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LARGE COMPLEX STRUCTURE LIMITS OF K3 SURFACES

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Abstract

Motivated by the picture of mirror symmetry suggested by Strominger, Yau and Zaslow, we make a conjecture concerning the Gromov-Hausdorff limits of Calabi-Yau n-folds (with Ricci-flat Kähler metric) as one approaches a large complex structure limit point in moduli; a similar conjecture was made independently by Kontsevich, Soibelman and Todorov. Roughly stated, the conjecture says that, if the metrics are normalized to have constant diameter, then this limit is the base of the conjectural special lagrangian torus fibrations associated with the large complex structure limit, namely an n-sphere, and that the metric on this S^n is induced from a standard (singular) Riemannian metric on the base, the singularities of the metric corresponding to the limit discriminant locus of the fibrations. This conjecture is trivially true for elliptic curves; in this paper we prove it in the case of K3 surfaces. Using the standard description of mirror symmetry for K3 surfaces and the hyperkähler rotation trick, we reduce the problem to that of studying Kähler degenerations of elliptic K3 surfaces, with the Kähler class approaching the wall of the Kähler cone corresponding to the fibration and the volume normalized to be one. Here we are able to write down a remarkably accurate approximation to the Ricci-flat metric: if the elliptic fibres are of area $\epsilon > 0$, then the error is $O(e^{-C/\epsilon})$ for some constant $C > 0$. This metric is obtained by gluing together a semi-flat metric on the smooth part of the fibration with suitable Ooguri-Vafa metrics near the singular fibres. For small ϵ, this is a sufficiently good approximation that the above conjecture is then an easy consequence.

0. Introduction

The notion of large complex structure limit plays a special role in the theory of mirror symmetry. If X is a Calabi–Yau manifold, a large complex structure limit point is a point in a compactified moduli space

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of complex structures $\overline{\mathcal{M}}_X$ on X which, in some sense, represents the "worst possible degeneration" of the complex structure. This notion was given a precise Hodge-theoretic meaning in [27]. The basic example to keep in mind of this sort of degeneration is the degeneration of a hypersurface of degree $n+1$ in $\mathbf{P}^n$ to a union of the $n+1$ coordinate hyperplanes. Mirror symmetry posits the existence of a mirror to X associated to each large complex structure limit point of X. To first approximation, this means that if $p \in \overline{\mathcal{M}}_X$ is a large complex structure limit point in a compactification of the complex moduli space of X, then there exists a mirror $\check{X}$ and an isomorphism between a neighbourhood of p in $\overline{\mathcal{M}}_X$ and the complexified Kähler moduli space of $\check{X}$ which preserves certain additional information, such as the Yukawa couplings (which will not concern us in this paper). This isomorphism is known as the mirror map.

Now the Strominger-Yau-Zaslow conjecture [32] suggests that mirror symmetry can be explained by the existence of a special Lagrangian fibration on X when the complex structure on X is near a large complex structure limit point. The mirror $\check{X}$ is then expected to be constructed as the dual of this special Lagrangian fibration. The notion of special Lagrangian is a metric one: it depends on both the complex structure (determined by a holomorphic n-form Ω on X, where $n = \dim_{\mathbf{C}} X$), and a Ricci-flat Kähler metric, determined by its Kähler form ω. Thus we expect the existence of special Lagrangian fibrations will depend a great deal on the metric properties of Calabi–Yau manifolds near large complex structure limit points.

The simplest example of such a situation occurs for elliptic curves. Consider the family of elliptic curves $E_\alpha = \mathbf{C}/\langle 1, i\alpha\rangle$, with $\alpha \to \infty$. We also choose a Ricci-flat, i.e., flat, metric g, which we will take to be the standard Euclidean metric. As $\alpha \to \infty$, the complex structure approaches the large complex structure limit point in the moduli space of elliptic curves; the period $i\alpha$ is approaching the cusp point of the compactification of $\mathcal{H}/SL_2(\mathbf{Z})$.

Now given the metric g, as $\alpha \to \infty$ it is clear that these elliptic curves converge to an infinitely long cylinder. However, if we rescale the metric, with $g_\alpha = g/\alpha$, then $Vol(E_\alpha) = 1$ in this metric. With this metric, we can instead view E_α as $\mathbf{C}/\langle 1/\sqrt{\alpha}, i\sqrt{\alpha}\rangle$ with the standard Euclidean metric, and then E_α converges to a line as $\alpha \to \infty$.

Finally, we may renormalize the metric again so that the diameter of E_α remains bounded, with the metric $g_\alpha = g/\alpha^2$. Then E_α can be identified with $\mathbf{C}/\langle 1/\alpha, i\rangle$ with the Euclidean metric, and E_α converges to a circle.

Of course, in this situation the special Lagrangian T^1-fibration on E_α is $E_\alpha \to S^1$ obtained by projection onto the imaginary axis. So with the second and third choices of normalization, the special Lagrangian fibres collapse.

This is a rather trivial example, but forms a good basis for speculating about what might happen in higher dimensions. Intuitively, if we normalize the metric so as to keep the volume of the manifold bounded, we expect to see the fibres of the hypothetical special Lagrangian fibration contracting down to points; if furthermore we normalize so as to have bounded diameter, we expect the Calabi–Yau manifold to "converge" (in a sense we will make more explicit in §6) to a sphere of dimension n.

To test this picture, and to improve our understanding of Ricci-flat metrics, we have chosen to study the metric on K3 surfaces approaching large complex structure limit points. This is made easier by the fact that special Lagrangian fibrations are known to exist on K3 surfaces by a standard trick of performing a hyperkähler rotation of the complex structure, so that one reduces the problem of finding a special Lagrangian fibration to that of finding an elliptic fibration. Using this, we show in §1 that after performing this hyperkähler rotation, approaching a large complex structure limit is more or less the same as fixing the complex structure on a K3 elliptic fibration $f : X \to \mathbf{P}^1$, and letting the Kähler form ω on X vary in such a way that the area of the fibres approaches zero. Thus we ask the question: what does a Ricci-flat metric on an elliptic K3 surface look like when the area of the fibres is very small?

This is an interesting question even if one is not interested in mirror symmetry. In [1], M. Anderson studied degenerations of Ricci-flat metrics on K3 surfaces. If the volume of the surface is fixed and the diameter remains bounded, then the metrics converge to an orbifold metric (corresponding to degeneration to a K3 with rational double points). This picture of the moduli space of K3 surfaces with orbifold metric was originally studied in [21]. If the diameter is unbounded, Anderson proved collapsing must occur, but gave no more detailed information. The case under consideration in this paper can be considered to be the most extreme degeneration of metric. In particular, the orbifold case and the elliptic fibration case are the only Kähler degenerations, in which the complex structure of the K3 surface is held fixed. We will in fact consider a slightly more general situation, where the complex structure still varies to some extent. This is described more clearly in §1.

We assume the generic case, so that f has 24 singular fibres, each of Kodaira type I_1 (a pinched torus). If X_0 denotes the complement of these 24 singular fibres, then it is possible to write down a family of explicit Ricci-flat metrics which we refer to as *semi-flat*: these metrics are in fact flat when restricted to the fibres. The semi-flat metric was first introduced in [12]. There, it was used to get a first approximation to a complete Ricci-flat metric on the complement of a fibre of a rational elliptic surface. In [12], an arbitrary metric was then glued in to take care of the singular fibres so that techniques of [33], [34] could be applied to obtain a complete Ricci-flat metric on this manifold. While we follow this idea in spirit, we have here a new ingredient we can take advantage of. There is an explicit Ricci-flat metric defined in a neighbourhood of each singular fibre, first written down by Ooguri and Vafa in [29]. It is not semi-flat, but it in fact decays to a semi-flat metric exponentially. We can glue 24 copies of the Ooguri–Vafa metric in to the semi-flat metric, and thereby obtain a metric which is remarkably close to being Ricci-flat: in fact, the Ricci curvature is bounded in absolute value by $O(e^{-C/\epsilon})$, where ϵ denotes the area of a fibre. Thus as $\epsilon \to 0$, the Ricci curvature of this glued metric approaches zero very rapidly.

We then follow standard techniques to show that the genuine Ricci-flat metric representing the same Kähler class is very close to the glued metric, hence showing the explicit metric we constructed is a very good approximation to the genuine metric. We follow the proof of Kobayashi in [20], based on the original methods of Yau [35] — cf. also [7], [33], [34]. In [20] Kobayashi proves that near a Kummer surface, the Ricci-flat metric on a K3 surface is close to the flat orbifold metric on the Kummer surface. While the techniques are the same, it is perhaps surprising that they apply in our circumstances. Indeed, if the volume of the K3 surface is held fixed, then as $\epsilon \to 0$, the diameter of our metric approaches ∞. Thus the relevant Sobolev constant approaches zero, and so it will be important to control this precisely. It turns out that everything works because the starting glued metric is already extremely close to being Ricci-flat.

More explicitly, for Kähler classes $[\omega_\epsilon]$ on X, where ϵ denotes the volume of a fibre of f, we construct a representative Kähler metric ω_ϵ with very small Ricci curvature. Yau's proof [35] of the Calabi conjecture yields a solution u_ϵ to the equations

$$(\omega_\epsilon + i\partial\bar{\partial}u_\epsilon)^2 = e^{F_\epsilon}\omega_\epsilon^2$$

$$\int_X u_\epsilon \omega_\epsilon^2 = 0$$

with $F_\epsilon = \log\left(\frac{\Omega\wedge\bar{\Omega}/2}{\omega_\epsilon^2}\right)$. The metric $\omega_\epsilon + i\partial\bar{\partial}u_\epsilon$ is the desired Ricci-flat metric. We obtain a global C^2-estimate (Lemma 5.3), namely that for some positive constant C,

$$C^{-1}\omega_\epsilon \leq \omega_\epsilon + i\partial\bar{\partial}u_\epsilon \leq C\omega_\epsilon.$$

Moreover, the main theorem of the paper (Theorem 5.6) states that for any simply connected open set $U \subset B$ whose closure is disjoint from the discriminant locus of f, and for any $k \geq 2$, $0 < \alpha < 1$, there exist positive constants C_1, C_2, ϵ_0 such that, for all $\epsilon < \epsilon_0$,

$$\|u_\epsilon\|_{C^{k,\alpha}} \leq C_1 e^{-C_2/\epsilon},$$

where the $C^{k,\alpha}$ norm is on the set $f^{-1}(U)$. Thus, away from the singular fibres, ω_ϵ is a very good approximation to the actual Ricci-flat metric. See Theorem 5.6 for a more precise statement, which requires some care in the choice of the Kähler class $[\omega_\epsilon]$.

The information obtained gives a clear picture of the metric behaviour as $\epsilon \to 0$. Using the above results, we prove the fibres are collapsing to points, and that away from the singular fibres, the metric approaches the semi-flat metric. In fact we will compute the Gromov–Hausdorff limit of a sequence of K3 surfaces with $\epsilon \to 0$ and the metrics renormalized so that the diameter remains bounded. This limit is indeed an S^2, but the metric on the S^2 is singular at precisely 24 points corresponding to the singular fibres. See §6 for more precise statements. There, we state a conjecture, also made independently by Kontsevich, Soibelman, and Todorov, about the Gromov–Hausdorff limit of Calabi–Yau manifolds approaching large complex structure limit points. The above results prove this conjecture in the two dimensional case.

The structure of the paper is as follows. In §1 we briefly review mirror symmetry for K3 surfaces, so as to reduce the problem to one of understanding elliptic fibrations. In §§2 and 3, we introduce various ways of thinking about Ricci-flat metrics on elliptic fibrations, and then discuss required properties of the semi-flat and Ooguri–Vafa metrics. In §4, we build the glued metric. In §5, we run through the standard program to obtain estimates for Ricci-flat metrics, proving the main result of the paper, Theorem 5.6. Finally, in §6, we relate these results to Gromov–Hausdorff convergence, and speculate as to what kind of results in this direction might be expected and useful in higher dimensions.

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1. Identification of large complex structure limits

There are a number of variants of mirror symmetry for K3 surfaces: see especially [10] for mirror symmetry between algebraic families of K3 surfaces and [4] for a more general version. We will use an intermediate version here, following [14], §7, which highlights the role of the special Lagrangian fibration. See also [17], §1. We review this point of view here. This will serve as motivation for Question 1.2 below, which will be addressed in the remainder of the paper. However, the setup of mirror symmetry will not be used again in this paper.

Let L be the K3 lattice, $L = H^2(X, \mathbf{Z})$ for X a K3 surface. Fix a sublattice of L isomorphic to the hyperbolic plane H generated by E and σ_0, with $E^2 = 0, \sigma_0^2 = -2$, and $E.\sigma_0 = 1$. We will view mirror symmetry as an involution acting on the moduli space of triples $(X, \mathbf{B} + i\omega, \Omega)$ where X is a marked K3 surface, Ω is the class of a holomorphic 2-form on X, $\omega \in E^\perp \otimes \mathbf{R}$ a Kähler class on X, and the B-field $\mathbf{B}$ lies in $E^\perp/E \otimes \mathbf{R}$. In addition Ω is normalised so that $\operatorname{Im}\Omega \in E^\perp \otimes \mathbf{R}$ and $\omega^2 = (\operatorname{Re}\Omega)^2 = (\operatorname{Im}\Omega)^2$. Mirror symmetry interchanges $(X, \mathbf{B} + i\omega, \Omega)$ with $(\check{X}, \check{\mathbf{B}} + i\check{\omega}, \check{\Omega})$, where $\check{X}$ denotes a marked K3 surface with the following data:

$$\begin{aligned}
\check{\Omega} &\equiv (E.\operatorname{Re}\Omega)^{-1}(\sigma_0 + \mathbf{B} + i\omega) \mod E\\
\check{\mathbf{B}} &\equiv (E.\operatorname{Re}\Omega)^{-1}\operatorname{Re}\Omega - \sigma_0 \mod E\\
\check{\omega} &\equiv (E.\operatorname{Re}\Omega)^{-1}\operatorname{Im}\Omega \mod E.
\end{aligned}$$

The actual classes of $\check{\Omega}$ and $\check{\omega}$ are determined completely by the relations $(\operatorname{Re}\check{\Omega})^2 = (\operatorname{Im}\check{\Omega})^2 = \check{\omega}^2$ and $\check{\omega}.(\operatorname{Re}\check{\Omega}) = \check{\omega}.(\operatorname{Im}\check{\Omega}) = (\operatorname{Re}\check{\Omega}).(\operatorname{Im}\check{\Omega}) = 0$.

We can now identify the large complex structure limit of $\check{X}$. This limit is mirror to the large Kähler limit of X. In the latter limit, we keep the complex structure on X fixed but allow the Kähler form to go to infinity. More precisely, if $\{\mathbf{B}_l + it_l\omega\}$ is a sequence of complexified Kähler forms on X with $t_l > 0$, $t_l \to \infty$, then we say $\{\mathbf{B}_l + it_l\omega\}$ are approaching the large Kähler limit in the complexified Kähler moduli space of X.

We will now take, for our purposes,

Definition 1.1. For each l, let $\check{X}_l$ be the K3 surface given by the data $(\check{X}_l, \check{\mathbf{B}}_l + i\check{\omega}_l, \check{\Omega}_l)$ mirror to $(X, \mathbf{B}_l + it_l\omega, t_l\Omega)$. The sequence of surfaces $\{\check{X}_l\}$ is said to approach a large complex structure limit point.

We will take this as the starting point of our analysis, and will not prove here that this is equivalent to other reasonable definitions of large complex structure limits found in the literature (but see discussions in [10]).

The reader will note that we are cheating to some extent here, by only approaching the large Kähler limit along a ray. The more general approach might be to allow a more general sequence of Kähler forms. However, this is more difficult to deal with because the elliptic fibration which arises below will be varying. We will ignore this difficulty in this paper, as it obscures our main objectives.

Note that

$$\check{\Omega}_l = (t_l E.\operatorname{Re}\Omega)^{-1}(\sigma_0 + \mathbf{B}_l + it_l\omega) \mod E$$

and

$$\check{\omega}_l = (E.\operatorname{Re}\Omega)^{-1}\operatorname{Im}\Omega \mod E.$$

More precisely, if a representative $\mathbf{B}_l$ for $\mathbf{B}_l \bmod E$ is chosen in $E^\perp \otimes \mathbf{R}$ with the property that $\mathbf{B}_l \cdot \sigma_0 = 0$, then the requirement that $\check{\Omega}_l^2 = 0$ yields

$$\check{\Omega}_l = (t_l E.\operatorname{Re}\Omega)^{-1}\left(\sigma_0 + (\mathbf{B}_l + it_l\omega) + \left(\frac{t_l^2\omega^2 - \mathbf{B}_l^2}{2} + 1 - it_l\omega.(\sigma_0 + \mathbf{B}_l)\right)E\right).$$

Furthermore the requirement that $\check{\omega}_l.\check{\Omega}_l = 0$ yields

$$\check{\omega}_l = (E.\operatorname{Re}\Omega)^{-1}(\operatorname{Im}\Omega - (\operatorname{Im}\Omega.(\sigma_0 + \mathbf{B}_l))E).$$

The Kähler class $\check{\omega}_l$ is represented by a Ricci-flat metric $\check{g}_l$, and we would like to understand the behaviour of this metric as $t_l \to \infty$. It is convenient to perform a hyperkähler rotation, i.e., $\check{g}_l$ is also a Kähler metric on the K3 surface $\check{X}_{l,K}$ with

$$\begin{aligned}\check{\Omega}_{l,K} &= \operatorname{Im}\check{\Omega}_l + i\check{\omega}_l\\ \check{\omega}_{l,K} &= \operatorname{Re}\check{\Omega}_l.\end{aligned}$$

This equality holds on the level of forms. Explicitly, in cohomology,

$$\begin{aligned}\check{\Omega}_{l,K} &= (E.\operatorname{Re}\Omega)^{-1}(\omega + i\operatorname{Im}\Omega - ((\omega + i\operatorname{Im}\Omega).(\sigma_0 + \mathbf{B}_l))E)\\ \check{\omega}_{l,K} &= (t_l E.\operatorname{Re}\Omega)^{-1}(\sigma_0 + \mathbf{B}_l) \mod E.\end{aligned}$$

We will assume that, for all l, E represents the class of a fibre of an elliptic fibration $f_l : \check{X}_{l,K} \to \mathbf{P}^1$. This elliptic fibration coincides with a special Lagrangian T^2-fibration on $\check{X}_l$. For general choice of data, such elliptic fibrations with fibre class E automatically exist, since then $\mathrm{Pic}\check{X}_{l,K} = \mathbf{Z}E$ and $E^2 = 0$. For any choice of data, there always exists an elliptic fibration on $\check{X}_{l,K}$, but the class of the fibre might only be the image of E under reflections by -2 curves in $\mathrm{Pic}\check{X}_{l,K}$. (See [17], §1 for further details.)

Note that the area of the fibre of f_l under the metric $\check{g}_l$ is $(t_l E.\,\mathrm{Re}\,\Omega)^{-1}$, which goes to zero as $t_l \to \infty$.

Now $\check{\Omega}_{l,K}$ depends on l, but these classes only differ by the pull-back of a class from $\mathbf{P}^1$. This in fact tells us the elliptic K3 surfaces $\check{X}_{l,K}$ are closely related. Indeed, if $f : X \to \mathbf{P}^1$ is an elliptic K3 surface, with holomorphic 2-form Ω, then whenever α is a 2-form on $\mathbf{P}^1$, $\Omega' = \Omega + f^*\alpha$ satisfies $\Omega' \wedge \Omega' = 0$ as forms, and thus Ω' induces another complex structure on X such that f remains a holomorphic elliptic fibration in this new complex structure. All the surfaces $\check{X}_{l,K}$ are clearly related in this way. In particular, all these elliptic surfaces have the same jacobian $\check{J}_K$, which is the unique elliptic K3 surface with a holomorphic section with complex structure induced by $\check{\Omega}_{l,K} + f_l^*\alpha$ for some α.

