Stability and wall-crossing in algebraic and differential geometry

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1. Introduction

General principle

General principle (after Narasimhan-Seshadri, Atiyah-Bott, Hitchin-Kobayashi, Donaldson, Yau, Tian...)

Solutions to natural *PDEs* in complex geometry

$$\Updownarrow$$

Natural objects in complex *algebraic* geometry

In terms of families of solutions and objects or *moduli spaces*:

$$\mathcal{M}_{\mathrm{PDE}} \cong \mathcal{M}_{\mathrm{alg}}$$

General principle

This point of view has been spectacularly successful.

The aim of our project is to **attack a number of open problems which fit in this context.**

At the same time we will **expand this principle** bringing in **new insights and problems coming from recent work in the geometry of quantum field theories.**

Total duration: **4 years.**

Examples: gauge theory

- **Hermitian Yang-Mills connections:** (X, ω) a *Kähler* manifold (i.e. complex manifold with Riemannian metric which is perfectly adapted to complex structure).

  $E \to X$ holomorphic vector bundle; A on E compatible connection $\Rightarrow$ a complex analogue of usual fields in physical gauge theory.

  Field-strength = F_A = curvature = $dA + [A, A]$, 2-form with values in ad(E).

  Equation of motion: $\mathrm{Tr}\ F_A = \lambda I$.

Thm (Hitchin-Kobayashi, Donaldson, Uhlenbeck-Yau)

A solution exists iff E satisfies a purely algebro-geometric condition, *slope polystability*. This is a constraint on *all the holomorphic subsheaves*.

Examples: gauge theory

- **Higgs bundles:** Σ a Riemann surface, $E \to \Sigma$ rank 2 complex vector bundle with structure group $SU(2)$.

  A a compatible connection.

  Φ a Higgs field: 1-form with values in $\mathrm{ad}(E) \otimes \mathbb{C}$.

  Equations of motion are Hitchin's equations:

$$F_A + [\Phi, \Phi^*] = 0$$
$$\overline{\partial}_A \Phi = 0.$$

  Theorem (Hitchin): solution iff $(E, \overline{\partial}_A)$ *holomorphic*, (E, Φ) *stable*.

(Open) problem (Hitchin, Simpson...):

Understand (hyperkähler) space of solutions $(\mathcal{M}, g)$ (general gauge groups, singular fields...).

Examples: Kähler geometry

- **Kähler-Einstein metrics:** (X, g) Kähler; ω_g Kähler differential 2-form.

  Ricci form: $\mathrm{Ric}(\omega_g) = -\partial\overline{\partial} \log \det g_{k\bar{l}}$.

  Einstein's equation in the Kähler world:

  $$\mathrm{Ric}(\omega_g) = \lambda \omega_g.$$

  Topological constraint: $c_1(X) = \lambda[\omega_g]$.

  Aubin-Calabi-Yau Theorem: If $c_1(X) \leq 0$ this is the only constraint.

Kähler-Einstein (open) problem (Calabi, Yau, Tian, Donaldson...)

Solve the equation for positive Ricci curvature. **Main conjecture (Yau-Tian-Donaldson): this is a purely algebro-geometric problem.**

Examples: Kähler geometry

- **CscK and extremal metrics:** prescribe *scalar curvature* to be constant, or as constant as possible:

  Scalar curvature: $s(g) = -g^{i\bar{j}}\partial_i\overline{\partial}_j \log\det g_{k\bar{l}}$.

  Csck equation and extremal equations:

$$s(g) = \widehat{s} = \text{a topological constant.}$$

$$\nabla^{1,0}s(g) = \chi = \text{ess. unique holomorphic vector field.}$$

CscK (open) problem (Calabi, Yau, Tian, Donaldson...)

Which manifolds have a cscK (or extremal) metric? **Main conjecture (Yau-Tian-Donaldson): when $[\omega] = c_1(L)$ for $L \to X$ ample, this is a purely algebro-geometric problem.**

2. Stability and canonical metrics

K-stability

- **K-(semi, poly)stability:** $[\omega_g] = c_1(L)$. Embed $X \hookrightarrow \mathbb{P}^{N_k}$ using powers L^k. K-(semi,poly)stability is a constraint on *all* degenerations $\mathcal{X}$ of (X, L) induced by flowing under a $\mathbb{C}^* \curvearrowright \mathbb{P}^{N_k}$, $F(\mathcal{X}) > 0$ (≥ 0).

Theorem (Donaldson)

$\omega_g \in c_1(L)$ cscK $\Rightarrow (X, L)$ K-*semi*stable.

Theorem (S.): "blow-up method"

If moreover $\mathrm{Aut}(X, L)$ discrete $\Rightarrow$ K-*stable*. In the extremal case get *relative* K-*poly*stability (with Székelyhidi).

Conjecture (Yau-Tian-Donaldson)

In general, cscK $\Leftrightarrow$ K-*poly*stable.

K-stability: problems and limitations

- **No uniform control:** for analytic purposes need uniform bound on $F(\mathcal{X})/\|\mathcal{X}\|$, but it's lacking.
- **Might not be sufficient:** conjectural counterexample by Apostolov, Calderbank, Gauduchon and Tonnesen-Friedman.
- **Not natural:** natural statements like stability $\Rightarrow$ reductivity, Zariski openness, Atiyah-Bott type theorems on worst degenerations are all *very hard* conjectures.
- **Not compatible with Gromov-Hausdorff limits:** in the approach of Donaldson (also with Chen, ...) Gromov-Hausdorff limits lead to a different notion: **b-stability**.

