb{A}^1 \times \mathbb{C}^*)$$

$$\cong \mathbb{A}^2 \setminus \{0, 0\}.$$

This implies the statement about fibres, in the complete case we get by glueing, eg

$$\mathbb{V}_n^\circ := \mathbb{V}_n \setminus \{\underline{0}\}.$$

The topological fact about fibres implies the statement on global functions, by

<u>Fact</u>. $\pi : \mathfrak{X} \to S$ flat family of surfaces, $\mathfrak{X}_s$ "S_2" $\forall s \in S$.

$\exists\ \mathfrak{X}^\circ \hookrightarrow \mathfrak{X}$ open with ... depth$(\mathcal{O}_{\mathfrak{X},x})$ ...

$\pi\big|_{\mathcal{X}\setminus\mathcal{X}^\circ} \longrightarrow S$ <u>FINITE</u>. $\geq \dim(\mathcal{O}_{X,x})$

Then $i_* \mathcal{O}_{\mathcal{X}^\circ} = \mathcal{O}_{\mathcal{X}}$ $\square$

$\square$

The upshot of Lemma are

1) top description of fibres.
   i.e. $X^\circ_{I,\mathfrak{D}} \longrightarrow \operatorname{Spec} \mathbb{C}[P]/I$ is "missing the origin"! Need to partially compactify.

2) In the toric case, with $\mathfrak{D} = \{0\}$, the correct partial compactification is

$$X_I := \operatorname{Spec} H^0(X^\circ_I, \mathcal{O}_{X^\circ_I})$$

<u>We would like this to be true in the general case!</u>

What is the correct scattering diagram (i.e. def. of complex structure) for this to work?! (If any!)

$$\boxed{\text{SYMPLECTIC HEURISTIC}}$$

<u>SYZ in the complement</u> of an <u>anticanonical</u>
divisor (Auroux).

Mirror (Auroux):

(Y, D) Looij pair.

ω symplectic on Y,

Ω holom. sympl. on $U := Y \setminus D$.

$f : U \to B$ SLag fibration by tori
(general fibre L satisfies
$\operatorname{Im} \Omega|_L = \omega|_L = 0$)

SYZ mirror:
"dual torus fibration"
$\check{f} : X \to B$ (generically).

$= \{$moduli space of pairs $(L, \nabla)\}$
L = a general fibre
∇ = $U(1)$ connection / gauge
$\sim \mathrm{hol}_\nabla : H_1(L, \mathbb{Z}) \to U(1) \subset \mathbb{C}^*$.

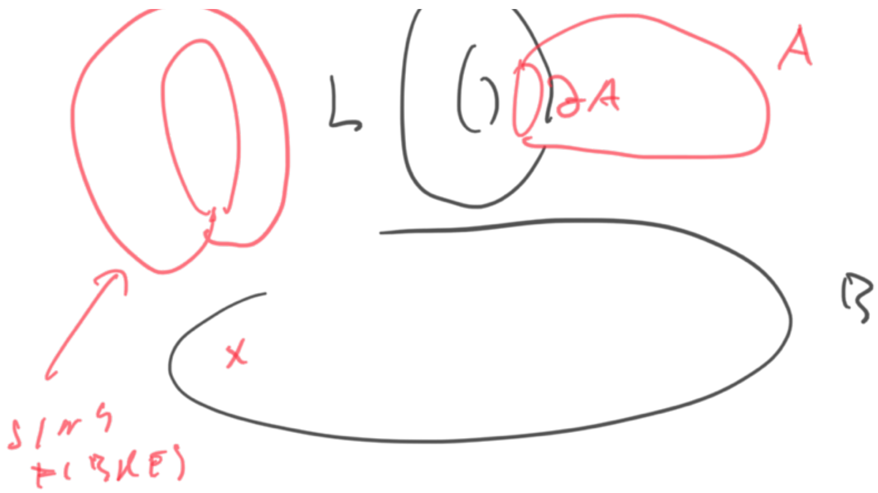
Problem: cplx structure on X, compactification.

Local holomorphic coords on X: $(A \in H_2(Y, L, \mathbb{Z}))$

$$z^A := \exp\left(-2\pi \int_A \omega\right) \mathrm{hol}_\nabla(\partial A) : X \to \mathbb{C}^*,$$

$A \in H_1(L, \mathbb{Z}) \hookrightarrow H_2(Y, L, \mathbb{Z})$ by
$\cong \mathbb{Z}^2$

choosing some splitting of
$H_2(Y, L, \mathbb{Z}) \to H_1(L, \mathbb{Z}) \to 0$.