This now leads us to the following question:

Question 1.2. Let $j : J \to \mathbf{P}^1$ be an elliptic K3 surface with a section, and let $f_l : X_l \to \mathbf{P}^1$ be a sequence of elliptic K3 surfaces with jacobian $j : J \to \mathbf{P}^1$. Let ω_l be a Ricci-flat Kähler metric on X_l with $Vol(X_l)$ independent of l. Let $\epsilon_l = Area_{\omega_l}(f_l^{-1}(y))$ for any point $y \in \mathbf{P}^1$, and suppose $\epsilon_l \to 0$ as $l \to \infty$. Describe the behaviour of the metric ω_l as $l \to \infty$.

We will solve this question in this paper in the case that the map j has 24 Kodaira type I_1 fibres. This is true for the generic K3 elliptic fibration.

We end this section with a few additional important comments about this setup.

First, it is often convenient to identify the underlying differentiable manifold of an elliptic K3 surface $f : X \to B$ with that of its jacobian. This can be done in a reasonably canonical fashion by choosing a C^∞ section $\sigma_0 : B \to X$ of f. If Ω_X is a holomorphic 2-form on X, then $\Omega_J = \Omega_X - f^*\sigma_0^*\Omega_X$ defines a new complex structure on X, in which $\sigma_0(B)$ is a holomorphic section. This new complex structure yields the jacobian.

Another important point is that once a C^∞ zero-section σ_0 for $f : X \to B$ is chosen, we obtain a group structure on the non-singular part of each fibre of f. Let $X^0 \subseteq X$ be obtained by taking the union of the identity components of each fibre. Then given a holomorphic 2-form Ω on X, we can construct a map from the holomorphic cotangent bundle $\mathcal{T}^*_B$ to X^0, taking the zero section of $\mathcal{T}^*_B$ to $\sigma_0(B)$, and with the property that the pull-back of Ω to $\mathcal{T}^*_B$ is a form $\Omega_{can} + \alpha$, where α is a 2-form pulled back from the base and Ω_{can} is the canonical holomorphic symplectic 2-form on $\mathcal{T}^*_B$. (See [14], §§2 and 7 for further details of this map.) The canonical holomorphic symplectic 2-form can be defined in local coordinates. If y is a local holomorphic coordinate on the base B, we can take x to be the corresponding canonical coordinate on the fibres of $\mathcal{T}^*_B$, so that the coordinate (x_0, y_0) represents the 1-form $x_0 dy$ at the point in B with coordinate y_0. The pair x, y are called *holomorphic canonical coordinates*. Then the canonical 2-form on $\mathcal{T}^*_B$ is $dx \wedge dy$ in these coordinates.

The map $\mathcal{T}^*_B \to X^0$ also gives an exact sequence

$$0 \to R^1 f_* \mathbf{Z} \to \mathcal{T}^*_{\mathbf{P}^1} \to X^0 \to 0.$$

$R^1 f_* \mathbf{Z}$ gives a degenerating family of lattices in the fibres of the complex line bundle $\mathcal{T}^*_B$. Thus working on the cotangent bundle of B gives useful coordinates for X away from the singular fibres, and these coordinates will be used repeatedly in later sections.

2. Equations for Ricci-flatness

In this section we will discuss equations for Ricci-flatness in different coordinate systems. We are interested in the behaviour of the metric on an elliptic K3 fibration, and this metric behaves in radically different ways away from the singular fibres as opposed to a neighbourhood of the singular fibres. In these two different cases, it will be useful to have two different coordinate systems to study the metrics.

For studying the metric away from the singular fibres, we adopt the set-up from the previous section, with $\pi : \mathcal{T}^*_B \to B$ where B is an open subset of $\mathbf{C}$. We are actually working on $X = \mathcal{T}^*_B/\Lambda$, where Λ is a holomorphically varying family of lattices in $\mathcal{T}^*_B$. We will assume in this section that the zero section is holomorphic, so that the holomorphic 2-form on X is induced by $\Omega = dx \wedge dy$ on $\mathcal{T}^*_B$, where $y = y_1 + iy_2$ and $x = x_1 + ix_2$ are holomorphic canonical coordinates on $\mathcal{T}^*_B$. The Kähler form in these coordinates takes the form

$$\begin{aligned}\omega =& \frac{i}{2} W(dx \wedge d\bar{x} + \bar{b}\, dx \wedge d\bar{y} + b\, dy \wedge d\bar{x} + (W^{-2} + |b|^2)\, dy \wedge d\bar{y}) \\ =& \frac{i}{2}(W(dx + b\, dy) \wedge \overline{(dx + b\, dy)} + W^{-1}\, dy \wedge d\bar{y}).\end{aligned}$$

Here W and b are defined by the above expression, and the coefficient of $dy \wedge d\bar{y}$ is chosen to ensure the normalisation $\omega^2 = (\operatorname{Im}\Omega)^2$. The function W is real-valued and the function b is complex-valued. The Kähler condition is now $d\omega = 0$. This equation can be written as

$$\begin{aligned}\partial_y W &= \partial_x(Wb) \\ \partial_y(W\bar{b}) &= \partial_x(W(W^{-2} + |b|^2)).\end{aligned}$$

Note that expanding the second equation out gives

$$W\partial_y \bar{b} + \bar{b}\partial_y W = -W^{-2}\partial_x W + (\partial_x W)|b|^2 + W(b\, \partial_x \bar{b} + \bar{b}\, \partial_x b).$$

Using the first equation to replace $\partial_y W$ and simplifying gives the above two equations being equivalent to

$$(\partial_y - b\partial_x)\bar{b} = -W^{-3}\partial_x W \tag{2.1}$$

$$(\partial_y - b\partial_x)W = W\partial_x b. \tag{2.2}$$

Define the vector fields

$$\begin{aligned}\partial_{\mathrm{v}} &= W^{-1}\partial_x \\ \partial_{\mathrm{h}} &= \partial_y - b\partial_x\end{aligned}$$

and denote by $\bar{\partial}_{\mathrm{v}}$ and $\bar{\partial}_{\mathrm{h}}$ the complex conjugate vector fields. The subscripts v and h denote the vertical and horizontal vector fields respectively. Let ϑ_{v} and ϑ_{h} denote the dual frame of one-forms, i.e.,

$$\begin{aligned}\vartheta_{\mathrm{v}} &= W(dx + bdy) \\ \vartheta_{\mathrm{h}} &= dy.\end{aligned}$$

Then

$$\omega = \frac{i}{2} W^{-1}(\vartheta_{\mathrm{v}} \wedge \bar{\vartheta}_{\mathrm{v}} + \vartheta_{\mathrm{h}} \wedge \bar{\vartheta}_{\mathrm{h}}).$$

In addition, Equations (2.1) and (2.2) take the simpler form

$$\partial_{\mathrm{h}}\bar{b} = \partial_{\mathrm{v}} W^{-1} \tag{2.1$'$}$$

$$-\partial_{\mathrm{h}} W^{-1} = \partial_{\mathrm{v}} b. \tag{2.2$'$}$$

Remark 2.1. While we don't use this here, one can calculate that the holomorphic curvature $\Theta = (\Theta_{ij})_{1\le i,j\le 2}$ of this metric is given by

$$\begin{aligned}\Theta_{11} = -\Theta_{22} &= \partial W \wedge \bar\partial W^{-1} + W\partial\bar\partial W^{-1} + W^2 \partial\bar b \wedge \bar\partial b\\ \Theta_{21} = -\bar\Theta_{12} &= -W^{-1}\partial(W^2\bar\partial b).\end{aligned}$$

Example 2.2. The standard semi-flat metric. We call a metric *semi-flat* if it restricts to a flat metric on each elliptic fibre. As above, let $B \subseteq \mathbf{C}$ an open subset, y the coordinate on $\mathbf{C}$. Let $\tau_1(y), \tau_2(y)$ be two holomorphic functions on B such that $\tau_1(y)dy, \tau_2(y)dy$ generate a lattice $\Lambda(y) \subseteq \mathcal{T}^*_{B,y}$ for each $y \in B$, giving us the holomorphically varying family of lattices $\Lambda \subseteq \mathcal{T}^*_B = B \times \mathbf{C}$. Typically, we may allow τ_1 and τ_2 to be multi-valued. Assuming without loss of generality that $\mathrm{Im}(\bar\tau_1\tau_2) > 0$, then a Ricci-flat metric on $X = (B \times \mathbf{C})/\Lambda$ is given by the data

$$\begin{aligned}W &= \frac{\epsilon}{\mathrm{Im}(\bar\tau_1\tau_2)}\\ b &= -\frac{W}{\epsilon}[\mathrm{Im}(\tau_2\bar x)\partial_y\tau_1 + \mathrm{Im}(\bar\tau_1 x)\partial_y\tau_2].\end{aligned}$$

It is easy to check that these satisfy the equations (2.1) and (2.2). This metric, a priori defined on $\mathcal{T}^*_B$, descends to a metric on X, and the area of a fibre of $f: X \to B$ is ϵ. We call this metric on X *the standard semi-flat metric*, with Kähler form ω_{SF}.

The reader may check explicitly that this metric is independent of the particular choice of generators for Λ, so that multi-valuedness of τ_1 and τ_2 do not cause a problem. Furthermore, the metric is independent of the choice of the coordinate y (keeping in mind that a change of the coordinate y necessitates a change of the canonical coordinate x, and hence the functions τ_1, τ_2). This may also be seen as follows: The inclusion $R^1 f_*\mathbf{Z} \cong \Lambda \subseteq \mathcal{T}^*_B$ allows one to identify $(R^1 f_*\mathbf{R}) \otimes C^\infty(B)$ with the underlying C^∞ vector bundle $\mathcal{T}^*_B$, along with the Gauss-Manin connection ∇_{GM} on $\mathcal{T}^*_B$, the flat connection whose flat sections are sections of $R^1 f_*\mathbf{R}$. The standard semi-flat metric is the unique semi-flat Ricci-flat Kähler metric satisfying the conditions:

(1) The area of each fibre is ϵ;

(2) $\omega_{SF}^2 = (\mathrm{Re}\,\Omega)^2 = (\mathrm{Im}\,\Omega)^2$;

(3) The orthogonal complement of each vertical tangent space is the horizontal tangent space of ∇_{GM} at that point.

This metric was described in [12], and in the more general context of special Lagrangian fibrations in [19], as well as [14], Example 6.4.

The reader should be aware however that if $T_\sigma : X \to X$ denotes translation by a holomorphic section σ, then $T_\sigma^* \omega_{SF}$ may give rise to a different semi-flat metric, satisfying conditions (1) and (2) but not (3). However, if σ is not only holomorphic but a flat section with respect to the Gauss-Manin connection (so that $\sigma(y) = a_1\tau_1(y) + a_2\tau_2(y)$ for constants a_1, a_2) then T_σ is an isometry and $T_\sigma^*\omega_{SF} = \omega_{SF}$, $T_\sigma^*\Omega = \Omega$.

It will also be useful to have the Kähler potential for the metric. This is a function φ such that $\omega = \frac{i}{2}\partial\bar\partial\varphi$. Let ϕ_1 and ϕ_2 be anti-derivatives of τ_1 and τ_2 respectively. Then we can take

$$\varphi = \frac{\epsilon}{\mathrm{Im}(\bar\tau_1\tau_2)}\left(-\frac{\bar x^2}{2}\frac{\tau_1}{\bar\tau_1} + |x|^2 - \frac{x^2}{2}\frac{\bar\tau_1}{\tau_1}\right) + \frac{i}{2\epsilon}(\phi_1\bar\phi_2 - \bar\phi_1\phi_2).$$

This is well-defined on subsets $\mathcal{T}_U^* \subseteq \mathcal{T}_B^*$ for U simply connected, but not on $\mathcal{T}_B^*/\Lambda$.

Construction 2.3. The Gibbons–Hawking Ansatz. We now describe the system of coordinates which is most suited to studying the hyperkähler metric in a neighbourhood of a singular fibre of the elliptic fibration. This system of coordinates goes under the name of the Gibbons–Hawking Ansatz, and the description in terms of a connection form on an S^1-bundle explained below is essentially the same as that given in [2], which in turn is based on earlier work of Gibbons and Hawking, Hitchin, and others.

Let $U \subseteq \mathbf{R}^3$ be an open set with the Euclidean metric, with coordinates u_1, u_2, u_3. Let $\pi : X \to U$ be a principal S^1 bundle, with S^1 action $S^1 \times X \to X$ written as $(e^{it}, x) \mapsto e^{it}\cdot x$. Let θ be a connection 1-form on X, i.e., a $\mathfrak{u}(1) = i\mathbf{R}$-valued 1-form invariant under the S^1-action and such that $\theta(\partial/\partial t) = i$. The curvature of the connection θ is $d\theta = \pi^*\alpha$ for a 2-form α on U, and $i\alpha/2\pi$ represents the first Chern class of the bundle (see [8], Appendix). Suppose V is a positive real function on U

satisfying $*dV = \alpha/2\pi i$. Let

$$\begin{aligned}\omega_1 &= du_1 \wedge \theta/2\pi i + V du_2 \wedge du_3\\ \omega_2 &= du_2 \wedge \theta/2\pi i + V du_3 \wedge du_1\\ \omega_3 &= du_3 \wedge \theta/2\pi i + V du_1 \wedge du_2.\end{aligned}$$

Then $\omega_1^2 = \omega_2^2 = \omega_3^2$ is nowhere zero, and $\omega_i \wedge \omega_j = 0$, for $i \neq j$. Furthermore, $*dV = \alpha/2\pi i$ implies $d\omega_i = 0$ for all i, since for instance

$$d\omega_1 = -du_1 \wedge d\theta/2\pi i + dV \wedge du_2 \wedge du_3 = -du_1 \wedge *dV + dV \wedge du_2 \wedge du_3 = 0.$$

Therefore $\omega_1, \omega_2, \omega_3$ define a hyperkähler metric on X. Note V is harmonic, since $d\alpha = 0$ implies that $*d * dV = 0$.

Let θ_0 denote the real 1-form $\theta/2\pi i$, and observe that

$$-\omega_1 - i\omega_2 = (\theta_0 - iV du_3) \wedge (du_1 + idu_2).$$

By taking this to be the (holomorphic) 2-form Ω on X, this determines an integrable almost complex structure on X, where $du_1 + idu_2$ and $\theta_0 - iV du_3$ span the holomorphic cotangent space inside the complexified cotangent space. It follows that the (integrable) almost complex structure J on the cotangent space is given by

$$J(du_1) = -du_2, \quad J(du_3) = -V^{-1}\theta_0.$$

Thus, if we consider the Kähler form $\omega = \omega_3$ as an alternating tensor, and use the relation that if g is the Riemannian metric, then $g(\zeta, \xi) = \omega(\zeta, J\xi)$, we obtain an expression for the metric

$$ds^2 = V d\mathbf{u} \cdot d\mathbf{u} + V^{-1}\theta_0^2.$$

Usually, we shall in fact start from a positive harmonic function V on U such that $- * dV$ represents the Chern class of the bundle. Then we can always find a connection 1-form θ with $d\theta/2\pi i = *dV$, such a θ being uniquely determined up to pull-backs of closed 1-forms from U, and hence we obtain hyperkähler metrics as above.

Remark 2.4. We will need to calculate some information about the curvature of this metric. We can work locally, and therefore take the orthonormal moving coframe given by $V^{1/2}du_1, V^{1/2}du_2, V^{1/2}du_3$ and $V^{-1/2}\theta_0$. We can moreover write the connection form locally as

$$\theta_0 = \frac{dt}{2\pi} + A_1 du_1 + A_2 du_2 + A_3 du_3,$$

where $\nabla V = \nabla \times \mathbf{A}$. To calculate the curvature, we may then apply Cartan's method. We obtain

$$\|R\|^2 = 12V^{-6}|\nabla V|^4 + V^{-4}\Delta(|\nabla V|^2) - 6V^{-5}(\nabla V)\cdot(\nabla(|\nabla V|^2)).$$

Using the fact that V is harmonic, we then recover the compact formula given in Equation (32) of [28] that

$$\|R\|^2 = \frac{1}{2}V^{-1}\Delta\Delta(V^{-1}).$$

Example 2.5. If we consider the natural map $\mathbf{C}^2 \setminus (0,0) \to \mathbf{P}^1(\mathbf{C}) = S^2$, and restrict to $S^3 \subset \mathbf{C}^2$, we easily check that the image of $(z_1, z_2) \in S^3$ is

$$(2\,\mathrm{Re}(z_1\bar{z}_2), 2\,\mathrm{Im}(z_1\bar{z}_2), |z_1|^2 - |z_2|^2).$$

This is the standard Poincaré map. The formula also defines a map $X = \mathbf{C}^2 \setminus (0,0) \to \mathbf{R}^3 \setminus (0,0,0)$; we compose this map with complex conjugation on z_2 to obtain a map $p : X = \mathbf{C}^2 \setminus (0,0) \to \mathbf{R}^3 \setminus (0,0,0)$, given by

$$p(z_1, z_2) = (2\,\mathrm{Re}(z_1 z_2), 2\,\mathrm{Im}(z_1 z_2), |z_1|^2 - |z_2|^2).$$

This map exhibits X as an S^1-bundle over $\mathbf{R}^3 \setminus (0,0,0)$, with Chern class ± 1. The action of S^1 on X is given by $e^{it}\cdot(z_1, z_2) = (e^{it}z_1, e^{-it}z_2)$. Note also that if we compose p with projection onto the first two factors, we obtain the map sending (z_1, z_2) to $2z_1z_2$, holomorphic with respect to the standard complex structures.

We now choose a positive harmonic function V on $\mathbf{R}^3 \setminus (0,0,0)$ such that

$$-\int_{S^2} *dV = \int_{S^2} i\alpha/2\pi = \pm 1,$$

i.e., the Chern number is correct. The particular examples of such V we consider are

$$V = e + \frac{1}{4\pi|\mathbf{u}|} = e + \frac{1}{4\pi\sqrt{u_1^2 + u_2^2 + u_3^2}},$$

where $e \geq 0$. The integral

$$\int_{S^2} *d\left(\frac{1}{4\pi\sqrt{u_1^2 + u_2^2 + u_3^2}}\right)$$

is easily seen to be ± 1 (depending on the orientation of the sphere).

Now we take as connection form

$$\theta = i\mathrm{Im}(\bar{z}_1 dz_1 - \bar{z}_2 dz_2)/(|z_1|^2 + |z_2|^2).$$

Then

$$d\theta/2\pi i = \frac{-(u_1 du_2 \wedge du_3 + u_2 du_3 \wedge du_1 + u_3 du_1 \wedge du_2)}{4\pi(u_1^2 + u_2^2 + u_3^2)^{3/2}} = *dV$$

as required. We therefore obtain hyperkähler metrics on X, which, for all $e \geq 0$, extend to metrics on $\mathbf{C}^2$. In fact, such metrics are ALF (asymptotically locally flat), approaching a flat metric when $|\mathbf{u}| \to \infty$, whilst being periodic in t. When $e = 1$, the metric obtained is the Taub-NUT metric, and when $e = 0$, it is just a flat metric on $\mathbf{C}^2$. To prove the assertions for $e = 0$, straightforward calculations show that, with $z_j = x_j + iy_j$,

$$\begin{aligned}
\omega_1 &= \frac{1}{\pi}(dx_2 \wedge dy_1 - dx_1 \wedge dy_2) \\
\omega_2 &= \frac{1}{\pi}(dx_1 \wedge dx_2 - dy_1 \wedge dy_2) \\
\omega_3 &= \frac{1}{\pi}(dx_1 \wedge dy_1 + dx_2 \wedge dy_2).
\end{aligned}$$

So $\omega_1, \omega_2, \omega_3$ extend to $\mathbf{C}^2$, and yield a flat metric, as claimed.