Our main objectives

General objective

Go beyond K-stability. Need **much more flexible** notion with respect to natural algebro-geometric operations, but **still necessary for cscK** (extremal). Study its **relation to K-stability** and asymptotic Chow stability.

Conjecture A and similar problems

Prove naturality properties for our new notion, and a posteriori for K-stability: **equivariance** and **compatibility to maximal tori**, **Zariski openness**, **reductivity** of Aut(X)...

b-stability

Obtain a detailed understanding of Donaldson's b-stability (the algebraic counterpart of Gromov-Hausdorff limits). **Prove that a cscK manifold is b-stable**.

Sketch of methodology

Candidate: stability by filtrations

Encode degenerations by filtrations of $\bigoplus_k H^0(X, L^k)$, not necessarily finitely generated (Witt-Nystrom, Székelyhidi). Better behaved, but all the main problems still open. **Bring in methods from birational geometry and the analysis of ample linear series.**

Variations of the blow-up method

Donaldson made progress on cscK $\Rightarrow$ b-stable using blow-up method, in the equivariant case. **Develop a non-equivariant version for the blow-up method.**

Approximation

Develop an **approximation theory for Donaldson's general families** in b-stability by one dimensional objects.

3. Hyperkähler metrics and wall-crossing

Seiberg-Witten and Higgs bundles

- **Seiberg-Witten:** Study $\mathcal{N} = 2$ Yang-Mills theories on $\mathbb{R}^3 \times S^1_R \Rightarrow$ a σ-model $\mathbb{R}^3 \to$ hyperkähler $\mathcal{M}$.
- **Higgs bundles:** Claim $(\mathcal{M}, g)$ is a moduli space of solutions to $F_A + R[\Phi, \Phi^*] = \overline{\partial}_A \Phi = 0$, with prescribed singularities!

Study the **geometry of** moduli of **Higgs bundles** $(\mathcal{M}, g)$ (especially Hitchin fibration $\mathcal{M} \to \mathcal{B}$) **via** the structure of $\mathcal{N} = 2$ **Yang-Mills** (i.e. its operators, spectrum...).

BPS states and Gaiotto-Moore-Neitzke

- **BPS spectrum:** states in $\mathcal{N} = 2$ Yang-Mills killed by half the supersymmetry operators.

Find mathematical framework (Donaldson-Thomas theory?): Bridgeland, Smith...

- **Wall-crossing:** the number of BPS states is a function $\Omega(\gamma, u)$ of charge and coupling constant $u \in \mathcal{B}$.

$\Omega(\gamma, u)$ jump when u crosses critical locus: **wall-crossing** formulae as in Joyce-Song, Kontsevich-Soibelman.

Gaiotto-Moore-Neitzke conjecture(s)

$\Omega(\gamma, u)$ **completely determines** $(\mathcal{M}, g)$ by (nonperturbative) **instanton corrections** in R through **integral equation.** Wall-crossing reflects **continuity** of g.

Our main objectives

General objective

Study the conjecture(s) of Gaiotto-Moore-Neitzke for a large class of $(\mathcal{M}, g)$. Prove **existence and uniqueness of solutions** for their integral equation.

Asymptotic expansion

Study the **convergence of the natural asymptotic expansion** for solution emerging from GMN.

Additional structures

Study natural additional structures on $(\mathcal{M}, g)$: **hyperholomorphic connections**. What is the **mirror** for these constructions?

Sketch of methodology

Finite BPS spectra

Concentrate initially on examples where $\Omega(\gamma, u)$ is **everywhere finite**.

Comparisons

Establish **precise comparisons** with algebro-geometric work of Joyce and Bridgeland, Toledano-Laredo. Conjecture (S.): asymptotic expansion recovers Joyce's theory (checked in many cases).

Nahm-type equations

Recast as **infinite dimensional Nahm-type equations.** Bring in recent ideas of Donaldson on infinite dimensional Nahm and geodesics in the space of Kähler potentials.

4. The team

- **Jacopo Stoppa** (Università di Pavia and Trinity College, Cambridge) - Principal Investigator. *CscK, K-stability, DT theory, quivers and tropical vertex, GMN theory*.

- **Gabor Székelyhidi** (University of Notre Dame) - Team Member. *CscK and extremal metrics, Monge-Ampère equations, Kahler-Ricci flow, Sasaki geometry*.
- **Two postdocs** ($2 + 2$ years). With suitable expertise in algebraic or differential geometry.

- **One doctoral student** (3 years). Initially working in low-dimensional examples for b-stability or finite spectrum examples in GMN theory.

5. Timeliness and flexibility

Very active research areas: recent or forthcoming works of Donaldson (also with Chen, Sun); Bridgeland, Smith...

Forthcoming **Junior Research Trimester** at HIM, Bonn, Sep-Dec 2012 (P.I. will be group leader for “BPS states” group).

Project is **flexible** and can adapt to the area of expertise of highly qualified post-doctoral members.

Good range of **PhD problems** to attract an excellent candidate.