<u>But</u>, Z^A only local because $H_2(Y, L, \mathbb{Z})$ is only a local system on $B_0 \subset B$ as L varies!

Discontinuity in Z^A along codim$_{\mathbb{R}} = 1$ <u>walls</u> in B, for which L_0 bounds a holomorphic disc $\subset U$,

$$(*) \quad Z^A \mapsto Z^A \cdot h(Z^\alpha)^{[\partial A]\cdot[\partial \alpha]}$$

$\alpha \in H_2(Y, L_0, \mathbb{Z})$ class of some holomorphic disc with bdry on L_0.

$h(Z^\alpha) :=$ the generating function "counting" such discs.

Idea: cut B_0 along walls, use nontrivial glueing $(*)$ to define a new "instanton corrected" cplx str on $X \setminus \{$sing fibres$\}$.

Now, this new cplx mfld should come with GLOBAL holomorphic functions

$$“\ \vartheta_i := \sum_{\beta \in H_2(Y,L,\mathbb{Z})} n_\beta z^\beta \ ”$$

$n_\beta :=$ # of "Maslov index 2 (holomorphic) discs with bdry on L, class β, and intersecting D_i in 1 reduced pt.

The ϑ_i should define the compactification of our moduli space!

In GHK algebraic description:

so far we have given a model for walls on B_0 and the glueing (*).

We will

give an algebro-geometric model for "$\vartheta_i = \sum_\beta n_\beta z^\beta$", for a diagram $\mathfrak{D}$ satisfying the "global consistency condition"

prove that for the canonical Mumford degeneration

Modified

$$\{ P_{\rho, d_i} := \eta([D_i]) \}$$

Such a "globally consistent diagram" exists and is essentially unique,

$$\underbrace{\mathfrak{D} = \mathfrak{D}^{can}}$$

defined by counting rational curves in toric blowups of (Y, D).

Tropical analogues of holomorphic discs with bdry on L : broken lines

Recall integral pts $B_0(\mathbb{Z})$ are well def because of $SL(2,\mathbb{Z})$ transition functions.

$\boxed{\text{Broken line for } q \in B_0(\mathbb{Z}) \text{ with end } Q \in B_0 :}$

proper, C^0, piecewise integral affine

$$\gamma : (-\infty, 0] \longrightarrow B_0$$

decorated with a monomial

$$m_L = c_L z^{q_L},$$

$$c_L \in \mathbb{C}, \quad q_L \in \mathbb{C}[\Gamma(L, \gamma^{-1}(\mathcal{P})|_L)],$$

$\forall\, L \subset (-\infty, 0]$ max connected domain of linearity

(recall $\mathfrak{D} := \pi_* \Lambda_{\mathbb{P}_0}$),

with constraints:

(1) as $t \to -\infty$, $\gamma(t) \to \infty$ inside $\sigma \in \Sigma_{max}$, with $q \in \Sigma$, and $m_L = z^{\varphi_\sigma(q)}$;

(2) $\forall L$, $\forall t$ $\; -r(q_L) = \gamma'(t)$;
$\gamma(0) = Q \in B_0$;

(3) $\forall t$, let $\gamma(t) \in \tau \in \Sigma$ (1d or 2d) A BEND;
$\partial_1, \dots, \partial_p \in \mathfrak{D}$ the rays containing $\gamma(t)$;
expand $\prod_{j=1}^{p} f_{\partial_j}^{\langle n, r(q_L)\rangle} \in \widehat{\mathbb{C}[P_{\varphi_\tau}]}$.

Then $\; m_{L'} = m_L \cdot (\underbrace{c z^q}_{\text{some monomial here}})$.

(L, L' consecutive domains of linearity)

<u>Rmk.</u> We have $\langle n, r(q_L)\rangle > 0$ because
$$-r(q_L) = \gamma'(t).$$