Construction 2.6. Gibbons–Hawking versus holomorphic coordinates. In the Gibbons–Hawking Ansatz, we consider the case when $U = B \times \mathbf{R}$, with B a contractible open subset of $\mathbf{R}^2$ — in particular, the S^1-bundle X over U is topologically trivial. Set $y_1 = u_1$, $y_2 = u_2$, so then $y = y_1 + iy_2$ is a complex coordinate on B. We will see below how the hyperkähler structure on X gives rise to a complex structure on X under which the function y is holomorphic, i.e., the map $X \to B$ is holomorphic. Moreover, if we pass to the universal cover $\tilde{X}$ on X, we can construct a holomorphic coordinate x (depending on a choice of holomorphic section of $\tilde{X}$ over B) such that the holomorphic 2-form is just $dx \wedge dy$. This in turn enables us to identify $\tilde{X}$ with $\mathcal{T}_B^*$ over B, with x, y then being holomorphic canonical coordinates on $\mathcal{T}_B^*$, where the identification depends on our choice of holomorphic section. The S^1-action on X yields an $\mathbf{R}$-action on $\mathcal{T}_B^*$, which we shall see is just translation on $x_1 = \mathrm{Re}\, x$, and so X is isomorphic to $\mathcal{T}_B^*/\mathbf{Z}$. The Kähler form provided by the Gibbons–Hawking Ansatz yields a Kähler form ω on $\mathcal{T}_B^*$, corresponding of course to a Ricci-flat metric, and for which the functions W and b are independent of x_1. The Kähler form therefore descends to $\mathcal{T}_B^*/\mathbf{Z}$, and is invariant under the obvious S^1-action.

Conversely, we shall see that any Ricci flat, S^1-invariant Kähler structure on $\mathcal{T}_B^*/\mathbf{Z}$ of the above type (i.e., we have x, y holomorphic canonical coordinates on $\mathcal{T}_B^*$ over B, for which W and b are independent of x_1) does in fact arise from the Gibbons–Hawking Ansatz in the way that has just been described. Moreover, Gibbons–Hawking coordinates u_1, u_2, u_3 and the connection form θ on X may be recovered from the holomorphic canonical coordinates x, y on $\mathcal{T}_B^*$. Here we have $u_1 = y_1$, $u_2 = y_2$, and u_3 determined up to a constant.

We now give the details for the construction. We have $U = B \times \mathbf{R}$, with B a contractible open subset of $\mathbf{R}^2$, and we set $y = y_1 + iy_2$, a complex coordinate on B. Then $dy_1 + idy_2 = dy$, and from the Gibbons–Hawking Ansatz equations we observe that $dy \wedge d(\theta_0 - iV du_3) = 0$. By the theorem on integrability of almost complex structures, $\Omega = (\theta_0 - iV du_3) \wedge dy$ is a holomorphic 2-form for an integrable almost complex structure J on X, and locally there exists a holomorphic coordinate z such that $dz = (\theta_0 - iV du_3) \mod dy$. Moreover it is then clear that z is determined up to a holomorphic function of y, and that locally the holomorphic coordinates recover the (integrable) complex structure. We now pass to the universal cover $\tilde{X}$ of X, topologically $B \times \mathbf{R}^2$, together with its integrable complex structure $\tilde{J}$ obtained from J (from now on, we shall work on $\tilde{X}$, but omit tildes from forms and functions pulled back from X). We note that the complex structure is invariant under the $\mathbf{R}$-action on $\tilde{X}$ induced from the given S^1-action on X. The (global) form $\theta_0 - iV du_3$ restricts down to a holomorphic 1-form on each fibre, locally just dz. Therefore, by integrating $\theta_0 - iV du_3$ along paths in the fibre from some fixed point, we obtain a holomorphic coordinate on the fibre, which locally (up to a constant depending on the choice of base point) will coincide with z.

In order to get a global holomorphic coordinate x on $\tilde{X}$, we choose a holomorphic section of $\tilde{X}$ over B (such sections always exist), which will then be regarded as giving the required base point in each fibre for the path integration. In this way, we obtain a global holomorphic function x on $\tilde{X}$ such that x, y are holomorphic coordinates everywhere, and where x is uniquely determined up to a holomorphic function of y (corresponding to the choice of holomorphic section). By construction, the global holomorphic coordinates x, y on $\tilde{X}$ realize the almost complex structure, with y a holomorphic coordinate on the base and x a holomorphic coordinate on the fibres. Moreover $\Omega = -\omega_1 - i\omega_2 = dx \wedge dy$, and so we can identify $\tilde{X} \to B$ with $\mathcal{T}_B^* \to B$ (with holomorphic canonical coordinates, as described in §1), where the chosen holomorphic section of $\tilde{X}$ over B is identified the zero section of the holomorphic cotangent bundle. Choosing a section of $\tilde{X}$ over $U = B \times \mathbf{R}$ enables us to consider the coordinate t on S^1 as a coordinate on the fibres; the above derivation of the holomorphic coordinate x then shows that its real part $x_1 = \frac{t}{2\pi} + g(y_1, y_2, u_3)$, for some function g, and that the action of $\mathbf{R}$ on $\tilde{X}$ is the obvious one given by translation on x_1. Explicitly X is obtained as a quotient of $\tilde{X}$ under the action of $\mathbf{Z}$ given by $x_1 \mapsto x_1 + 1$.

Since $dx = \theta_0 - iV du_3 \mod dy$, there exists a complex-valued function b on $\tilde{X}$ such that $dx + bdy = \theta_0 - iV du_3$. Also

$$(dx + bdy) \wedge \overline{(dx + bdy)} = 2iV\theta_0 \wedge du_3.$$

We now set $W = V^{-1}$ and calculate the Kähler form ω_3 in terms of the holomorphic coordinates:

$$\omega_3 = du_3 \wedge \theta_0 + V du_1 \wedge du_2 = \frac{i}{2}(W(dx+bdy) \wedge \overline{(dx + bdy)} + W^{-1} dy \wedge d\bar{y}),$$

which we observe has the same form as our original general formula for ω in holomorphic canonical coordinates. Since we started with a Ricci-flat metric, the previous equations for Ricci-flatness (2.1) and (2.2) which we derived are then automatically satisfied.

The next point is to observe that W and b are independent of x_1, the real part of x. To see this, recall now that

$$\vartheta_{\mathrm{v}} = W(dx + bdy) = (V^{-1}\theta_0 - i du_3)$$

and $\vartheta_{\mathrm{h}} = dy$ is a globally defined coframe for the holomorphic cotangent bundle of $\tilde{X}$. In particular, since the imaginary part of ϑ_{v} is $-du_3$, we will have that the imaginary part of $d\vartheta_{\mathrm{v}}$ is zero. We first calculate

$$\partial\vartheta_{\mathrm{v}} = W^{-1}\partial W \wedge \vartheta_{\mathrm{v}} + W\partial b \wedge \vartheta_{\mathrm{h}} = (W^{-1}\partial_{\mathrm{h}} W - W\partial_{\mathrm{v}} b)\, \vartheta_{\mathrm{h}} \wedge \vartheta_{\mathrm{v}}. \tag{2.3}$$

From this it is seen that Equation (2.2′) is just the statement that $\partial\vartheta_{\mathrm{v}} = 0$. We next calculate

$$\begin{aligned} \bar\partial\vartheta_{\mathrm{v}} &= W^{-1}\bar\partial W \wedge \vartheta_{\mathrm{v}} + W\bar\partial b \wedge \vartheta_{\mathrm{h}} \\ &= W^{-1}\bar\partial_{\mathrm{v}} W\ \bar\vartheta_{\mathrm{v}} \wedge \vartheta_{\mathrm{v}} + W^{-1}\bar\partial_{\mathrm{h}} W\ \bar\vartheta_{\mathrm{h}} \wedge \vartheta_{\mathrm{v}} \\ &\quad + W\bar\partial_{\mathrm{v}} b\ \bar\vartheta_{\mathrm{v}} \wedge \vartheta_{\mathrm{h}} + W\bar\partial_{\mathrm{h}} b\ \bar\vartheta_{\mathrm{h}} \wedge \vartheta_{\mathrm{h}}. \end{aligned} \tag{2.4}$$

If then Equation (2.1′) also holds, it is easily checked that the imaginary part of $\bar{\partial}\vartheta_{\mathrm{v}}$ is zero if and only if $\partial_{\mathrm{v}}W = -\bar{\partial}_{\mathrm{v}}W$ and $\partial_{\mathrm{v}}b = -\bar{\partial}_{\mathrm{v}}b$, that is W and b are independent of x_1, the real part of x. Thus b is invariant under the $\mathbf{R}$-action, that is b is the pull-back of a function from U.

Conversely, if we start from a Ricci-flat, S^1-invariant Kähler metric on $\mathcal{T}_B^*/\mathbf{Z}$ of the above type (i.e., we have x, y holomorphic canonical coordinates on $\mathcal{T}_B^*$ over B, for which W and b are independent of x_1), we can pass to the universal cover $\tilde{X} = \mathcal{T}_B^*$ over B. The above construction then reverses. We set ϕ to be the imaginary part of $\vartheta_{\mathrm{v}} = W(dx + bdy)$. Clearly ϕ is invariant under the given $\mathbf{R}$-action on $\tilde{X}$. Reversing the derivation of the previous paragraph ensures that $d\phi = 0$ on $\tilde{X}$, and so there is a global function u_3 with $\phi = -du_3$, where u_3 is invariant under the action of $\mathbf{R}$, and is determined up to a constant. We set $V = W^{-1}$, $\theta_0 = (dx + bdy) - iV\phi$ and $\theta = 2\pi i\theta_0$; thus both V and θ are also invariant under the action of $\mathbf{R}$. It is straightforward now to verify that we get back the above form of the Gibbons–Hawking Ansatz, with $u_1 = y_1$ and $u_2 = y_2$, and where $U = B \times \mathbf{R}$ is the quotient of $\tilde{X}$ by the $\mathbf{R}$-action. The periodicity of this $\mathbf{R}$-action then yields an S^1-bundle X over U (to which V and θ descend, and on which the corresponding S^1-action leaves V and θ invariant).

Finally, we calculate (for use in §4) the differential p_*, where $p : \tilde{X} \to U = B \times \mathbf{R}$ is the natural projection. Using the expression $du_3 = \frac{i}{2}W(dx - d\bar{x}) + \frac{i}{2}W(bdy - \bar{b}d\bar{y})$, we obtain

$$\begin{aligned} p_*\partial_x &= \frac{iW}{2}\partial_{u_3} \\ p_*\partial_{\bar{x}} &= \frac{-iW}{2}\partial_{u_3} \\ p_*\partial_y &= \partial_y + \frac{iW}{2}b\partial_{u_3} \\ p_*\partial_{\bar{y}} &= \partial_{\bar{y}} - \frac{iW}{2}\bar{b}\partial_{u_3}. \end{aligned} \tag{2.5}$$

Thus $p_*\partial_{\mathrm{h}} = \partial_y$, $p_*\partial_{\mathrm{v}} = \frac{i}{2}\partial_{u_3}$.

Also, as W and b can be thought of as functions on $B \times \mathbf{R}$, being independent of x_1, the formula (2.2′) translates into

$$-\partial_y V = \frac{i}{2}\partial_{u_3} b. \tag{2.6}$$

Thus b can be calculated as

$$b(y, u_3) = \sigma(y) + \int 2i\partial_y V du_3 \tag{2.7}$$

where $\sigma(y)$ is some constant of integration.

S^1-invariant Ricci flat metrics on elliptic fibrations

We shall be most interested in the transformation described above when V and θ are themselves periodic in $u = u_3$. The hyperkähler metric descends to one on the corresponding S^1-fibration over $Y = B \times S^1$ if and only if the three 2-forms $\omega_1, \omega_2, \omega_3$ are invariant under changing u by a period, which in turn is saying that the periodicity in u is independent of y. We shall now change notation and denote this $S^1 \times S^1$ fibration over B by X (the universal cover $\tilde{X}$ being the same as before). Since the restriction of the Kähler form ω_3 to a fibre X_y is just $du \wedge \theta_0 = du \wedge dt/2\pi$, the volume of any fibre is just the periodicity in u. Changing coordinates to the holomorphic coordinates of Construction 2.6, we obtain a holomorphic map $f : X \to B$ to a contractible open subset B of $\mathbf{C}$, whose fibres are elliptic curves. Having chosen a holomorphic section, we obtain holomorphic canonical coordinates x, y on the corresponding line bundle $\tilde{X}$ over B, where the holomorphic 2-form $\Omega = dx \wedge dy$, and where the Kähler form ω (as defined by the usual formula) determines a hyperkähler metric on X. Moreover, both W and b are independent of x_1.

The periods of the above elliptic fibration have a basis $\{1, \tau(y)\}$, for some holomorphic function τ of y. If we wish to have an explicit formula for $\tau(y)$, we take a basis of homology $\{\gamma_1, \gamma_2\}$, where γ_1 is an S^1 in a fibre X_y of $X \to B$ given by the orbit of the S^1-action, and γ_2 is an S^1 in X_y mapping isomorphically to $\{y\} \times S^1 \subset Y$. Restricted to the fibre X_y, we have $dx = \theta_0 - iV\,du_3$; one of the periods is then

$$\int_{\gamma_1} dx = \int_{\gamma_1} \theta_0 = 1,$$

as already observed, whilst the other period

$$\tau(y) = \int_{\gamma_2} dx = \int_{\gamma_2} \theta_0 - i \int_{\gamma_2} V\,du_3.$$

By choosing the appropriate orientation for γ_2, we may also assume that $\operatorname{Im}\tau(y) > 0$.

If we have such a holomorphic elliptic fibration $f : X \to B$ and Ricci-flat metric (independent of x_1), we shall refer to it as an *S^1-invariant Ricci-flat metric* (*on X*) *in canonical form.*

Conversely, if we are given such an S^1-invariant Ricci-flat metric on X, we saw above how this does indeed arise from the Gibbons–Hawking Ansatz. Moreover, in this case, we also have that V and θ are periodic in u, with the period in u being constant, namely the volume of the elliptic fibres of $f : X \to B$.

Remark 2.7. A particular case of an S^1-invariant Ricci-flat metric in canonical form is a semi-flat metric: Given, locally, two periods τ_1 and τ_2, these should be interpreted as 1-forms on B, i.e., are $\tau_1 dy$, $\tau_2 dy$. We can then locally replace y with a holomorphic function g on an open set U such that $dg = \tau_1 dy$, and thus can assume $\tau_1 = 1$. Then in these coordinates, the semi-flat metric coincides with the Gibbons–Hawking metric obtained by taking $V = \operatorname{Im}\tau_2/\epsilon$ on $U \times \mathbf{R}/\epsilon\mathbf{Z}$. We can then use the formula of Remark 2.4 to compute $\|R\|^2$ for a semi-flat metric (which will coincide with the value calculated via Remark 2.1). Thus

$$\|R\|^2 = \frac{1}{2}V^{-1}\Delta\Delta V^{-1} = \frac{\epsilon^2}{2}(\operatorname{Im}\tau_2)^{-1}\Delta\Delta(\operatorname{Im}\tau_2)^{-1}.$$

In particular, $\|R\|^2 \to 0$ as $\epsilon \to 0$.

Returning now to the set-up in Question 1.2; away from the singular fibres, we expect that, as the volume ϵ of the fibres tends to zero, the metric (suitably normalized) will approach a semi-flat one. This expectation is motivated by the following result, which proves a slightly weaker version of the expected convergence for the S^1-invariant Ricci-flat case, purely by local considerations, as a consequence of Harnack's inequality for harmonic functions. Whilst we don't expect a purely local proof of convergence in general (i.e., not assuming the S^1-invariance of the metrics), the main result of this paper (Theorem 5.6) will prove a very strong form of the expected convergence to a semi-flat metric (locally over the base) by means of global methods.

Proposition 2.8. *Let $\pi : X \to B$ be an elliptic fibration with periods $\{1, \tau(y)\}$, over the open disc B of radius R in $\mathbf{C}$, with $\operatorname{Im}\tau(y) > 0$, and let $B_0 \subset B$ denote a smaller disc of radius $R_0 < R$. Suppose we have a sequence on X of S^1-invariant Ricci-flat metrics g_i in canonical form (and with constant volume form), for which the volume $\epsilon_i := \epsilon(g_i)$ of the fibres tends to zero as $i \to \infty$. Then on $\pi^{-1}(B_0)$ we have $W_i := W(g_i) \to 0$ uniformly as $i \to \infty$. On a fixed fibre, with periods $\{1, \tau\}$, we have the stronger statement that $\epsilon_i^{-1} W_i \operatorname{Im}\tau \to 1$ uniformly as $i \to \infty$.*

Proof. Our assumption that the volume form is constant ensures that we can fix the holomorphic canonical coordinates x, y and the holomorphic 2-form $\Omega = dx \wedge dy$ independent of i. We now transform the coordinates to Gibbons–Hawking coordinates; the first claim is equivalent to $W_i \to 0$ uniformly on $B_0 \times \mathbf{R}$. If the volume of the fibres is ϵ_i, then the periodicity in u is ϵ_i. Fix R_1 with $R_0 < R_1 < R$; then for $\epsilon_i << 1$, the ball $\tilde{B}_1$ in $U = B \times \mathbf{R}$ with centre the origin and radius R_1 will contain the set $B_0 \times [0, \epsilon_i]$, so it will suffice to show that $W_i \to 0$ uniformly on $\tilde{B}_1$ as $i \to \infty$. Fix i for the moment so that ϵ_i is sufficiently small as above, and drop the subscript for convenience. Let $\tilde{B}$ denote the ball of radius R, with centre the origin. Recall now that $W = V^{-1}$. Given that V is harmonic on $\tilde{B}$, this is precisely the situation in which we can apply the strong form of Harnack's inequality, as stated in Problem 2.6 on page 29 of [11], namely that for any point $P \in \tilde{B}_1$,

$$\frac{(1 - R_1/R)}{(1 + R_1/R)^2} \leq \frac{V(P)}{V(0)} \leq \frac{(1 + R_1/R)}{(1 - R_1/R)^2}.$$

Thus, for $P \in \pi^{-1}(B_0)$, the ratio $W(P)/W(0)$ is bounded above and below by appropriate positive constants. For each $y \in B_0$, we can calculate the volume of the fibre X_y as

$$\epsilon = \int_{X_y} W dx_1 \wedge dx_2 = \int_0^{\operatorname{Im}\tau(y)} W dx_2.$$

For R_0 fixed and for $y \in B_0$, we also have that $\operatorname{Im}\tau(y)$ is bounded above and below by appropriate positive constants. On any fibre X_y with $y \in B_0$, we can find a point at which W takes the average value on the fibre, namely $\epsilon/\operatorname{Im}\tau(y)$. Putting all these facts together yields the claim that $W_i \to 0$ uniformly on $\tilde{B}_1$ as $i \to \infty$.

For the stronger statement on a fixed fibre, we can assume that the fibre is X_0, and that W takes the average value $\epsilon/\operatorname{Im}\tau$ at the centre 0 of the ball $\tilde{B}$. If we take a concentric ball $\tilde{B}(r)$ of small radius r, it will still contain all of $\{0\} \times [0, \epsilon]$, provided $\epsilon < r$. Harnack's inequality then yields

$$\frac{(1 - r/R)}{(1 + r/R)^2} \leq \frac{W(0)}{W(P)} \leq \frac{(1 + r/R)}{(1 - r/R)^2}$$

for all $P \in X_0$. By taking r arbitrarily small, these upper and lower bounds are arbitrarily close to 1, and hence $\epsilon_i^{-1} W_i \operatorname{Im}\tau \to 1$ uniformly on X_0. q.e.d.

3. The Ooguri–Vafa metric

The aim of this section is to describe a certain hyperkähler metric on a neighbourhood of each singular fibre in our elliptically fibred K3 surface, and to derive various estimates associated with this metric. If the fibres are assumed to have volume ϵ, then away from the singular fibre, this metric decays very rapidly, for small ϵ, to a semi-flat metric. We shall assume throughout that we only have singular fibres of Kodaira type I_1, and so locally around the singular fibre, one of the periods is invariant under monodromy (and in fact, by an appropriate choice of holomorphic coordinate y on the base, may be taken to be constant, value 1), whilst the other period will be multivalued and tend to infinity. The metric we define will be an S^1-invariant metric (as described in the previous section) on the smooth part of the fibration, and will be most conveniently described in the Gibbons–Hawking coordinates.

The metric we describe was first written down (in a slightly different form) by Ooguri and Vafa [29], and so will be referred to as the *Ooguri–Vafa* metric. In §4, we shall start with an Ooguri–Vafa metric in a neighbourhood of each singular fibre; by appropriately twisting these metrics, we'll show that they may be glued with a semi–flat metric away from the singular fibres, hence obtaining a global metric, which is Ricci-flat away from the gluing regions, and which represents the correct Kähler class. For small ϵ, it is these metrics which approximate very accurately the global Ricci flat metric with the given Kähler class.

Before launching into the technical details, we shall briefly describe the basic idea behind the construction of the Ooguri–Vafa metric, which, given the description of the Gibbons–Hawking Ansatz in §2, should strike the reader as very natural. The harmonic function V we use will be periodic in u of period ϵ (the volume of the fibres), but have Taub–NUT type singularities on the fibre $y = 0$ at the points $u \in \epsilon\mathbf{Z}$.

We take $U = D \times \mathbf{R} \setminus \{0\} \times \epsilon\mathbf{Z}$, or more precisely its quotient by $\epsilon\mathbf{Z}$, where $D \subset \mathbf{C}$ is an open disc centred at the origin. We denote by y_1, y_2 the coordinates on $D \subset \mathbf{C}$, and by u the coordinate on $\mathbf{R}$. We want to write down V harmonic on U, periodic in u and with singularities of the correct type at the points $\{0\} \times \epsilon\mathbf{Z}$. For instance, around zero, V should behave like a harmonic function plus a term $\frac{1}{4\pi|\mathbf{x}|}$, from which it will follow that the total space X of the S^1-fibration over U extends (by adding a single point) to a manifold $\bar{X}$ mapping onto $\bar{U} = D \times \mathbf{R}/\epsilon\mathbf{Z}$. In addition, the hyperkähler metric extends to $\bar{X}$. We are led therefore to take $V = V_0 + f(y_1, y_2)$, where f is a harmonic function in y_1, y_2 on

D, and

$$V_0 = \frac{1}{4\pi}\sum_{-\infty}^{\infty}\left(\frac{1}{\sqrt{(u+n\epsilon)^2+y_1^2+y_2^2}} - a_{|n|}\right),$$

where $a_n = \frac{1}{n\epsilon}$ $(n > 0)$, thus ensuring appropriate convergence, and a_0 is chosen appropriately to ensure that the periods do not change as we change ϵ — that is, we are defining metrics on a fixed elliptic fibration. This choice of a_0 also ensures, on a fixed annulus in D, that $\epsilon V_0 \sim -\frac{1}{2\pi}\log r$ as $\epsilon \to 0$, where $r^2 = y_1^2 + y_2^2$. In general, the periods around an I_1 fibre may be assumed to be 1 and $\tau(y) = \frac{1}{2\pi i}\log y + ih(y)$, where h is holomorphic in $y = y_1 + iy_2$, and these may be achieved in our construction by taking $V = V_0 + f(y_1, y_2)$, where f denotes the real part of h.

We now give the technical details.

Lemma 3.1. *Let*

$$T_j = \frac{1}{4\pi}\sum_{n=-j}^{j}\left(\frac{1}{\sqrt{(u+n\epsilon)^2+y_1^2+y_2^2}} - a_{|n|}\right)$$

where

$$a_n = \begin{cases} 1/n\epsilon & n \neq 0 \\ 2(-\gamma + log(2\epsilon))/\epsilon & n = 0 \end{cases}$$

and γ is Euler's constant. Then:

(a) *The sequence $\{T_j\}$ converges uniformly on compact sets in $D \times \mathbf{R} - \{0\} \times \epsilon\mathbf{Z}$ to a harmonic function V_0. Here $D \subseteq \mathbf{C}$ is the unit disc centred at the origin.*

(b) *V_0 has an expansion, valid when $|y| \neq 0$,*

$$V_0 = -\frac{1}{4\pi\epsilon}\log|y|^2 + \sum_{\substack{m=-\infty \\ m\neq 0}}^{m=\infty}\frac{1}{2\pi\epsilon}e^{2\pi i m u/\epsilon}K_0(2\pi|my|/\epsilon)$$

where $y = y_1 + iy_2$ and K_0 is the modified Bessel function. (*See* [3], *p.* 374.)

(c) *There exists a constant C such that for any $0 < r_0 < 1$, there exists an $\epsilon_0 > 0$ such that for all $\epsilon < \epsilon_0$, $|y| > r_0$,*

$$\left|V_0 + \frac{1}{4\pi\epsilon}\log|y|^2\right| \leq \frac{C}{\epsilon}e^{-2\pi|y|/\epsilon}.$$

(d) *If $r \leq 1$, and f is a harmonic function on the disc D_r of radius r such that $f(y) - \frac{1}{4\pi}\log|y|^2 > 0$ for $|y| \leq r$, then there exists an ϵ_0 such that for all $\epsilon < \epsilon_0$,*

$$V_0 + f(y)/\epsilon > 0$$

in $D_r \times \mathbf{R}$.

Proof. (a) Let p be the smallest integer greater than $\epsilon^{-1}\sqrt{1+\epsilon^2}$. Then for $0 \leq u \leq \epsilon$, $y_1^2 + y_2^2 \leq 1$, we have

$$\frac{1}{\sqrt{(u+n\epsilon)^2 + y_1^2 + y_2^2}} > a_{|n|+p}$$

for all n. Let

$$R_j = \frac{1}{4\pi}\sum_{n=-j}^{j}\left(\frac{1}{\sqrt{(u+n\epsilon)^2 + y_1^2 + y_2^2}} - a_{|n|+p}\right).$$

Then for $j > 2p$,

$$T_j - R_j = \frac{1}{4\pi}\left(-a_p - a_0 - 2\sum_{n=1}^{p-1} a_n + 2\sum_{n=j-p+1}^{j} a_{n+p}\right).$$

Put $C(\epsilon) = \frac{1}{4\pi}(-a_p - a_0 - 2\sum_{n=1}^{p-1} a_n)$. Note that $\sum_{n=j-p+1}^{j} a_{n+p} \to 0$ as $j \to \infty$, so if R_j converges uniformly on compact sets to a harmonic function R, then T_j converges to a harmonic function $R + C(\epsilon)$. Now for $0 < u < \epsilon$, $y_1^2 + y_2^2 < 1$, R_j is a monotonically increasing sequence of harmonic functions (since all terms are positive). Furthermore, it is easy to check that, say, the sequence R_j is bounded at $u = \epsilon/2, y_1 = y_2 = 0$. Thus by the Harnack convergence theorem, (Theorem 2.9, [11]), the R_j converge uniformly on compact subsets to a harmonic function R, and $V_0 = R + C(\epsilon)$. Since R is positive, we see $V_0 > C(\epsilon)$. For $u = 0, \epsilon$, we merely omit the term which blows up and then repeat the previous argument.

(b) The part which requires care is the constant term of the Fourier expansion, i.e., computing $\frac{1}{\epsilon}\int_0^\epsilon V_0 du$. To do so, consider the following variant on the T_j:

$$S_j = \frac{1}{4\pi}\sum_{n=-j}^{j}\left(\frac{1}{\sqrt{(u+n\epsilon)^2 + y_1^2 + y_2^2}} - b_{|n|}\right)$$

where

$$b_n = \begin{cases} (\log(n+1) - \log n)/\epsilon & n \neq 0 \\ 0 & n = 0. \end{cases}$$

Then

$$T_j - S_j = \frac{1}{4\pi}\left(\frac{2}{\epsilon}\log(j+1) - a_0 - \frac{2}{\epsilon}\sum_{n=1}^{j}\frac{1}{n}\right).$$

As $j \to \infty$, this converges to

$$\frac{1}{4\pi}(-\frac{2\gamma}{\epsilon} - a_0) = -\frac{1}{2\pi\epsilon}\log(2\epsilon).$$

Now we calculate

$$\begin{aligned}
4\pi\int_0^\epsilon S_j du &= \sum_{n=-j}^{j}\int_0^\epsilon\left(\frac{1}{\sqrt{(u+n\epsilon)^2+|y|^2}} - b_{|n|}\right)du \\
&= \sum_{\substack{n=-j \\ n\neq 0}}^{j}(-\log(|n|+1) + \log|n|) \\
&\quad + \sum_{n=-j}^{j}\int_{n\epsilon}^{(n+1)\epsilon}\frac{1}{\sqrt{u^2+|y|^2}}du \\
&= \int_\epsilon^{(j+1)\epsilon}\left(\frac{1}{\sqrt{u^2+|y|^2}} - \frac{1}{u}\right)du \\
&\quad + \int_{-j\epsilon}^{0}\left(\frac{1}{\sqrt{u^2+|y|^2}} + \frac{1}{u-\epsilon}\right)du \\
&\quad + \int_0^\epsilon \frac{1}{\sqrt{u^2+|y|^2}}du \\
&= \log\left(u^{-1}\left(u+\sqrt{u^2+|y|^2}\right)\right)\Big|_\epsilon^{(j+1)\epsilon} \\
&\quad + \log\left(|u-\epsilon|\left(u+\sqrt{u^2+|y|^2}\right)\right)\Big|_{-j\epsilon}^{0} \\
&\quad + \log\left(u+\sqrt{u^2+|y|^2}\right)\Big|_0^\epsilon.
\end{aligned}$$

Evaluating this and letting $j \to \infty$, one obtains

$$\frac{1}{\epsilon}\lim_{j\to\infty}\int_0^\epsilon S_j du = \frac{1}{4\pi\epsilon}(\log 2 + 2\log\epsilon - \log\frac{|y|^2}{2})$$

from which we conclude that

$$\frac{1}{\epsilon}\int_0^\epsilon V_0 du = -\frac{1}{4\pi\epsilon}\log|y|^2.$$

To compute the other terms in the Fourier expansion, we just need to calculate

$$\begin{aligned}
\frac{1}{\epsilon}\int_0^\epsilon V_0 e^{2\pi imu/\epsilon} du &= \frac{1}{4\pi\epsilon}\int_{-\infty}^{\infty}\frac{e^{2\pi imu/\epsilon}}{\sqrt{u^2+|y|^2}}du \\
&= \frac{1}{2\pi\epsilon}\int_0^\infty \frac{\cos(2\pi mu/\epsilon)}{\sqrt{u^2+|y|^2}}du \\
&= \frac{1}{2\pi\epsilon}\int_0^\infty \frac{\cos(2\pi|my|v/\epsilon)}{\sqrt{v^2+1}}dv \\
&= \frac{1}{2\pi\epsilon}K_0(2\pi|my|/\epsilon).
\end{aligned}$$

The last equality follows from [3], page 376, formula 9.6.21.

(c) By [3], 9.8.6, there exists a constant C_1 such that $\sqrt{x}e^x K_0(x) \leq C_1$ for $x \geq 2$. (In fact $C_1 \leq 2$). In particular, $K_0(x) \leq C_1 e^{-x}$ for $x \geq 2$. Thus

$$\begin{aligned}
\left|\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\frac{1}{2\pi\epsilon}e^{2\pi imu/\epsilon}K_0(2\pi|my|/\epsilon)\right| &\leq \frac{C_1}{\pi\epsilon}\sum_{m=1}^{\infty}e^{-2\pi|my|/\epsilon} \\
&= \frac{C_1}{\pi\epsilon}\frac{e^{-2\pi|y|/\epsilon}}{1-e^{-2\pi|y|/\epsilon}}
\end{aligned}$$

for $2\pi|y|/\epsilon \geq 2$. From this follows (c).

(d) By the maximum principal, the minimum value M of f occurs on the boundary of D_r. On the other hand, for fixed u, it is clear V_0 is monotonically decreasing in $|y|$. Thus the minimum value of $V_0 + f/\epsilon$ must occur on $(\partial D_r) \times \mathbf{R}$. But taking $r_0 < r$, by (c) there exists an ϵ_0 such that for all $\epsilon < \epsilon_0$,

$$\left|V_0 + \frac{1}{4\pi\epsilon}\log|y|^2\right| < -\frac{1}{4\pi\epsilon}\log r^2 + M/\epsilon$$

whenever $|y| = r$. Thus $V_0 + f/\epsilon$ is positive on $\partial D_r \times \mathbf{R}$ for $\epsilon < \epsilon_0$, hence $V_0 + f/\epsilon$ is positive on $D_r \times \mathbf{R}$. q.e.d.

With this rather technical lemma out of the way, we may now proceed to the construction of our metric, using the Gibbons–Hawking

Ansatz formalism, as developed in §2. Suppose $D_r \subset \mathbf{C}$ is the disc of radius $r < 1$, centre the origin, and $f : \bar{X} \to D_r$ an elliptic fibration, with singular fibre over the origin of type I_1. Let $\bar{Y} = D_r \times \mathbf{R}/\epsilon\mathbf{Z}$ and $Y = (D_r \times \mathbf{R} - \{0\} \times \epsilon\mathbf{Z})/\epsilon\mathbf{Z}$. It is straightforward to check that there is an induced map $\bar{\pi} : \bar{X} \to \bar{Y}$ of C^∞ manifolds, which restricts to an S^1-bundle $\pi : X \to Y$ with Chern class ± 1, the sign dependent on the choice of orientation for the fibre. For further justification of these statements, the reader is referred to [15], Example 2.6 (1). The plan now is to define a hyperkähler metric on X via the Gibbons–Hawking Ansatz applied to $\pi : X \to Y$, and then check that it extends to a hyperkähler metric on $\bar{X}$.

Proposition 3.2. *With the notation as above, let*

$$h(y) = f(y_1, y_2) + ig(y_1, y_2)$$

be a holomorphic function on D_r, so that

$$-\frac{1}{4\pi} \log |y|^2 + f(y_1, y_2) > 0$$

on D_r. Let V_0 be the harmonic function on Y defined in Lemma 3.1, and $V = V_0 + f(y_1, y_2)/\epsilon$, with ϵ chosen small enough so that $V > 0$ on Y. Then there exists a connection 1-form θ on X such that $d\theta/2\pi i = *dV$, and this defines a hyperkähler metric on X with

$$\begin{aligned} -\operatorname{Re}\Omega &= dy_1 \wedge \theta/2\pi i + V dy_2 \wedge du \\ -\operatorname{Im}\Omega &= dy_2 \wedge \theta/2\pi i + V du \wedge dy_1 \\ \omega &= du \wedge \theta/2\pi i + V dy_1 \wedge dy_2. \end{aligned}$$

These forms extend to $\bar{X}$, giving a hyperkähler metric on $\bar{X}$, and a holomorphic elliptic fibration $\bar{X} \to D_r$ with periods 1 and $\frac{1}{2\pi i} \log y + ih(y) + C$, for some real constant C. By appropriate choice of θ, this constant C may be taken to be zero.

Proof. Since V is harmonic, recall that $*dV$ is closed. Taking a sphere S^2 of radius $< \epsilon$ centred at $0 \in D_r \times \mathbf{R}$, we have

$$\int_{S^2} *dV = \int_{S^2} *d\left(\frac{1}{4\pi\sqrt{u^2 + y_1^2 + y_2^2}}\right)$$

since all other terms in $*dV$ are defined at 0, and hence are exact on an ϵ-ball around 0, and therefore do not contribute to the integral. In

Example 2.5, it was however observed that this latter integral is ± 1 (depending on the orientation of the sphere). Thus, since a connection form θ can be found such that $id\theta/2\pi$ is any desired representative of c_1, we can find a connection form θ such that $d\theta/2\pi i = *dV$. Applying now the Gibbons–Hawking Ansatz construction described in §2, we obtain a hyperkähler metric on X, with the forms $\operatorname{Re}\Omega$, $\operatorname{Im}\Omega$ and ω as described in the Proposition.

To see that these forms extend to $\bar{X}$, focus on an $\epsilon/2$-ball B around 0 in $\bar{Y}$. Then $\bar{\pi}^{-1}(B) \to B$ can be identified with the map given in Example 2.5, restricted to the inverse image of the $\epsilon/2$-ball in $\mathbf{C}^2$. Let θ' be the connection form given in that example. Now $d(\theta) - d(\theta')$ is the pull-back of an *exact* form on B, since all other terms of V besides the $n = 0$ term are defined on B. Thus on $\bar{\pi}^{-1}(B - \{0\})$ we can write $\theta = \theta' + \bar{\pi}^*\beta$ for a form β defined on all of B. Now consider for example on $\bar{\pi}^{-1}(B - \{0\})$

$$\begin{aligned}\omega &= du \wedge \theta/2\pi i + V dy_1 \wedge dy_2 \\ &= du \wedge (\theta' + \bar{\pi}^*\beta)/2\pi i + \left(1/4\pi\sqrt{u^2 + y_1^2 + y_2^2} + V'\right) dy_1 \wedge dy_2\end{aligned}$$

where V' is a function defined everywhere on B. Thus we obtain

$$du \wedge \frac{\theta'}{2\pi i} + \left(1/4\pi\sqrt{u^2 + y_1^2 + y_2^2}\right) dy_1 \wedge dy_2 + du \wedge \frac{\beta}{2\pi i} + V' dy_1 \wedge dy_2.$$

The sum of the first two terms was seen to extend to all of $\bar{\pi}^{-1}(B)$ in Example 2.5, and the last two terms are defined everywhere on B, so ω extends to $\bar{X}$. Note that $\omega^2 \neq 0$ at the singular point of the singular fibre, because both $du \wedge \beta$ and $V' dy_1 \wedge dy_2$ vanish at that point.

Finally, we compute the periods. Referring back to our discussion of S^1-invariant Ricci-flat metrics in §2, one of the periods is constant, value 1. The other period $\tau(y)$ is locally holomorphic in y, and given by

$$\int_{\gamma_2} dx = \int_{\gamma_2} \theta_0 - i \int_{\gamma_2} V du,$$

where γ_2 is an S^1 in the fibre X_y mapping isomorphically to $\{y\} \times S^1 \subset Y$. Calculating the imaginary part of this,

$$\int_{\gamma_2} dx_2 = -\int_{\gamma_2} V du = \pm\Big(\frac{1}{4\pi} \log |y|^2 - f(y_1, y_2)\Big),$$

using the Fourier expansion for V_0 proved in Lemma 3.1 (b). We choose the orientation of γ_2 to obtain the choice of sign to be minus. Then

$\int_{\gamma_2} dx_1$ is necessarily locally a harmonic conjugate of $-\frac{1}{4\pi}\log|y|^2 + f(y_1, y_2)$, and so the period of γ_2 is

$$\frac{1}{2\pi i}\log y + ih(y) + C$$

for some real constant C. Now θ_0 may be modified by adding a term adu ($a \in \mathbf{R}$) without changing the fact that $d\theta_0 = *dV$. If θ_0 is changed in this way, we have

$$\begin{aligned}\int_{\gamma_2} dx_1 &= \int_{\gamma_2} \theta_0 + adu \\ &= a\epsilon + \int_{\gamma_2} \theta_0.\end{aligned}$$

We can therefore choose a suitably to obtain $C = 0$, and hence the periods as claimed. q.e.d.