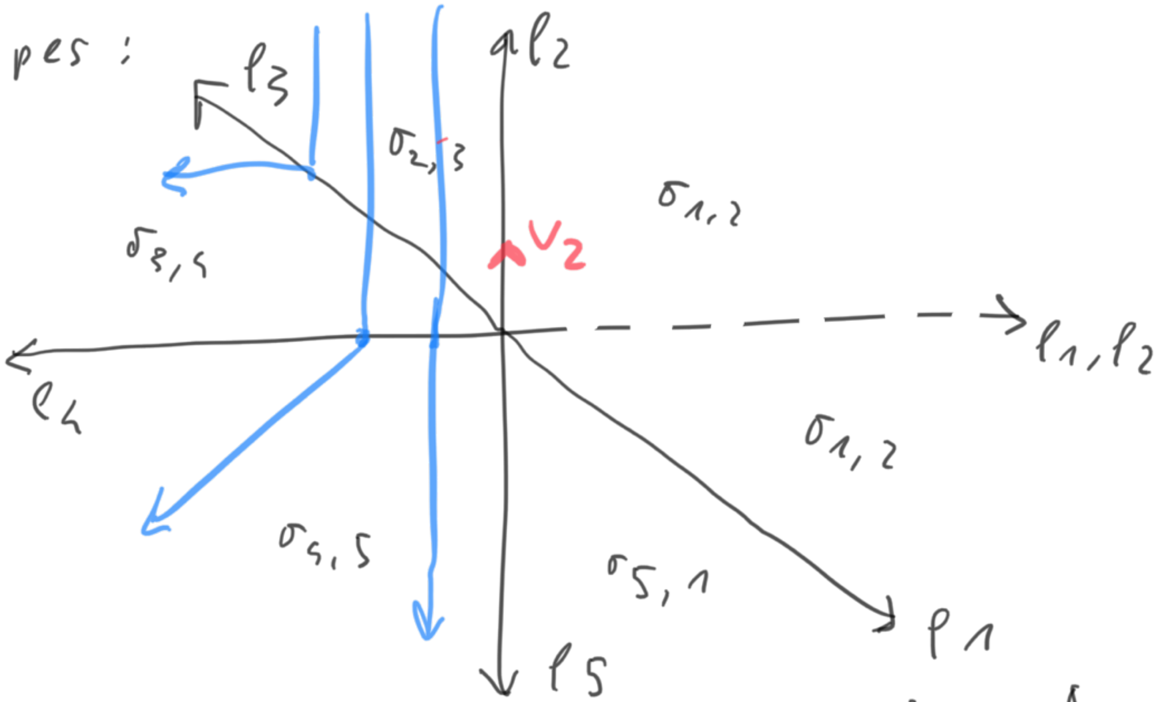
Eg/Exe.: Deg 5 del Pezzo.

$\quad \; ^{can} \quad \{ \quad \quad z^{[E_i] - \varphi_{\mathbb{P}_i}(v_i)}) \}$

Take $W = W \quad = \{(\rho_i, 1+z$

(Very important later!)

Prove that the broken lines for v_2 only have 3 types:



$[E_i] :=$ unique -1 curve $\not\subset D = D_1 + \dots + D_5$, meeting D_i transversally.

<u>Lemma.</u> Propagation along a broken line preserves the "Mumford monoids":

$$\gamma(t) \in \tau \Rightarrow q_L \in P_{\varphi_\tau} \subset P_\tau .$$

<u>Proof.</u> For $t \ll 0$ we have

$$m_L = z^{\varphi_\tau(q)} \in P_{\varphi_\sigma}$$

by definition.

We need to show the property is preserved

points.

at bending p...

Recall at a bend $\gamma(t)$ we have

$$(*)\ m_{L'} = m_L \cdot \underbrace{(c z^q)}_{}$$

a mono in $\prod_j f_{\partial_j}^{\langle n, r(q_L)\rangle}$,

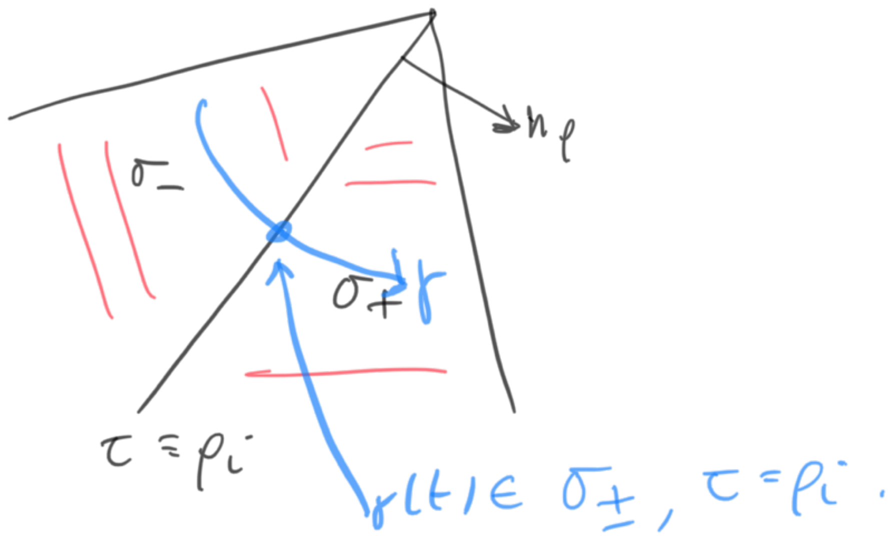
for $\gamma(t) \in \partial_j$, $j = 1 \dots p$.

If $\gamma(t) \in \sigma \in \Sigma_{max}$ (a 2d cone), then $(*)$ holds with both sides in $\mathbb{C}[P_{\varphi_\sigma}]$.

So the only nontrivial case is when $\gamma(t) \in \tau$ = a 1d cone in Σ, i.e. when a priori we have

$$q_L \in P_{\sigma_-}, \quad q_{L'} \in P_\tau,$$

where we have usual picture



Now if we regard $(*)$ as an identity of the...

holding in $\mathbb{C}[P_{\varphi_{\sigma_-}}]$, we get the weaker statement

$$q_{L'} \in P_{\varphi_{\sigma_-}} \ (\supsetneq P_{\varphi_\tau}),$$

so

$$q_{L'} = \varphi_{\sigma_-}(r(q_{L'})) + \underbrace{p}_{\in \underline{P}}.$$

On the other hand, using $-r(q_{L'}) = \gamma'(t)$, we see that

$$\varphi_{\sigma_+}(-r(q_{L'})) = \varphi_{\sigma_-}(-r(q_{L'})) + \underbrace{\langle n_\rho, -r(q_{L'})\rangle}_{>0} p_{\rho,\varphi}.$$

Combining the two identities we find

$$q_{L'} = \underbrace{\varphi_{\sigma_+}(r(q_{L'})) + \underbrace{p + \langle n_\rho, -r(q_{L'})\rangle p_{\rho,\varphi}}_{\in P}}_{P_{\varphi_{\sigma_+}}}$$

□

Notation:

- $\forall\, J \subset P,\ p \in J,\ \operatorname{ord}_J(p) := \max_k \{ p = p_1 + \dots + p_k,\ p_i \in J \}$ $(0 \text{ for } p \notin J)$;

$$\ell(\ ) := \operatorname{ord}\ (q\ -\varphi\ (r(q)))\ .$$

• $\mathrm{ord}_{J,\gamma}(t) := \mathrm{ord}_J(t_L$ [illegible]$)$,

• $\forall\, I \subset P$, $\sqrt{I} = J$, $\mathrm{Supp}_I(\mathfrak{D}) := \bigcup\limits_{f_\partial \not\equiv 1 \bmod I_{\tau_\partial, \tau_\partial}} \partial$;

• $\mathrm{Mono}(\gamma) :=$ the monomial attached to the last domain of linearity.

• $\forall\, q \in B_0(\mathbb{Z})$, $Q \in \tau \in \Sigma$, $Q \in B \setminus \mathrm{Supp}_I(\mathfrak{D})$

$$\mathrm{Lift}_Q(q) := \sum_{\substack{\{\gamma \text{ broken line for } q\} \\ \text{end pt}(\gamma) = Q}} \mathrm{Mono}(\gamma) .$$

<u>Lemma.</u> Suppose $P_{\rho,\sigma} \in J$ for at least one ray. Then:

Fix $Q \in \sigma \in \Sigma_{\max}$, $q \in B_0(\mathbb{Z})$. Then:

(1) The set of γ's as above such that

$$\mathrm{Mono}(\gamma) \notin I \cdot \mathbb{C}[P_{\varphi\sigma}]$$

is finite;