Remark 3.3. (1) There is still some remaining flexibility over choosing θ, as we can change θ by the pull-back of a closed form from $\bar{Y}$. This however need not worry us, since in order to perform the gluing in §4, we will in any case need to twist the Ooguri–Vafa metrics, the twist given as translation by an appropriate local section.

(2) Recall that in the holomorphic canonical coordinates x, y, the holomorphic 2-form Ω on X is just $dx \wedge dy$, and so the complex structure on X will be the one desired. This 2-form extends uniquely to give the correct complex structure on $\bar{X}$.

Remark 3.4. A useful transformation. As $\epsilon \to 0$, the behaviour near the singular fibre of the Ooguri–Vafa metric is understood best by making a change of variables. The periods may be assumed to be $1, \frac{1}{2\pi i}\log y + ih(y)$, as in Proposition 3.2, and we take $V = V_0 + f(y)/\epsilon$. We make the change of variables $s = u/\epsilon, v_1 = y_1/\epsilon, v_2 = y_2/\epsilon$. Thus the disc of radius ϵ in the complex y-plane corresponds to the unit disc in the complex v-plane. If we now consider V_0 as a function of these new variables, we observe that

$$\epsilon V_0 = \frac{1}{4\pi}\sum_{n=-\infty}^{\infty}\left(\frac{1}{\sqrt{(s+n)^2 + v_1^2 + v_2^2}} - c_{|n|}\right),$$

where $c_n = \frac{1}{n}$ ($n > 0$), and

$$c_0 = 2(-\gamma + \log(2\epsilon)) = 2(-\gamma + \log 2) + 2\log\epsilon.$$

So, if $\tilde{V}_0$ is the standard function V_0 in variables s, v_1, v_2 for $\epsilon = 1$, we deduce that

$$\epsilon V = \tilde{V}_0 - \frac{1}{2\pi}\log\epsilon + f.$$

Thus, if we start with an Ooguri–Vafa metric with fibres of volume ϵ over the disc of radius ϵ, make the change of variables described above, and then rescale the metric by ϵ^{-1}, we obtain the Ooguri–Vafa metric over the unit disc, with fibres of volume one, corresponding to the harmonic function $\tilde{V}_0 + f - \frac{1}{2\pi}\log\epsilon$. Thus the periods of the corresponding elliptic fibration are seen to be $1, \frac{1}{2\pi i}\log v + \frac{1}{2\pi i}\log\epsilon + ih(y)$. This transformation lies behind the various estimates for diameters and curvature we derive below. We note here that in fact the formula given in [29] was for ϵV_0, rather than V_0, except that the constant a_0 was not specified. The exact value for a_0 greatly influences the behaviour of the metric as $\epsilon \to 0$, so this is quite important.

To understand the metric for $|y| > \epsilon$, we can use the Fourier expansion for V_0 from Lemma 3.1 (b), and use the same change of variables as above. Thus

$$\epsilon V_0 = -\frac{1}{4\pi}\log(v_1^2 + v_2^2) - \frac{1}{2\pi}\log\epsilon + \sum_{\substack{m=-\infty \\ m\neq 0}}^{m=\infty} \frac{1}{2\pi} e^{2\pi i m s} K_0(2\pi|mv|)$$

where $v = v_1 + iv_2$.

Estimates for diameter and curvature

We now consider a fixed elliptic fibration $f : X' \to D'$ over a disc D' of radius $a' < 1$, with singular fibre of Type I_1 over the origin, and which we assume extends to an elliptic fibration over some larger disc. We assume that the periods are of the form $1, \tau(y)$, where $\tau(y) = \frac{1}{2\pi i}\log y + ih(y)$ as in Proposition 3.2. We then wish to study sequences of Ooguri–Vafa metrics yielding the correct holomorphic 2-form Ω, but with the volume ϵ of the fibres tending to zero — such metrics exist on X' for small enough ϵ by Proposition 3.2. We first ask about the diameters of the fibres.

Proposition 3.5. *There exists a positive constant C_1 (independent of ϵ) such that, for metrics as above with fibre volume ϵ, the diameters of the fibres over D' are bounded above by $C_1(\epsilon\,\log\epsilon^{-1})^{1/2}$. Moreover, there exists a second constant C_2 such that the diameter $d(\epsilon)$ of the singular fibre is at least $C_2(\epsilon\,\log\epsilon^{-1})^{1/2}$.*

Remark 3.6. In particular, it follows that $d(\epsilon) \to 0$ as $\epsilon \to 0$. If however we rescale the metric by ϵ^{-1} as in Remark 3.4 to obtain fibres of volume one, then the diameter of the singular fibre is of order $(\log \epsilon^{-1})^{1/2}$, and therefore becomes arbitrarily large as $\epsilon \to 0$. This then contrasts with the situation for a non-singular fibre, where for sufficiently small ϵ, the Ooguri–Vafa metric near this non-singular fibre is close to being semi-flat. Thus the diameter of the fibre in the rescaled metric remains bounded.

Proof. To calculate the diameter of a fibre, we recall from §2 the formula for ds^2 in the Gibbons–Hawking Ansatz, namely

$$ds^2 = V d\mathbf{u} \cdot d\mathbf{u} + V^{-1}\theta_0^2.$$

From this, it is clear that the diameter of a fibre is at least $\int_0^{\epsilon/2} V^{1/2} du = \frac{1}{2}\int_0^{\epsilon} V^{1/2} du$. Recall however that for all $y \neq 0$, there exists a point on the fibre over y at which $V = \operatorname{Im}\tau(y)/\epsilon$, where now $\operatorname{Im}\tau(y)$ is bounded below by a positive constant for $y \in D'$. For some constant C therefore, we have on each fibre $0 \neq y \in D'$, a point at which $V^{-1/2} \leq C\epsilon^{1/2}$; by continuity, this is also true for the singular fibre. Thus each fibre over D' contains an S^1 in the S^1-bundle (where u is constant) of length at most $C\epsilon^{1/2}$, and hence the diameter of the fibre is at most $\frac{1}{2}\int_0^{\epsilon} V^{1/2} du + C\epsilon^{1/2}$. Since $V = V_0 + f(y_1, y_2)/\epsilon$, and $|f|$ is bounded on D' by some constant $A > 0$, we have

$$\int_0^{\epsilon} V_0^{1/2} du - A^{1/2}\epsilon^{1/2} \leq \int_0^{\epsilon} V^{1/2} du \leq \int_0^{\epsilon} V_0^{1/2} du + A^{1/2}\epsilon^{1/2}.$$

Since $\int_0^{\epsilon} V_0^{1/2} du$ clearly takes its maximum when $y = 0$, we are reduced to estimating $\int_0^{\epsilon} V_0^{1/2} du$ on the singular fibre only, and showing that it is of order $(\epsilon \log \epsilon^{-1})^{1/2}$.

We now let $\bar{V}_0$ denote the restriction of V_0 to the singular fibre, that is we take $y = 0$. Making the substitution $s = u/\epsilon$ as above, we observe that, for $0 < s < 1$,

$$4\pi\epsilon\bar{V}_0 = \sum_{n=1}^{\infty}\left(\frac{1}{s+n} - \frac{1}{n}\right) + \sum_{n=1}^{\infty}\left(\frac{1}{-s+n} - \frac{1}{n}\right) + \frac{1}{s} + 2\gamma - 2\log 2\epsilon.$$

We now quote formula 6.3.16 from [3], for the fact that, for $0 < s < 1$,

$$-\sum_{n=1}^{\infty}\left(\frac{1}{s+n} - \frac{1}{n}\right) - \gamma = \psi(1+s),$$

where ψ denotes the psi function. Thus, for $0 < s < 1$,

$$4\pi\epsilon\bar{V}_0 = -\psi(1+s) - \psi(1-s) + \frac{1}{s} - 2\log 2\epsilon.$$

Using formula 6.3.15 from [3], we know that

$$-\left(\psi(1+s)+\psi(1-s)\right) = 2(1-s^2)^{-1} + 2\gamma - 2 + \sum_{n=1}^{\infty} 2(\zeta(2n+1)-1)s^{2n},$$

where ζ denotes the usual zeta function. Hence, for $0 < s < 1$,

$$4\pi\epsilon\bar{V}_0 = 2(1-s^2)^{-1} + \frac{1}{s} - 2\log\epsilon + G + 2g(s),$$

with $G = -2\log 2 + 2\gamma - 2$, and where

$$g(s) = \sum_{n=1}^{\infty}(\zeta(2n+1)-1)s^{2n}$$

has radius of covergence at least 2 (by inspection of the coefficients), and so defines a continuous (non-negative) function on $[0, 1]$. Now observe that $\int_0^\epsilon \bar{V}_0^{1/2} du = \epsilon^{1/2}\int_0^1 (\epsilon\bar{V}_0)^{1/2} ds$. The lower bound now follows immediately by ignoring the first two terms in the expression for $4\pi\epsilon\bar{V}_0$. The upper bound follows by using the elementary fact that for α, β non-negative real numbers, $(\alpha+\beta)^{1/2} \le \alpha^{1/2} + \beta^{1/2}$, along with the fact that the integrals $\int_0^1 s^{-1/2} ds$ and $\int_0^1 (1-s^2)^{-1/2} ds$ are finite. q.e.d.

Corollary 3.7. *With notation as in Proposition 3.5, we suppose $D \subset D'$ is a disc centred on the origin of radius $a \le a' < 1$, and let $Diam(\epsilon)$ denote the diameter of the total space of the elliptic fibration over D, under an Ooguri–Vafa metric on X' with fibre volume ϵ. There exists a constant C_3 (independent of both ϵ and a) such that, if $\epsilon \le a$, then*

$$Diam(\epsilon) < C_3 a^{1/2}\epsilon^{-1/2}.$$

Proof. Consider the slice $u = \epsilon/2$ of Y, and a radial curve γ from $y = 0$ to $y = ae^{i\theta}$ within this slice. There is a horizontal lift $\tilde{\gamma}$ of γ to X; recalling that

$$ds^2 = V d\mathbf{u}\cdot d\mathbf{u} + V^{-1}\theta_0^2,$$

we deduce that the length of $\tilde{\gamma}$ is just

$$\int_\gamma V(y,\epsilon/2)^{1/2}|dy| = \int_0^a V(re^{i\theta},\epsilon/2)^{1/2}dr.$$

Since by Proposition 3.5, the diameters of the fibres are bounded above by

$$C_1\epsilon^{1/2}(\log\epsilon^{-1})^{1/2} < C_1 a^{1/2}\epsilon^{-1/2},$$

if we can show that the latter integral is bounded above by $Ca^{1/2}\epsilon^{-1/2}$, for some constant C independent of both ϵ and a, then the desired bound for $Diam(\epsilon)$ will follow (to go between any two fibres, we can always take the route via the central fibre).

We estimate the above integral in two parts, from 0 to ϵ, and from ϵ to a. We can estimate the first of these integrals most easily by performing the useful transformation described in Remark 3.4. Recall that

$$\epsilon V = \tilde{V}_0 + f - \frac{1}{2\pi}\log\epsilon.$$

Now $\tilde{V}_0(|v|,1/2)$ is bounded above for $0 \le |v| \le 1$ by $\tilde{V}_0(0,1/2)$, and so $\epsilon V(re^{i\theta},\epsilon/2) \le A' - \frac{1}{2\pi}\log\epsilon$ for $0 \le r \le \epsilon$, where A' is some positive constant. Thus

$$\begin{aligned}\int_0^\epsilon V(re^{i\theta},\epsilon/2)^{1/2}dr &\le \epsilon^{-1/2}\int_0^\epsilon (A' - \frac{1}{2\pi}\log\epsilon)^{1/2}dr\\ &\le C'\epsilon^{1/2}(\log\epsilon^{-1})^{1/2},\end{aligned}$$

for some positive constant C' independent of ϵ (and of course a).

We therefore now need to demonstrate that

$$\int_\epsilon^a V(re^{i\theta},\epsilon/2)^{1/2}dr$$

has a bound of the desired type. To do this, we use the expression for V_0 given in Lemma 3.1(b). From the proof of Lemma 3.1(c), we deduce that, for $|y| \ge \epsilon/\pi$, we have

$$2\pi\epsilon V_0 < -\log|y| + 2C_1'\frac{e^{-2\pi|y|/\epsilon}}{1-e^{-2\pi|y|/\epsilon}}.$$

In particular, since the second term is decreasing in the range, we have, for $|y| \ge \epsilon$, that

$$2\pi\epsilon V_0 < -\log|y| + 2C_1'\frac{e^{-2\pi}}{1-e^{-2\pi}},$$

and hence that

$$2\pi\epsilon V < -\log|y| + C_2',$$

for some constant C_2' independent of ϵ and a. Using the assumption that $a \le a' < 1$, we have

$$\begin{aligned}\int_\epsilon^a V(re^{i\theta},\epsilon/2)^{1/2}dr &< (2\pi\epsilon)^{-1/2}\int_\epsilon^a (C_2' - \log r)^{1/2}dr\\ &< \epsilon^{-1/2}C_3'\int_\epsilon^a (\log r^{-1})^{1/2}dr\\ &< \epsilon^{-1/2}C_3'\int_\epsilon^a r^{-1/2}dr\\ &< 2C_3'\epsilon^{-1/2}a^{1/2},\end{aligned}$$

for an appropriate constant C_3', depending on a' but independent of ϵ and a. The result then follows immediately. q.e.d.

Proposition 3.8. *With notation as in Proposition 3.5, let $R(\epsilon)$ denote the curvature tensor of the total space X' of the elliptic fibration over D', under an Ooguri–Vafa metric on X' with fibre volume ϵ. Then there exists positive constants C_4, C_4' (independent of ϵ) such that, for all sufficiently small ϵ,*

$$C_4'\epsilon^{-1}\log(\epsilon^{-1})^{-2} < \|R(\epsilon)\|_{C^0} < C_4\epsilon^{-1}\log(\epsilon^{-1}),$$

where $\| \,.\, \|_{C^0}$ denotes the usual C^0-norm on X'.

Proof. Recall first from Remark 2.4 that

$$\|R\|^2 = 12V^{-6}|\nabla V|^4 + V^{-4}\Delta(|\nabla V|^2) - 6V^{-5}(\nabla V)\cdot(\nabla(|\nabla V|^2)).$$

We now perform our change of coordinates $s = u/\epsilon$, $v = y/\epsilon$. We recall that $V = V_0 + f(y_1,y_2)/\epsilon$ for some bounded harmonic function f defined over D', and that

$$\epsilon V = \tilde{V}_0 + f - \frac{1}{2\pi}\log\epsilon.$$

Also observe that $\nabla_{u,y_1,y_2} = \epsilon^{-1}\nabla_{s,v_1,v_2}$; from now on ∇ will denote ∇_{s,v_1,v_2}, and Δ will denote Δ_{s,v_1,v_2}. We set $V_1 = \epsilon V = \tilde{V}_0 + f - \frac{1}{2\pi}\log\epsilon$, considered as a function of s, v_1, v_2. Thus

$$\epsilon^2\|R\|^2 = 12V_1^{-6}|\nabla V_1|^4 + V_1^{-4}\Delta(|\nabla V_1|^2) - 6V_1^{-5}(\nabla V_1)\cdot(\nabla(|\nabla V_1|^2)).$$

We first prove the upper bound for $\|R(\epsilon)\|_{C^0}$, namely that

$$\|R(\epsilon)\| < C_4\epsilon^{-1}\log(\epsilon^{-1})$$

at all points of X'. The easy part of this is to deal with the points in the range $1/2 \leq |v| < a/\epsilon$ (where a now denotes the radius of D'), corresponding to $|y| \geq \epsilon/2$ in the disc D'. Here we use the Fourier expansion for V_1, namely

$$V_1 = -\frac{1}{2\pi}\log|v| + f - \frac{1}{2\pi}\log\epsilon + \sum_{\substack{m=-\infty \\ m\neq 0}}^{m=\infty} \frac{1}{2\pi}e^{2\pi ims}K_0(2\pi|mv|).$$

Recalling that $K_0(x)$ and its derivatives decay at least as fast as e^{-x} for large x, it is clear that $|\nabla V_1|^4$, $\Delta(|\nabla V_1|^2)$ and $(\nabla V_1)\cdot(\nabla(|\nabla V_1|^2))$ are bounded (independent of ϵ) for $1/2 \leq |v| < a/\epsilon$. Moreover, for ϵ sufficiently small,

$$\epsilon V = -\frac{1}{2\pi}\log|y| + f(y) + \sum_{\substack{m=-\infty \\ m\neq 0}}^{m=\infty} \frac{1}{2\pi}e^{2\pi imu/\epsilon}K_0(2\pi|my|/\epsilon)$$

is bounded below, over D', by some positive constant (independent of ϵ). Thus, V_1 is bounded below on $1/2 \leq |v| < a/\epsilon$, and hence $\epsilon\|R\|$ is bounded above on the given range by some constant, again independent of ϵ.

The trickier argument is of course for the range $0 \leq |v| \leq 1/2$, corresponding to $0 \leq |y| \leq \epsilon/2$. We assume that ϵ is small enough that $3\epsilon/4 \leq a$. We make our usual change of variables, so that

$$V_1 = \epsilon V = \tilde{V}_0 + f + \frac{1}{2\pi}\log(\epsilon^{-1})$$

defines an Ooguri–Vafa metric over the disc $|v| < 3/4$, fibres of volume one, and periods $\{1, (2\pi i)^{-1}\log v + (2\pi i)^{-1}\log\epsilon + ih\}$. Now choose $A \geq 0$ such that $f + A > 0$ whenever $|v| < 3/4$, and set $V_2 = \tilde{V}_0 + f + A$; V_2 then determines an Ooguri–Vafa metric over the disc $|v| < 3/4$, fibres of volume one, and periods $\{1, (2\pi i)^{-1}\log v + ih + iA\}$. We may obviously assume that $A < (2\pi)^{-1}\log(\epsilon^{-1})$. Let R_1, respectively R_2, denote the curvature tensors of the metrics determined by V_1, respectively V_2. Our aim now is to show that $\|R_1\|^2 < C\,\log(\epsilon^{-1})^2$ over the disc $|v| \leq 1/2$; if this is true, it follows from the above that $\|R\| < C_4\epsilon^{-1}\log(\epsilon^{-1})$ at all points over D', for some positive constant C_4.