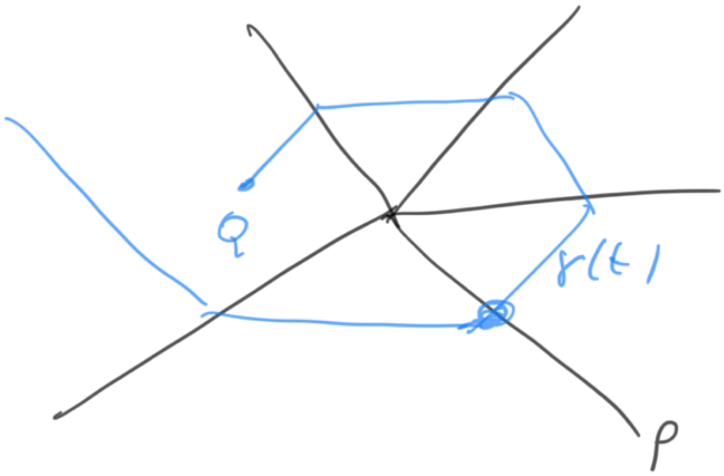
(2) If a ray $\rho \in \Sigma$ is a boundary ray of the connected component of $B \setminus \mathrm{Supp}_I(\mathfrak{D})$ containing Q, then the set of γ's such that

$$\mathrm{Mono}(\gamma) \notin I \cdot \mathbb{C}[P_{\varphi\rho}]$$

is finite.

Proof. (Sketch). Only prove (1).

By assumption, $\exists\, \rho$ a ray with $P_{\rho, q} \in J$, so $\mathrm{ord}_{J,\gamma}(t)$ increases when we cross this ray:



And we have $f_\gamma \equiv 1 \mod J$ if $\mathfrak{d} \in \sigma \in \Sigma_{max}$, so $\mathrm{ord}_{J,\gamma}$ increases when γ bends at such $\mathfrak{d}$.

Now $\sqrt{I} = J \Rightarrow \exists\, k$ s.t. $J^k \subset I$.

So $\exists$ a bound k on # times γ can cross ρ, or γ bends.

$\Rightarrow$ given q, Q $\exists$ # of γ's. $\square$

SYZ HEURISTICS:

$\gamma \sim$ a holomorphic disc in (some toric blup of) (Y, D), meeting bdry at dual to q, with bdry on Lagrangian Q;

$\mathrm{Mono}(\gamma) \sim$ the monomial z^{w},
$w \in H_2(Y, L)$ = relative hom class of disc.

$\boxed{\text{CONSISTENCY}}$

(AKA "GLOBAL CONSISTENCY")

We say $\mathfrak{D}$ is consistent if $\forall I$ the sums of monomials

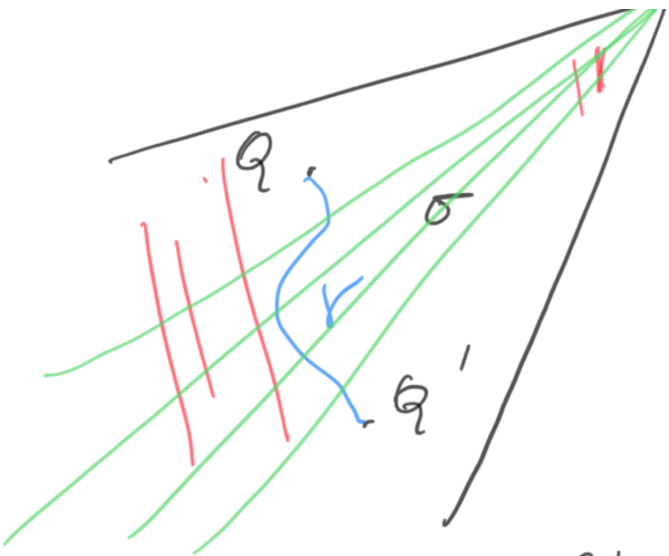
$$\mathrm{Lift}_Q(q) \in \mathbb{C}[P_{\varphi_I}]/I \cdot \mathbb{C}[P_{\varphi_I}]$$

for fixed $q \in {}_0B(\mathbb{Z})$ and suitable
$Q \in B \setminus \mathrm{Supp}_I(\mathfrak{D})$
patch up to globally defined functions on the "instanton corrected Mumford degener." $X^{\circ}_{I, \mathfrak{D}}$,

just as predicted by SYZ picture.