Since the metric determined by V_2 is independent of ϵ, it is clear that $\|R_2\|$ is bounded over $|v| \leq 1/2$, with the bound independent of ϵ. Hence

$$12V_2^{-6}|\nabla V_2|^4 + V_2^{-4}\Delta(|\nabla V_2|^2) - 6V_2^{-5}(\nabla V_2)\cdot(\nabla(|\nabla V_2|^2)) \tag{3.1}$$

is bounded independent of ϵ over the range in question, $|v| \leq 1/2$, which from now on will be taken as understood. We wish to show that

$$\begin{aligned}\|R_1\|^2 = {} & 12V_1^{-6}|\nabla V_2|^4 + V_1^{-4}\Delta(|\nabla V_2|^2) \\ & - \ 6V_1^{-5}(\nabla V_2)\cdot(\nabla(|\nabla V_2|^2)) \\ \leq {} & C(\log(\epsilon^{-1}))^2.\end{aligned}$$

Since $V_1 = V_2 + (2\pi)^{-1}\log(\epsilon^{-1}) - A \geq V_2$, it will be enough to prove the same bound for

$$\begin{aligned}& 12V_1^{-2}V_2^{-4}|\nabla V_2|^4 + V_2^{-4}\Delta(|\nabla V_2|^2) \\ & \qquad - 6V_1^{-1}V_2^{-4}(\nabla V_2)\cdot(\nabla(|\nabla V_2|^2)).\end{aligned} \tag{3.2}$$

By subtracting our previously bounded expression (3.1), we need then only show boundedness for

$$\begin{aligned}& 12V_1^{-2}V_2^{-6}(V_2^2 - V_1^2)|\nabla V_2|^4 \\ & \qquad - 6V_1^{-1}V_2^{-5}(V_2 - V_1)(\nabla V_2)\cdot(\nabla(|\nabla V_2|^2)).\end{aligned} \tag{3.3}$$

Expanding this latter expression out, we get

$$\begin{aligned}6\left((2\pi)^{-1}\log(\epsilon^{-1}) - A\right)V_1^{-1}\Bigg(& V_2^{-5}(\nabla V_2)\cdot(\nabla(|\nabla V_2|^2)). \\ & - 2V_2^{-6}(1 + \frac{V_2}{V_1})|\nabla V_2|^4\Bigg).\end{aligned} \tag{3.4}$$

We now claim that

$$\left|V_1^{-1}\left(V_2^{-5}(\nabla V_2)\cdot(\nabla(|\nabla V_2|^2)) - 4V_2^{-6}|\nabla V_2|^4\right)\right|$$

and

$$V_1^{-2}V_2^{-6}|\nabla V_2|^4$$

are bounded independent of ϵ. If this is true, then the latter bound will imply that

$$V_1^{-1}(1 - \frac{V_2}{V_1})V_2^{-6}|\nabla V_2|^4 \leq C'(\log(\epsilon^{-1}) - 2\pi A)$$

for some positive C', and then the former bound implies that the expression we are interested in has a bound of the form

$$B_1(\log(\epsilon^{-1}) - 2\pi A)^2 + B_2(\log(\epsilon^{-1}) - 2\pi A),$$

for suitable positive constants B_1, B_2. This then gives the required result.

To show boundedness for the two remaining quantities, it is sufficient to bound the functions

$$V_2^{-6}\left|4V_2^{-1}|\nabla V_2|^4 - (\nabla V_2)\cdot(\nabla(|\nabla V_2|^2))\right|$$

and

$$V_2^{-8}|\nabla V_2|^4.$$

Both these functions are defined away from $\{0\} \times \mathbf{Z}$ and are periodic in s; moreover, they plainly do not depend on ϵ. If we show that they are in fact both continuous at the origin ($v = 0, s = 0$), the existence of the required bounds will follow automatically.

We now write $4\pi V_2 = \rho^{-1} + w$, where $\rho = (s^2 + v_1^2 + v_2^2)^{1/2}$ and w is a harmonic function on a neighbourhood of the origin. Then we see that $(4\pi)^4|\nabla V_2|^4 = \rho^{-8} + O(\rho^{-6})$. Since $(4\pi V_2)^{-8} = \rho^8(1 + w\rho)^{-8}$, we deduce that $V_2^{-8}|\nabla V_2|^4$ is regular at the origin, taking the value $(4\pi)^4$ there. Moreover, it is easily checked that

$$(4\pi)^3(\nabla V_2)\cdot(\nabla(|\nabla V_2|^2)) = 4\rho^{-7} + O(\rho^{-5}),$$

and so in particular

$$4V_2^{-7}|\nabla V_2|^4 - V_2^{-6}(\nabla V_2)\cdot(\nabla(|\nabla V_2|^2))$$

is also regular at the origin, and vanishes there.

We now turn to the lower bound for $\|R(\epsilon)\|_{C^0}$. We work on the transformed elliptic fibration over the disc $|v| \le 1/2$, and let M denote the C^0-norm of the function given by (3.1).

From the above calculations, at all points P with sufficiently small value of ρ, we have

$$\left|V_2^{-6}(\nabla V_2)\cdot(\nabla(|\nabla V_2|^2) - 4V_2^{-7}|\nabla V_2|^4\right| < M \ , \quad V_2^{-7}|\nabla V_2|^4 > 2M.$$

We now fix such a point P; note that the coordinates s, v_1, v_2 are then taken to be fixed, and so this does not correspond to taking a fixed point (independent of ϵ) on our original family X'.

Observe now that

$$V_1/V_2 = 1 + \frac{(2\pi)^{-1}\log(\epsilon^{-1}) - A}{V_2};$$

so for P fixed, $V_1(P)/V_2(P) > 2$ for ϵ sufficiently small. From this it follows that, when evaluated at P,

$$\left| V_2^{-6}(\nabla V_2)\cdot(\nabla(|\nabla V_2|^2)) - 2V_2^{-7}(1+\frac{V_2}{V_1})|\nabla V_2|^4 \right| > M,$$

for ϵ sufficiently small. Hence, for ϵ sufficiently small, the modulus of (3.4) evaluated at P is at least $3M$ say, and thus the same is true of (3.3). From this, and our original choice for M, it follows that the modulus of (3.2) evaluated at P is at least $2M$. Therefore, for ϵ sufficiently small,

$$\|R_1(P)\|^2 > B(\log(\epsilon^{-1}))^{-4}$$

for some constant B independent of ϵ. Thus

$$\|R(\epsilon)\|_{C^0} > B^{1/2}\epsilon^{-1}(\log(\epsilon^{-1}))^{-2},$$

as required. q.e.d.

4. Almost Ricci-flat metrics on elliptic K3 surfaces

Our goal in this section is to construct Kähler metrics on elliptic K3 surfaces which are very close to being Ricci-flat by gluing the Ooguri–Vafa metric in neighbourhoods of singular fibres to the semi-flat metric away from the singular fibres.

We begin by producing one such metric on a Jacobian elliptic fibration. Fix a K3 surface X with a fixed holomorphic 2-form Ω and an elliptic fibration $f: X \rightarrow B = \mathbf{P}^1$, which we will take to have a holomorphic section σ_0. Furthermore, assume all singular fibres of f are of Kodaira type I_1; there will then be 24 such fibres. Let $p_1, \ldots, p_{24} \in B$ be those points for which $X_{p_i} = f^{-1}(p_i)$ is singular, $\Delta = \{p_1, \ldots, p_{24}\}$, $B_0 = B - \Delta$, $X_0 = f^{-1}(B_0)$, and $X^\# = X - Sing(f^{-1}(\Delta))$. There is an exact sequence, already mentioned in §1,

$$0 \longrightarrow R^1 f_*\mathbf{Z} \longrightarrow \mathcal{T}_B^* \xrightarrow{\phi} X^\# \longrightarrow 0,$$

with the property that ϕ maps the zero section of $\mathcal{T}_B^*$ to σ_0 and $\phi^*\Omega$ is the canonical holomorphic 2-form on $\mathcal{T}_B^*$, which is $dx\wedge dy$ if y is a coordinate on B and x a canonical fibre coordinate. (See [14], Proposition 7.2). Here $x = 0$ defines the zero section.

Given this data, by Example 2.2, for each ϵ, there exists a well-defined Ricci-flat metric on X_0, the standard semi-flat metric ω_{SF}, with the area of each fibre being ϵ. The reader should keep in mind the dependence of ω_{SF} on ϵ.

Now let y be a holomorphic coordinate on B defined in a neighbourhood U of $p \in \Delta$, U contractible with $U \cap \Delta = \{p\}$, and $y = 0$ at the point p. Let x be the corresponding canonical fibre coordinate. Let $U^* = U - \{p\}$, $X_{U^*} = f^{-1}(U^*)$. We can then choose over U^* holomorphic periods $\tau_1(y), \tau_2(y)$, representing possibly multi-valued holomorphic sections of $\mathcal{T}^*_{U^*}$ generating the period lattice. Because the monodromy about an I_1 fibre in a suitable basis is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, we can take one of these, say τ_1, to be single-valued, though τ_2 will be multi-valued around the I_1 fibre. We will always choose τ_1 and τ_2 so that $\mathrm{Im}(\bar\tau_1\tau_2) > 0$. Set

$$\begin{aligned} W_0(y) &= 1/\mathrm{Im}(\bar\tau_1\tau_2) \\ b_0(x,y) &= -\frac{\mathrm{Im}(\tau_2\bar x)\partial_y\tau_1 + \mathrm{Im}(\bar\tau_1 x)\partial_y\tau_2}{\mathrm{Im}(\bar\tau_1\tau_2)} \end{aligned}$$

and

$$\begin{aligned} \partial_{\mathrm{v}} &= W_0^{-1}\partial_x \\ \partial_{\mathrm{h}} &= \partial_y - b_0\partial_x \\ \vartheta_{\mathrm{v}} &= W_0(dx + b_0 dy) \\ \vartheta_{\mathrm{h}} &= dy \end{aligned}$$

as in §2. The latter two 1-forms are well-defined on X^*_U, so form a basis for $(1,0)$ forms. We denote by $\bar\partial_{\mathrm{v}}$ et cetera the complex conjugates of the above as usual.

Lemma 4.1. *Let ω be a real closed $(1,1)$ form on X_{U^*}, with*

$$\omega = \frac{i}{2}(\alpha\,\vartheta_{\mathrm{v}} \wedge \bar\vartheta_{\mathrm{v}} + \beta\,\vartheta_{\mathrm{h}} \wedge \bar\vartheta_{\mathrm{v}} + \bar\beta\,\vartheta_{\mathrm{v}} \wedge \bar\vartheta_{\mathrm{h}} + \gamma\,\vartheta_{\mathrm{h}} \wedge \bar\vartheta_{\mathrm{h}}).$$

There exists a function φ on X_{U^*} such that $\omega = i\partial\bar\partial\varphi$ if and only if ω represents the zero cohomology class on X_{U^*} and

$$\int_{X_b} \beta\ dx_1 \wedge dx_2 = 0$$

for all $b \in U^*$. Furthermore, for $0 < r_1 < r_2$, let $U_{r_1,r_2} = \{y \in U \mid r_1 < |y| < r_2\}$. If $r_1 < r_1' < r_2' < r_2$ and $\overline{U}_{r_1,r_2} \subseteq U^*$, then there exists a constant C depending only on r_1, r_2, r_1', r_2' and the periods of f over U_{r_1,r_2} such that φ can be chosen with

$$\|\varphi\|'_{C^{k+2,\alpha}} \leq C(\|\alpha\|_{C^{k,\alpha}} + \|\beta\|_{C^{k,\alpha}} + \|\gamma\|_{C^{k,\alpha}}).$$

Here, we compute the $C^{k,\alpha}$ norm of a function on $f^{-1}(U_{r_1,r_2})$ by thinking of them as functions on $\mathcal{T}^*_{U_{r_1,r_2}}$, which we embed in $\mathbf{C}^2$ by the coordinates x and y. We can then use the standard $C^{k,\alpha}$ norms on a bounded open set of $\mathcal{T}^*_{U_{r_1,r_2}}$ which contains a fundamental domain of each fibre. The norm $\|\cdot\|'_{C^{k,\alpha}}$ denotes the similar norm of a function over $U_{r_1',r_2'}$.

Remark 4.2. We note that the definition of the $C^{k,\alpha}$ norm given above depends on the choice of holomorphic coordinate y and the bounded open set, but any two such norms will be equivalent.

Proof. Before beginning the proof, we observe from (2.4) and (2.3) that

$$\begin{aligned}\partial\bar{\vartheta}_{\mathrm{v}} =&((\partial_{\mathrm{v}} W_0)\vartheta_{\mathrm{v}} + (\partial_{\mathrm{h}} W_0)\vartheta_{\mathrm{h}}) \wedge W_0^{-1}\bar{\vartheta}_{\mathrm{v}} \\ &+ W_0((\partial_{\mathrm{v}}\bar{b}_0)\vartheta_{\mathrm{v}} + (\partial_{\mathrm{h}}\bar{b}_0)\vartheta_{\mathrm{h}}) \wedge \bar{\vartheta}_{\mathrm{h}} \\ \bar{\partial}\bar{\vartheta}_{\mathrm{v}} =&0.\end{aligned} \tag{4.1}$$

Furthermore, locally for the base, a function on X_U^* can be expanded in a Fourier series on the fibres, yielding

$$f(x,y) = \sum_{n,m\in\mathbf{Z}} a_{n,m}(y)e^{2\pi i(n\mathrm{Im}(\tau_2\bar{x}) + m\mathrm{Im}(\bar{\tau}_1 x))/\mathrm{Im}(\bar{\tau}_1\tau_2)}.$$

A direct calculation shows that

$$\partial_{\mathrm{h}} f = \sum_{n,m\in\mathbf{Z}} \partial_y(a_{n,m})e^{2\pi i(n\mathrm{Im}(\tau_2\bar{x}) + m\mathrm{Im}(\bar{\tau}_1 x))/\mathrm{Im}(\bar{\tau}_1\tau_2)}.$$

If a φ exists, then of course ω represents the zero cohomology class on X_{U^*}. Also,

$$i\partial\bar{\partial}\varphi = i\partial((\bar{\partial}_{\mathrm{v}}\varphi)\bar{\vartheta}_{\mathrm{v}} + (\bar{\partial}_{\mathrm{h}}\varphi)\bar{\vartheta}_{\mathrm{h}}).$$

From (4.1) it then follows that if $\omega = i\partial\bar{\partial}\varphi$, then

$$\beta = 2(\partial_{\mathrm{h}}\bar{\partial}_{\mathrm{v}}\varphi + (\bar{\partial}_{\mathrm{v}}\varphi)W_0^{-1}\partial_{\mathrm{h}} W_0),$$

and then by looking at the constant term $a_{0,0}$ of the Fourier expansion of β, it is clear that $\int_{X_b} \beta dx_1 \wedge dx_2 = 0$.

Conversely, first suppose ω is cohomologically trivial. Then there exists a one-form ξ of type $(1,0)$ such that $\frac{i}{2}d(\bar{\xi}-\xi)=\omega$ (since ω is real). Necessarily $\partial\xi=\bar{\partial}\bar{\xi}=0$. Thus $\bar{\xi}$ represents a class in $H^{0,1}(X_{U^*})$. If this class is zero, then there exists a function φ such that $\bar{\partial}\varphi=\bar{\xi}$, and then $\partial\bar{\varphi}=\xi$, so

$$\omega=\frac{i}{2}(\partial\bar{\partial}\varphi-\bar{\partial}\partial\bar{\varphi})=\frac{i}{2}\partial\bar{\partial}(\varphi+\bar{\varphi})=i\partial\bar{\partial}\operatorname{Re}\varphi,$$

as desired. Thus we need to understand when $\bar{\xi}$ represents the zero class.

Now $H^{0,1}(X_{U^*})=H^1(X_{U^*},\mathcal{O}_{X_{U^*}})$, which by the Leray spectral sequence for f is isomorphic to $H^0(U^*,R^1f_*\mathcal{O}_{X_{U^*}})$, as $H^i(U^*,f_*\mathcal{O}_{X_{U^*}})=H^i(U^*,\mathcal{O}_{U^*})=0$ for $i\geq 1$. Thus $\bar{\xi}$ represents zero in $H^{0,1}(X_{U^*})$ if and only if $\bar{\xi}|_{X_b}$ represents the zero class in $H^{0,1}(X_b)$ for all $b\in U^*$. If we write $\bar{\xi}=\bar{g}\bar{\vartheta}_{\mathrm{v}}+\bar{h}\bar{\vartheta}_{\mathrm{h}}$, this is equivalent to the constant term in the Fourier expansion of $\bar{g}$ on the fibre being zero. Denote this constant term by $\bar{g}_0(y)$.

What kind of function is $\bar{g}_0$? Well, by (4.1),

$$0=\bar{\partial}\bar{\xi}=(\bar{\partial}_{\mathrm{h}}\bar{g}-\bar{\partial}_{\mathrm{v}}\bar{h})\bar{\vartheta}_{\mathrm{h}}\wedge\bar{\vartheta}_{\mathrm{v}}.$$

By looking at the constant term of the Fourier expansion of this coefficient, we see $\partial_{\bar{y}}\bar{g}_0=0$, so $\bar{g}_0$ is a holomorphic function on U^*. This function gives the section of $R^1f_*\mathcal{O}_{X_{U^*}}$ defined by $\bar{\xi}$.

Now let us compute the coefficient β of $\vartheta_{\mathrm{h}}\wedge\bar{\vartheta}_{\mathrm{v}}$ in ω in terms of g and h. From $\omega=\frac{i}{2}(\partial\bar{\xi}-\bar{\partial}\xi)$ and (4.1), it follows that

$$\beta=\partial_{\mathrm{h}}\bar{g}+\bar{\partial}_{\mathrm{v}}h+\bar{g}W_0^{-1}\partial_{\mathrm{h}}W_0+gW_0\bar{\partial}_{\mathrm{v}}b_0.$$

If β_0 is the constant term in the Fourier expansion of β, then we get, using (2.2′) for the second line,

$$\begin{aligned}\beta_0&=\partial_y\bar{g}_0+\bar{g}_0W_0^{-1}\partial_{\mathrm{h}}W_0+g_0W_0\bar{\partial}_{\mathrm{v}}b_0\\&=\partial_y\bar{g}_0+\bar{g}_0\partial_x b_0+g_0\partial_{\bar{x}}b_0\\&=\partial_y\bar{g}_0+\frac{\bar{g}_0(\bar{\tau}_2\partial_y\tau_1-\bar{\tau}_1\partial_y\tau_2)-g_0(\tau_2\partial_y\tau_1-\tau_1\partial_y\tau_2)}{\bar{\tau}_1\tau_2-\tau_1\bar{\tau}_2}\\&=\partial_y\bar{g}_0+b_0(\bar{g}_0,y).\end{aligned}\tag{4.2}$$

If we now assume in addition that $\int_{X_b}\beta dx_1\wedge dx_2=0$ for all $b\in U^*$, then $\beta_0=0$, so

$$\partial_y\bar{g}_0+b_0(\bar{g}_0,y)=0.$$

Now write $\bar{g}_0(y) = a_1(y)\tau_1(y) + a_2(y)\tau_2(y)$, where a_1, a_2 are real functions of y. Then $b_0(\bar{g}_0, y) = -a_1\partial_y\tau_1 - a_2\partial_y\tau_2$, so

$$0 = \partial_y\bar{g}_0 + b_0(\bar{g}_0, y) = (\partial_y a_1)\tau_1 + (\partial_y a_2)\tau_2.$$

But

$$0 = \partial_{\bar{y}}\bar{g}_0 = (\partial_{\bar{y}} a_1)\tau_1 + (\partial_{\bar{y}} a_2)\tau_2.$$

Thus combining these two equations gives

$$\begin{aligned}(\partial_{y_1} a_1)\tau_1 + (\partial_{y_1} a_2)\tau_2 = 0\\ (\partial_{y_2} a_1)\tau_1 + (\partial_{y_2} a_2)\tau_2 = 0,\end{aligned}$$

and by linear independence of τ_1 and τ_2 we see a_1 and a_2 are constant. Since $\bar{g}_0$ is well-defined and we are assuming τ_1 is the monodromy invariant period, we have $\bar{g}_0 = a\tau_1$, a a constant. Now a calculation shows that

$$\frac{i}{2}d(\tau_1\bar{\vartheta}_{\mathrm{v}} - \bar{\tau}_1\vartheta_{\mathrm{v}}) = 0,$$

so we can subtract $a\tau_1\bar{\vartheta}_{\mathrm{v}}$ from $\bar{\xi}$ without affecting $\frac{i}{2}d(\bar{\xi} - \xi) = \omega$. Thus we can assume $\bar{g}_0 = 0$, and then $\bar{\xi}$ represents the zero class in $H^{0,1}(X_{U^*})$, allowing us to complete the proof of the existence of φ.