Gluing conditions:

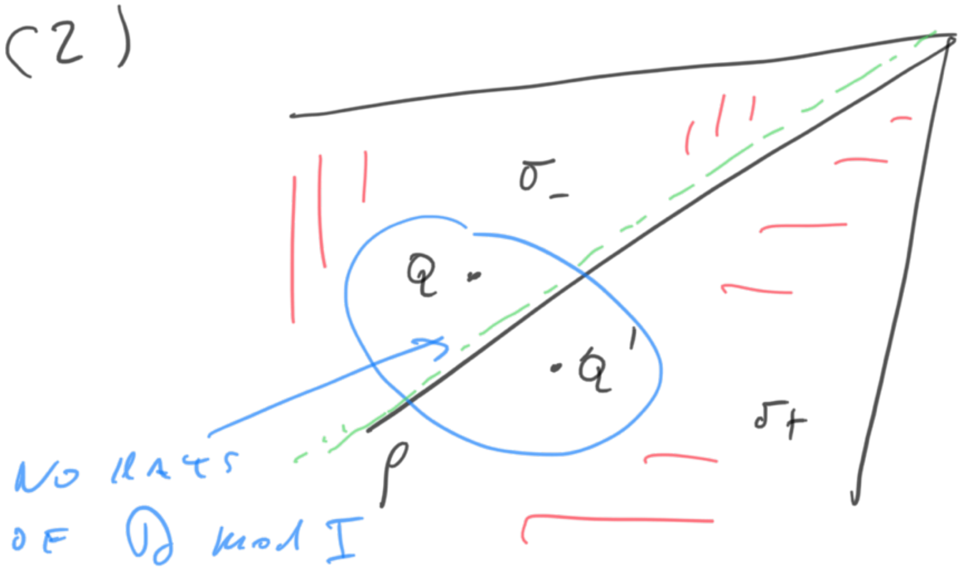
(1) $Q, Q' \in \sigma \in \Sigma_{max}$,



$$\Rightarrow \quad \mathrm{Lift}_{Q'}(q) = \Theta_{\gamma,\mathcal{D}}\left(\mathrm{Lift}_{Q}(q)\right)$$

in $R_{\sigma,\sigma,I}$;

(2)



$$\Rightarrow \left(\mathrm{Lift}_{Q}(q),\ \mathrm{Lift}_{Q'}(q)\right)$$

$$\mathrm{Lift}_{Q}(q) \in R_{\rho,\sigma_-,I}, \qquad \mathrm{Lift}_{Q'}(q) \in R_{\rho,\sigma_+,I}$$

lies in $R_{\rho,I}$.

Not at all clear that consistent $\mathcal{D}$ exist!

But if they do, they solve our par[illegible] compactification problem, and more:

Theorem. Usual technical assumptions on I, J, Σ, Σ_I : $\exists\, P_{\sigma,a} \in J$, Σ_I contains ≥ 2 rays, Σ_m contains ≥ 3 rays and has Delzant property.

Then:

1) $$X_I := \operatorname{Spec} \Gamma(X^{o}_{I,\emptyset}, \mathcal{O}_{X^{o}_{I,\emptyset}})$$
is affine scheme with flat morphism
$$f_I : X_I \longrightarrow \operatorname{Spec} \mathbb{C}[P]/I,$$
X_I contains $X^{o}_{I,\emptyset}$ as open subscheme, and fibre over torus fixed pt is
$$\mathbb{V}_n \,;$$

2) There is a canonical basis
$$\{ \vartheta_q : q \in B(\mathbb{Z}) \}$$ "theta functions"
for $\Gamma(X_I, \mathcal{O}_{X_I})$ as $\mathbb{C}[P]/I$-mod.

Proof (Sketch).
[illegible] scheme / $\operatorname{Spec} \mathbb{C}[P]/I$,

Since $X^\circ_{I,\mathcal{D}}$ is a
the same holds for $X_I := \operatorname{Spec}\Gamma(X^\circ_{I,\mathcal{D}}, \mathcal{O})$,
so the morphism

$$f_I : X_I \longrightarrow \operatorname{Spec}\mathbb{C}[P]/I$$

is given.

The fibre over torus fix. pt. is $\mathbb{V}_n$, by toric case we already discussed.

It remains to be seen f_I is flat. But this follows easily if we can show (2), i.e. that $\exists\, \vartheta_q \in \Gamma(X_I, \mathcal{O}_{X_I})$ such that map

$$\bigoplus_{q\in B(\mathbb{Z})} (\mathbb{C}[P]/I)\cdot\vartheta_q \longrightarrow \Gamma(X_I, \mathcal{O}_{X_I})$$

is $\cong$ of $\mathbb{C}[P]/I$-mod.

Strategy of proof:

(a) define $\vartheta_q \in \Gamma(\mathcal{O}_{X^\circ_I})$ for $q \in B(\mathbb{Z})$, using consistency;

(b) prove (2) for base ideal J, i.e. for

$$X^\circ_{J,\mathcal{D}} = X^\circ_J;$$

(c) use deformation theory to prove (2) for all $\sqrt{I} = J$.