Now we need to control the norm of φ. First note that

$$W_0^2\alpha = 2\partial_x\partial_{\bar{x}}\varphi = \frac{1}{2}\Delta_x\varphi,$$

where $\Delta_x = \partial_{x_1}^2 + \partial_{x_2}^2$ denotes the standard Laplacian on fibres. Writing $\varphi = \varphi_0 + \varphi_v$ where φ_0 is the pull-back of a function on U^* and $\int_{X_b}\varphi_v dx_1 dx_2 = 0$ for all $b \in U^*$, we have $W_0^2\alpha = \Delta_x\varphi_v/2$. It then follows that $|\varphi_v|$ is bounded on each fibre (being a torus) with the bound proportional to a bound for $|\alpha|$ on that fibre, with the constant of proportionality depending on the periods at that point. (To see this, one can just work with Fourier series). Thus

$$\|\varphi_v\|_{C^0} \le C_1\|\alpha\|_{C^0}$$

on U_{r_1,r_2}, where C_1 depends on the periods over U_{r_1,r_2}.

Next restrict φ and ω to the zero section of $f : X_U \to U$. On this zero-section,

$$\begin{aligned}\omega &= \frac{i}{2}\gamma\ \ dy \wedge d\bar{y} \\ &= i\partial\bar{\partial}\varphi \\ &= i\partial_y\partial_{\bar{y}}\varphi\ \ dy \wedge d\bar{y} \\ &= \frac{i}{4}\Delta_y\varphi\ \ dy \wedge d\bar{y}.\end{aligned}$$

so $\Delta_y\varphi = 2\gamma$ on the zero section, where Δ_y is the the standard Laplacian $\partial_{y_1}^2 + \partial_{y_2}^2$ on U^*.

Now let ψ be a harmonic function on U_{r_1,r_2} such that $\psi|_{\partial\overline{U}_{r_1,r_2}} = \varphi|_{\partial\overline{U}_{r_1,r_2}}$. (Here we are identifying U_{r_1,r_2} with its image under the zero section.) This function exists and is unique. Then

$$\Delta_y\left(\varphi - \psi + \frac{\|2\gamma\|_{C^0}}{4}(y_1^2 + y_2^2)\right) = 2\gamma + \|2\gamma\|_{C^0} \geq 0.$$

Thus by the maximum principal, $\varphi - \psi + \|2\gamma\|_{C^0}(y_1^2 + y_2^2)/4$ achieves its maximum when either $|y| = r_1$ or $|y| = r_2$, and since $\varphi - \psi = 0$ on the boundary of U_{r_1,r_2}, we have

$$\varphi - \psi \leq \|2\gamma\|_{C^0}\ r_2^2/4.$$

Similarly, $\psi - \varphi \leq \|2\gamma\|_{C^0}\ r_2^2/4$, so

$$\|\varphi - \psi\|_{C^0} \leq \|\gamma\|_{C^0}\ r_2^2.$$

This estimate holds on U_{r_1,r_2}, but from $\|\varphi_v\|_{C^0} \leq C_1\|\alpha\|_{C^0}$, it is clear that the oscillation of φ along the fibres is bounded by $C_1\|\alpha\|_{C^0}$, and thus on $f^{-1}(U_{r_1,r_2})$,

$$\|\varphi - \psi\|_{C^0} \leq C_2(\|\alpha\|_{C^0} + \|\gamma\|_{C^0})$$

for some constant C_2 depending on the periods, r_1, and r_2. Noting that $\partial\bar{\partial}\psi = 0$, we can replace φ by $\varphi - \psi$. Then the $C^{k+2,\alpha}$ estimates follow from the standard interior Schauder estimates for the Laplacian (see [11], problem 6.1.) This is because the ordinary Laplacian (in the coordinates x_1, x_2, y_1, y_2) of φ can be expressed in terms of α, β and γ.

q.e.d.

Lemma 4.3. *Let ω be a Kähler form on X_U, ω_{SF} the semi-flat Kähler form on X_0, such that*

$$\int_{X_b}\omega = \int_{X_b}\omega_{SF} = \epsilon.$$

Then $[\omega_{SF}-\omega]=0$ *in* $H^2(X_{U^*},\mathbf{R})$*, and furthermore, there exists a holomorphic section* σ *of* $f: X_U \to U$ *and a function* φ *on* X_{U^*} *such that*

$$\omega_{SF}-T_\sigma^*\omega = i\partial\bar\partial\varphi,$$

where T_σ *is translation by the section* σ*.*

Proof. To show the first part, we first observe that $H_2(X_{U^*},\mathbf{Z})$ is generated by the homology classes of two submanifolds: X_b for some $b\in U^*$, and T, where T is a torus fibred in circles over a simple closed loop $\gamma:[0,1]\to U^*$ generating $\pi_1(U^*)$, with the class of the fibre being the monodromy invariant cycle. To show $[\omega_{SF}-\omega]=0$, we just need $\int_{X_b}\omega_{SF}-\omega=0$, which is obvious, and $\int_T\omega_{SF}-\omega=0$. Now $\int_T\omega=0$ since ω is defined on X_U, where T is homologous to zero. On the other hand, if we describe T explicitly, parametrised by coordinates s,t with $\mu:[0,1]^2\to X_{U^*}$ given by

$$\mu(s,t)=(x(s,t),y(s,t))=(s\tau_1(\gamma(t)),\gamma(t)),$$

then a calculation shows that $\mu^*\omega_{SF}=0$, and hence $\int_T\omega_{SF}=0$. Thus $[\omega_{SF}-\omega]=0$.

As in Lemma 4.1, write, for each section σ of $f:X_{U^*}\to U^*$,

$$\omega_{SF}-T_\sigma^*\omega=\frac{i}{2}(\alpha_\sigma\vartheta_{\rm v}\wedge\bar\vartheta_{\rm v}+\beta_\sigma\vartheta_{\rm h}\wedge\bar\vartheta_{\rm v}+\cdots).$$

Let σ_0 be the zero section, so that T_{σ_0} is the identity. We showed in (4.2) that the function β_0, the constant term in the Fourier expansion of β_{σ_0}, was of the form

$$\beta_0=\partial_y k+b_0(k,y)$$

where $k(y)$ is a holomorphic function on U^*.

Now write

$$\omega=\frac{i}{2}(WW_0^{-2}\vartheta_{\rm v}\wedge\bar\vartheta_{\rm v}+\beta_\omega\vartheta_{\rm h}\wedge\bar\vartheta_{\rm v}+\cdots)$$

where necessarily the constant term of W is

$$\begin{aligned}(\mathrm{Im}\bar\tau_1\tau_2)^{-1}\int_{X_b}W dx_1\wedge dx_2 &= (\mathrm{Im}\bar\tau_1\tau_2)^{-1}\int_{X_b}\frac{i}{2}WW_0^{-2}\vartheta_{\rm v}\wedge\bar\vartheta_{\rm v}\\ &=\epsilon/\mathrm{Im}(\bar\tau_1\tau_2).\end{aligned}$$

We calculate $T_\sigma^*\omega$. First note that

$$\begin{aligned} T_\sigma^*(\vartheta_\mathrm{v}) &= W_0(d(x+\sigma(y)) + b_0(x+\sigma(y),y)\,dy) \\ &= W_0(dx + b_0\,dy) + W_0(\partial_y\sigma(y) + b_0(\sigma(y),y))\,dy \\ &= \vartheta_\mathrm{v} + W_0(\partial_y\sigma + b_0(\sigma(y),y))\,\vartheta_\mathrm{h}. \end{aligned}$$

Thus the coefficient of $\vartheta_\mathrm{h}\wedge\bar\vartheta_\mathrm{v}$ in $T_\sigma^*(\omega)$ is

$$\frac{i}{2}\left(\beta_\omega\circ T_\sigma + \frac{W}{W_0}(\partial_y\sigma + b_0(\sigma(y),y))\right).$$

On the other hand, $\omega_{SF} = \frac{i}{2}W_0^{-1}(\epsilon\,\vartheta_\mathrm{v}\wedge\bar\vartheta_\mathrm{v} + \epsilon^{-1}\,\vartheta_\mathrm{h}\wedge\bar\vartheta_\mathrm{h})$. Thus $\beta_{\sigma_0} = -\beta_\omega$, and

$$\beta_\sigma = \beta_{\sigma_0}\circ T_\sigma - \frac{W}{W_0}(\partial_y\sigma + b_0(\sigma(y),y)).$$

So

$$\begin{aligned} &(\mathrm{Im}\bar\tau_1\tau_2)^{-1}\int_{X_b}\beta_\sigma\;dx_1\wedge dx_2 \\ &\quad= (\mathrm{Im}\bar\tau_1\tau_2)^{-1}\bigg(\int_{X_b}\beta_{\sigma_0}\circ T_\sigma\;dx_1\wedge dx_2 \\ &\qquad\qquad - W_0^{-1}(\partial_y\sigma + b_0(\sigma(y),y))\int_{X_b}W\;dx_1\wedge dx_2\bigg) \\ &\quad= \beta_0 - \epsilon(\partial_y\sigma + b_0(\sigma(y),y)). \end{aligned}$$

If we take $\sigma(y) = k(y)/\epsilon$, this will yield zero. So for this choice of σ, $\omega_{SF} - T_\sigma^*\omega = i\partial\bar\partial\varphi$ for some function φ on X_{U^*}.

Note that a holomorphic section of f over U^* always extends to a holomorphic section of f on U. q.e.d.

Theorem 4.4. *Let $f: X\to\mathbf{P}^1$ be an elliptically fibred* K3 *surface with a holomorphic section and* 24 *singular fibres over $\Delta = \{p_1,\dots,p_{24}\}$ as above. Then there exists open sets $U_1^i\subseteq U_2^i\subseteq\mathbf{P}^1$, $i=1,\dots,24$, each diffeomorphic to a disc, $U_j^i\cap\Delta = \{p_i\}$, positive constants $D_1,\dots,D_6$ and ϵ_0 such that, for all $\epsilon<\epsilon_0$, there exists a Kähler metric ω_ϵ on X with the following properties:*

(1) $$\int_X\omega_\epsilon^2 = \int_X(\mathrm{Re}\,\Omega)^2 = \int_X(\mathrm{Im}\,\Omega)^2.$$

(2) $$\int_{X_b}\omega_\epsilon = \epsilon.$$

(3) $$\omega_\epsilon|_{f^{-1}(\mathbf{P}^1\setminus\bigcup_i U_2^i)} = \omega_{SF}.$$

(4) $\omega_\epsilon|_{f^{-1}(U_1^i)} = T_{\sigma_i}^*\omega_{OV}$, *where* ω_{OV} *is an Ooguri–Vafa metric and* T_{σ_i} *denotes translation by some holomorphic section* σ_i.

(5) *If* $F_\epsilon = \log\left(\frac{\Omega\wedge\bar\Omega/2}{\omega_\epsilon^2}\right)$, *then*

$$\|F_\epsilon\|_{C^0} \le D_1 e^{-D_2/\epsilon}$$

and

$$\|\Delta F_\epsilon\|_{C^0} \le D_1 e^{-D_2/\epsilon}$$

where Δ *denotes the Laplacian with respect to the metric* ω_ϵ.

(6) $$inf_v\{Ric(v,v)\,|\,|v|_{\omega_\epsilon}=1\} \ge -D_3 e^{-D_4/\epsilon}.$$

(7) *With the Riemannian metric induced by* ω_ϵ, $Diam(X) \le D_5\epsilon^{-1/2}$.

(8) *If* R *denotes the Riemann curvature tensor, then*

$$\|R\|_{C^0} \le D_6\epsilon^{-1}\log\epsilon^{-1},$$

$$\|R\|_{C^0} \to \infty \text{ as } \epsilon \to 0,$$

and on any non-singular fibre, there exists a constant C *depending on the fibre such that*

$$\|R\| \le C\epsilon.$$

Proof. Let $p \in \Delta$; we fix our attention near this point. Choosing a holomorphic coordinate y in a neighbourhood of p, we can express the holomorphic periods of f as $\tau_1(y), \tau_2(y)$, where τ_1 is taken to be single valued. In $\mathcal{T}_B^*$, this coincides with the holomorphic differential $\tau_1(y)dy$. Locally, there exists a function $g(y)$ with $dg = \tau_1(y)dy$; since $\tau_1(p) \ne 0$, we can use g as a local holomorphic coordinate in a neighbourhood of p. Replacing y by g, we can then assume that $\tau_1(y) = 1$ and also that $y = 0$ at p. By results of §3, we can then construct for all ϵ less than some ϵ_0, a metric ω_{OV} on $f^{-1}(U)$, for some $U = \{y\,|\,|y| < r\}$, for some r which only depends on the period τ_2 and ϵ_0, but not ϵ. Fix $r_1 < r_2 < r$, and let $U_i = \{y\,|\,|y| < r_i\}$. If $p = p_j$, we set $U_i^j = U_i$.

Remaining focused near p, let $\psi : (0, (r_2+\delta)^2) \to [0,1]$ be a fixed C^∞ cut-off function, with $\psi(r^2) = 1$ for $r \le r_1$, $\psi(r^2) = 0$ for $r \ge r_2$.

Now apply Lemma 4.3 with $\omega = \omega_{OV}$. Then there exists a holomorphic section σ of f over U, such that

$$\omega_{SF} - T_\sigma^* \omega_{OV} = i\partial\bar{\partial}\varphi$$

for some function φ on X_{U^*}. We can then glue $T_\sigma^*\omega_{OV}$ and ω_{SF} by

$$\omega_{\text{new}} = \omega_{SF} - i\partial\bar{\partial}(\psi(|y|^2)\varphi).$$

For $|y| \geq r_2$, ω_{new} coincides with ω_{SF}; for $|y| \leq r_1$, ω_{new} coincides with $T_\sigma^*\omega_{OV}$. This can be done at each singular fibre, obtaining a global closed real $(1,1)$ form ω_{new}.

We still need to check ω_{new} is positive. One calculates that on X_{U^*},

$$\begin{aligned}\omega_{\text{new}} =& (1-\psi(|y|^2))\omega_{SF} + \psi(|y|^2)T_\sigma^*\omega_{OV} - i\Big(\psi'(|y|^2)\bar{y}\; dy\wedge\bar{\partial}\varphi \\ &+ \psi'(|y|^2)y\;\partial\varphi\wedge d\bar{y} + \psi''(|y|^2)|y|^2\varphi\; dy\wedge d\bar{y}\Big).\end{aligned}$$

The sum of the first two terms is positive, so we need to make sure the last three terms are small. Thus we need to control the size of φ. To do so, we need to show $\omega_{SF} - T_\sigma^*\omega_{OV}$ is small. Now

$$\omega_{SF} = \frac{i}{2}W_0^{-1}(\epsilon\;\vartheta_{\mathrm{v}}\wedge\bar{\vartheta}_{\mathrm{v}} + \epsilon^{-1}\;\vartheta_{\mathrm{h}}\wedge\bar{\vartheta}_{\mathrm{h}}).$$

On the other hand, we can assume σ is the zero section by having chosen the right holomorphic section in Construction 2.6 to perform the transformation between coordinates, and write, with $W = V^{-1}$,

$$\begin{aligned}\omega_{OV} =& \frac{i}{2}(W(dx + b\,dy)\wedge\overline{(dx+b\,dy)} + W^{-1}\;dy\wedge d\bar{y}) \\ =& \frac{i}{2}\Big(W(dx+b_0\,dy)\wedge\overline{(dx+b_0\,dy)} + W(b-b_0)\;dy\wedge\overline{(dx+b_0\,dy)} \\ &+ W(\bar{b}-\bar{b}_0)(dx+b_0\,dy)\wedge d\bar{y} + (W|b-b_0|^2 + W^{-1})\;dy\wedge d\bar{y}\Big) \\ =& \frac{i}{2}\Big(WW_0^{-2}\;\vartheta_{\mathrm{v}}\wedge\bar{\vartheta}_{\mathrm{v}} + \frac{W}{W_0}(b-b_0)\;\vartheta_{\mathrm{h}}\wedge\bar{\vartheta}_{\mathrm{v}} \\ &+ \frac{W}{W_0}(\bar{b}-\bar{b}_0)(\vartheta_{\mathrm{v}}\wedge\bar{\vartheta}_{\mathrm{h}}) + (W|b-b_0|^2 + W^{-1})\;\vartheta_{\mathrm{h}}\wedge\bar{\vartheta}_{\mathrm{h}}\Big).\end{aligned}$$

Thus we are applying Lemma 4.1 with $\alpha = \epsilon W_0^{-1} - WW_0^{-2}$, $\beta = \frac{W}{W_0}(b_0 - b)$, and $\gamma = \epsilon^{-1}W_0^{-1} - W^{-1} - W|b-b_0|^2$. Now we work in Gibbons–Hawking coordinates using the fact that α, β and γ are invariant under the action $x_1 \mapsto x_1 + t$. So if we bound the C^k norm of α, β

and γ as functions on $U_{r_1,r_2} \times \mathbf{R}/\epsilon\mathbf{Z}$, with coordinates y and u, we can apply (2.5) to bound the C^k norms of α, β, γ with respect to the coordinates x and y. The interpolation inequalities then give $C^{k',\alpha'}$ bounds for any $k' < k$.

First look at α. Now

$$\begin{aligned} V &= V_0 + \frac{1}{4\pi\epsilon}\log|y|^2 + \epsilon^{-1}\operatorname{Im}(\tau_2) \\ &= \epsilon^{-1}\operatorname{Im}(\tau_2) + g(u,y) \\ &= \epsilon^{-1}W_0^{-1} + g(u,y), \end{aligned}$$

where $g(u,y)$ is, by Lemma 3.1 (c), a harmonic function on $U_{r_1,r_2} \times \mathbf{R}/\epsilon\mathbf{Z}$ with $\|g\|_{C^0}$ being $O(e^{-C/\epsilon})$. It then follows from [11], Theorem 2.10, that for each k, $\|g\|_{C^k}$ is also $O(e^{-C/\epsilon})$. Thus

$$\begin{aligned} \alpha &= \epsilon W_0^{-1} - WW_0^{-2} \\ &= \frac{\epsilon}{W_0} - \frac{\epsilon}{W_0 + \epsilon g W_0^2} \\ &= \frac{\epsilon^2 g W_0^2}{W_0(W_0 + \epsilon g W_0^2)}. \end{aligned}$$

Now using the fact that the denominator is bounded above and below, and observing that any derivative of α will have, in the numerator, only terms which include factors of g or its derivatives, we see that for each k, $\|\alpha\|_{C^k}$ is $O(e^{-C/\epsilon})$.

Next look at β. By construction,

$$0 = \int_{X_b} \beta dx_1 \wedge dx_2 = \int_{X_b} \beta\theta_0 \wedge V du = \int_0^\epsilon W_0^{-1}(b - b_0)du.$$

Thus $b - b_0$, which is a function on $U_{r_1,r_2} \times S^1(\epsilon)$ (even though b and b_0 are not) has no constant term in its Fourier expansion. Both b and b_0, however, are quasi-periodic in u, i.e., consist of a linear plus periodic term. Let $\tilde{b}$ and $\tilde{b}_0$ denote the periodic part (not including the constant term) of b and b_0 respectively. Then $b - b_0 = \tilde{b} - \tilde{b}_0$, and we can bound the C^k norm of $\tilde{b}$ and $\tilde{b}_0$ separately. For example, by (2.6), $\partial_u \tilde{b} = 2i\partial_y g(u,y)$, which is $O(e^{-C/\epsilon})$, and then the Poincaré inequality implies $\|\tilde{b}\|_{C^0}$ is $O(e^{-C/\epsilon})$. Similar arguments apply to $\|\tilde{b}_0\|_{C^0}$, using the explicit form for b_0, and from this one obtains $O(e^{-C/\epsilon})$ bounds on $\|\beta\|_{C^k}$ for each k. Similar arguments apply for $\gamma = -g(u,y) - W|b - b_0|^2$.

Thus the last three terms of ω_{new} are $O(e^{-C/\epsilon})$, and since the sum of the first two terms have eigenvalues $O(\epsilon^{-1})$ and $O(\epsilon)$, it is clear that for sufficiently small ϵ, $\omega_{\rm new}$ is positive definite. On the other hand, it is not clear that $\int_X \omega_{\rm new}^2 = \int_X (\mathrm{Re}\,\Omega)^2$. However, by construction $\omega_{\rm new}^2 = (\mathrm{Re}\,\Omega)^2$ outside of $f^{-1}(U_{r_1,r_2})$, and $\omega_{\rm new}^2$ and $(\mathrm{Re}\,\Omega)^2$ differ only by $O(e^{-C/\epsilon})$ on $f^{-1}(U_{r_1,r_2})$, so $\int_X \omega_{\rm new}^2 - \int_X (\mathrm{Re}\,\Omega)^2 = O(e^{-C/\epsilon})$. Now noting that

$$([\omega_{\rm new}] + aE)^2 = [\omega_{\rm new}]^2 + 2a\epsilon,$$

we can find a two-form α on B supported on U_{r_1,r_2} with C^k norm $O(e^{-C/\epsilon})$ such that $\int_X (\omega_{\rm new} + f^*\alpha)^2 = \int_X (\mathrm{Re}\,\Omega)^2$. Set $\omega_\epsilon = \omega_{\rm new} + f^*\alpha$. Because α is still small, ω_ϵ is still positive and defines the desired Kähler metric. Properties (1)–(4) are then satisfied by construction. Note that $F_\epsilon = \log(\Omega \wedge \bar\Omega / 2\omega_\epsilon^2)$ is zero outside of $f^{-1}(U_{r_1,r_2})$, and ω_ϵ is within $O(e^{-C/\epsilon})$ of ω_{SF} on $f^{-1}(U_{r_1,r_2})$. Thus $\|F_\epsilon\|_{C^0}$ is $O(e^{-C/\epsilon})$. The same is true of $\|F_\epsilon\|_{C^2}$, and since the coefficients of the metric are at worst $O(\epsilon)$ or $O(\epsilon^{-1})$ in $f^{-1}(U_{r_1,r_2})$, $\|\Delta F_\epsilon\|_{C^0}$ is also $O(e^{-C/\epsilon})$. Furthermore, the Ricci form is $i\partial\bar\partial F_\epsilon$, which is $O(e^{-C/\epsilon})$. This gives (5) and (6).

To bound the diameter of X with the metric ω_ϵ, first restrict the metric to the zero section σ_0 of f. Identifying σ_0 with the base B, we note that on $B \setminus \bigcup U_2^i$, the Kähler form of this restricted metric is $\frac{i}{2}(\epsilon W_0)((\epsilon W_0)^{-2} + |b_0|^2) dy \wedge d\bar y$. But $b_0 = 0$ on σ_0, so this is just $\frac{i}{2}\epsilon^{-1} W_0^{-1} dy \wedge d\bar y$. Let D be the diameter of $B \setminus \bigcup U_2^i$ under the metric $\frac{i}{2} W_0^{-1} dy \wedge d\bar y$; this is independent of ϵ. Thus $Diam(B \setminus \bigcup U_2^i) = D\epsilon^{-1/2}$ under the metric induced by ω_ϵ. On the other hand, the diameter of each fibre over $B \setminus \bigcup U_2^i$ is bounded by some constant times $\epsilon^{1/2}$, so $Diam(f^{-1}(B \setminus \bigcup U_2^i)) \le D'\epsilon^{-1/2}$ for sufficiently small ϵ. Then applying Corollary 3.7 to each $f^{-1}(U_2^i)$, (keeping in mind that the changes to the metric in the gluing area are negligible for small ϵ), we see in fact that

$$Diam(X) \le D'\epsilon^{-1/2} + D''\epsilon^{-1/2},$$

which we can always bound by $D_5\epsilon^{-1/2}$ for some constant D_5. Hence (7).

Finally, (8) follows immediately from Proposition 3.8, Remark 2.7, and the fact that any non-singular fibre has a neighbourhood in which ω_ϵ is arbitrarily close to the semi-flat metric for ϵ sufficiently small.

q.e.d.

Theorem 4.5. *Let $j : J \to \mathbf{P}^1$ be an elliptically fibred* K3 *surface with section and* 24 *singular fibres over $\Delta = \{p_1, \dots, p_{24}\}$ as above. Then there exists open sets $U_1^i \subseteq U_2^i \subseteq \mathbf{P}^1$, $i = 1, \dots, 24$, each diffeomorphic to a disc, $U_j^i \cap \Delta = \{p_i\}$, positive constants D_1, D_2, D_3, D_4, D_5 and ϵ_0 such that, for all $\epsilon < \epsilon_0$, for any elliptic fibration $f : X \to \mathbf{P}^1$*

with Jacobian $j : J \to \mathbf{P}^1$ *with holomorphic 2-form* Ω *with* $[\mathrm{Re}\,\Omega]^2 = [\mathrm{Re}\,\Omega_J]^2$*, and for any Kähler class* $[\omega_\epsilon]$ *on* X *with* $[\omega_\epsilon].X_b = \epsilon$ *and* $[\omega_\epsilon]^2 = [\mathrm{Re}\,\Omega]^2 = [\mathrm{Im}\,\Omega]^2$*, there exists a Kähler metric* ω_ϵ *representing* $[\omega_\epsilon]$ *on* X *with the following properties:*

(1) $\omega_\epsilon|_{f^{-1}(\mathbf{P}^1 \setminus \bigcup_i U_2^i)}$ *is a semi-flat metric (not necessarily the standard one).*

(2) $\omega_\epsilon|_{f^1(U_1^i)} = T_{\sigma_i}^* \omega_{OV}$*, where* ω_{OV} *is an Ooguri–Vafa metric and* T_{σ_i} *denotes translation by a (not necessarily holomorphic) section.*

(3) *If* $F_\epsilon = \log\left(\frac{\Omega\wedge\bar{\Omega}/2}{\omega_\epsilon^2}\right)$*, then*

$$\|F_\epsilon\|_{C^0} \leq D_1 e^{-D_2/\epsilon}$$

and

$$\|\Delta F_\epsilon\|_{C^0} \leq D_1 e^{-D_2/\epsilon},$$

where Δ *denotes the Laplacian with respect to* ω_ϵ*.*

(4)
$$inf_v\{Ric(v,v) \mid |v|_{\omega_\epsilon} = 1\} \geq -D_3 e^{-D_4/\epsilon}.$$

(5) *With the Riemannian metric induced by* ω_ϵ*,* $Diam(X) \leq D_5 \epsilon^{-1/2}$*.*

(6) *If* R *denotes the Riemann curvature tensor, then*

$$\|R\|_{C^0} \leq D_6 \epsilon^{-1} \log \epsilon^{-1},$$

$$\|R\|_{C^0} \to \infty \text{ as } \epsilon \to 0,$$

and on any non-singular fibre, there exists a constant C *depending on the fibre such that*

$$\|R\| \leq C\epsilon.$$

Proof. First note that as in §1, we think of X as a K3 surface obtained from J simply by altering the holomorphic 2-form Ω_J on J to $\Omega_J + j^*\alpha$, for some 2-form α on $\mathbf{P}^1$. Thus it is natural to identify the underlying manifolds X and J, and we are only changing the complex structure. So we can think of $[\omega_\epsilon] \in H^2(J, \mathbf{R})$, and in particular, in the notation of §1, we can write

$$[\omega_\epsilon] = \epsilon(\sigma_0 + \mathbf{B}) \bmod E$$

for some $\mathbf{B} \in E^{\perp}/E \otimes \mathbf{R}$. Furthermore, given the values of the classes $[\omega_\epsilon]$ and $[\Omega]$ modulo E, and given that $[\omega_\epsilon], [\operatorname{Re}\Omega], [\operatorname{Im}\Omega]$ form a hyperkähler triple, the classes $[\omega_\epsilon]$ and $[\Omega]$ are completely determined. Thus the choice of $\mathbf{B}$ uniquely determines the Kähler class and complex structure.

We modify the role of $\mathbf{B}$ slightly. Let ω_ϵ^0 be the Kähler form on J provided by Theorem 4.4. Then in fact we can write

$$[\omega_\epsilon] - [\omega_\epsilon^0] = \epsilon\mathbf{B} \bmod E$$

for some $\mathbf{B} \in E^{\perp}/E \otimes \mathbf{R}$. The class $\mathbf{B}$ still determines all data. So fix this class in $E^{\perp}/E \otimes \mathbf{R}$. This latter vector space is naturally identified with $H^1(\mathbf{P}^1, R^1 j_* \mathbf{R})$. Consider the exact sequence

$$0 \to R^1 j_* \mathbf{R} \to C^\infty(\mathcal{T}_B^*) \to \mathcal{F} \to 0.$$

Here $C^\infty(\mathcal{T}_B^*)$ denotes the sheaf of C^∞ sections of $\mathcal{T}_B^*$, and the first map is induced by tensoring the inclusion $R^1 j_* \mathbf{Z} \hookrightarrow \mathcal{T}_B^*$ with $\mathbf{R}$. This gives a surjection $H^0(\mathbf{P}^1, \mathcal{F}) \to H^1(\mathbf{P}^1, R^1 j_* \mathbf{R})$. Now a section of $\mathcal{F}$ is given by an open covering $\{U_i\}$ of $\mathbf{P}^1$ and sections $\sigma_i \in \Gamma(U_i, C^\infty(\mathcal{T}_B^*))$ with $\sigma_i - \sigma_j \in \Gamma(U_i, R^1 j_* \mathbf{R})$. This open covering $\{U_i\}$ can always be chosen with the following properties:

(1) Each U_i contains at most one point of Δ, and if $p_j \in U_i$, then $\overline{U_2^j} \subseteq U_i$.

(2) Each U_i is convex with respect to some metric on $\mathbf{P}^1$, so that all multiple intersections of the U_i's are contractible.

(3) If $U_i \cap \Delta = \phi$, then $U_i \cap \bigcup_j U_2^j = \phi$.

In fact, fixing one such open covering, all sections of $\mathcal{F}$ can be represented over this open covering.

Now represent $\mathbf{B}$ by (U_i, σ_i), and let $T_{\sigma_i} : f^{-1}(U_i) \to f^{-1}(U_i)$ denote translation by the section σ_i. Now consider the forms $T_{\sigma_i}^* \omega_\epsilon^0$ and $T_{\sigma_i}^* \Omega_J$. On $f^{-1}(U_i \cap U_j)$, ω_ϵ^0 is the standard semi-flat metric, by condition (3) on the open covering above, and since $\sigma_j - \sigma_i$ is a flat section with respect to the Gauss-Manin connection, $T_{\sigma_j - \sigma_i}$ is an isometry, i.e., $T_{\sigma_j-\sigma_i}^* \omega_\epsilon^0 = \omega_\epsilon^0$, $T_{\sigma_j-\sigma_i}^* \Omega_J = \Omega_J$ (see Example 2.2). Thus

$$T_{\sigma_i}^* \omega_\epsilon^0 = T_{\sigma_i}^* T_{\sigma_j-\sigma_i}^* \omega_\epsilon^0 = T_{\sigma_j}^* \omega_\epsilon^0,$$

and similarly $T^*_{\sigma_i}\Omega_J = T^*_{\sigma_j}\Omega_J$. Thus these forms glue, to give global forms ω_ϵ, Ω on the manifold J. The 2-form Ω satisfies $\Omega\wedge\Omega = 0$, and thus induces a new complex structure on J. An easy local calculation shows that $\Omega = \Omega_J + j^*\alpha'$, for some 2-form α' on $\mathbf{P}^1$. Furthermore, it is clear that

$$\int_{J_b}\omega_\epsilon = \epsilon,\ \int_J \omega_\epsilon\wedge\Omega = 0, \text{ and } \int_J \omega_\epsilon^2 = \int_J(\operatorname{Re}\Omega)^2 = \int_J(\operatorname{Im}\Omega)^2.$$

Thus the cohomology classes $[\omega_\epsilon]$, $[\operatorname{Re}\Omega]$ and $[\operatorname{Im}\Omega]$ form a hyperkähler triple. If we show that

$$[\omega_\epsilon] - [\omega_\epsilon^0] = \epsilon\mathbf{B} \bmod E,$$

then we have constructed a Kähler form in the desired class (deducing moreover that the new complex structure is just that obtained from X).

To see the required identity, observe that we have an exact sequence

$$H^2_{f^{-1}(\Delta)}(X,\mathbf{R})\xrightarrow{\varphi}H^2(X,\mathbf{R})\longrightarrow H^2(X_0,\mathbf{R})\longrightarrow H^3_{f^{-1}(\Delta)}(X,\mathbf{R}).$$

Now $H^2_{f^{-1}(\Delta)}(X,\mathbf{R}) = H^0(f^{-1}(\Delta),\mathbf{R}) = \mathbf{R}^{24}$, and the image of φ is just the one-dimensional subspace of $H^2(X,\mathbf{R})$ spanned by $[E]$, the class of a fibre. Thus it is enough to show that

$$[\omega_\epsilon|_{X_0}] - [\omega_\epsilon^0|_{X_0}] = \epsilon\mathbf{B} \in H^2(X,\mathbf{R})/E \subseteq H^2(X_0,\mathbf{R}).$$

Now on X_0, ω_ϵ^0 is cohomologous to ω_{SF} by construction, and ω_ϵ is cohomologous to a Kähler form ω'_{SF} obtained in the same way as ω_ϵ via translation and gluing, but starting from ω_{SF} rather than ω_ϵ^0. Thus it is enough to show that on X_0

$$[\omega'_{SF}] - [\omega_{SF}] = \epsilon\mathbf{B}.$$

Over an open set U_i, write

$$\omega_{SF} = \frac{i}{2}W_0^{-1}(\epsilon\,\vartheta_{\rm v}\wedge\bar\vartheta_{\rm v} + \epsilon^{-1}\,\vartheta_{\rm h}\wedge\bar\vartheta_{\rm h}).$$

Now

$$T^*_{\sigma_i}(\vartheta_{\rm v}) = \vartheta_{\rm v} + W_0(\partial_y\sigma_i + b_0(\sigma_i(y),y))\,\vartheta_{\rm h} + W_0(\partial_{\bar y}\sigma)\,\bar\vartheta_{\rm h},$$

so over U_i

$$\begin{aligned}\omega'_{SF} - \omega_{SF} =& T^*_{\sigma_i}\omega_{SF} - \omega_{SF}\\ =& \frac{\epsilon i}{2}\Big((\partial_y \sigma_i + b_0(\sigma_i(y), y))\, \vartheta_{\mathrm{h}} \wedge \bar\vartheta_{\mathrm{v}} + \partial_{\bar y}\sigma_i \;\; \bar\vartheta_{\mathrm{h}} \wedge \bar\vartheta_{\mathrm{v}}\\ &+ \partial_y \bar\sigma_i \;\; \vartheta_{\mathrm{v}} \wedge \vartheta_{\mathrm{h}}\\ &+ (\partial_{\bar y}\bar\sigma_i + \bar b_0(\sigma_i(y), y))\, \vartheta_{\mathrm{v}} \wedge \bar\vartheta_{\mathrm{h}}\Big) \quad \bmod \vartheta_{\mathrm{h}} \wedge \bar\vartheta_{\mathrm{h}}\\ =& \frac{\epsilon i}{2} d(\sigma_i \bar\vartheta_{\mathrm{v}} - \bar\sigma_i \vartheta_{\mathrm{v}}) \quad \bmod \vartheta_{\mathrm{h}} \wedge \bar\vartheta_{\mathrm{h}}\end{aligned}$$

as can be easily seen using (4.1) and a calculation similar to that of (4.2).

How does the two-form $\omega'_{SF} - \omega_{SF}$ determine an element of $H^1(B_0, R^1 f_{0*}\mathbf{R})$? Given the open covering $\{U_i\}$ of B_0, if $\omega'_{SF} - \omega_{SF}$ is an exact form on each $f^{-1}(U_i)$, we can write $\omega'_{SF} - \omega_{SF} = d\alpha_i$ for some 1-form α_i on $f^{-1}(U_i)$. Then on $f^{-1}(U_i \cap U_j)$, $\alpha_i - \alpha_j$ is closed, and hence determines an element of $H^1(f^{-1}(U_i \cap U_j), \mathbf{R}) = \Gamma(U_i \cap U_j, R^1 f_{0*}\mathbf{R})$ for our choice of open covering. Now we have found such α_i modulo $\vartheta_{\mathrm{h}} \wedge \bar\vartheta_{\mathrm{h}}$, so $\omega'_{SF} - \omega_{SF}$ is represented by a Čech cocycle for $R^1 f_{0*}\mathbf{R}$ given by

$$(U_i \cap U_j, \frac{\epsilon i}{2}((\sigma_i - \sigma_j)\bar\vartheta_{\mathrm{v}} - (\bar\sigma_i - \bar\sigma_j)\vartheta_{\mathrm{v}})).$$

By integrating this one-form over the periods, one sees this is precisely the section of $\Gamma(U_i \cap U_j, R^1 f_{0*}\mathbf{R})$ given by $\epsilon(\sigma_i - \sigma_j)$. Thus $\omega'_{SF} - \omega_{SF}$ represents the class $\epsilon\mathbf{B}$.

Finally, properties (1)–(4) and (6) follow immediately from Theorem 4.4, (3)–(6) and (8). On the other hand, the diameter of $f^{-1}(U_i)$ with respect to ω_ϵ is the same as the diameter of $f^{-1}(U_i)$ with respect to the metric ω^0_ϵ. Since there are a fixed number of U_i's, the estimate on the diameter continues to hold from Theorem 4.4, (7). q.e.d.

Remark 4.6. In the construction of the proof of Theorem 4.5, we may sometimes want to be able to control the sections σ_i we use to represent the class $\mathbf{B}$. This can be done as follows. The class of $\mathbf{B}$ depends on the choice of the zero section σ_0. Changing the class of the zero section changes $\mathbf{B}$ by an element of $E^\perp/E$. Thus $\mathbf{B}$ really should be thought of as living in $E^\perp/E \otimes \mathbf{R}/\mathbf{Z}$. (See [13] or [14], §7.) Thus in some cases we might want to choose a compact set F in $E^\perp/E \otimes \mathbf{R}$ containing a fundamental domain for $E^\perp/E$. We can then choose for each $\mathbf{B} \in F$ a representative (σ_i) of $\mathbf{B}$ with various norms, as required, bounded by constants independent of $\mathbf{B} \in F$. We will say we are choosing the B-field $\mathbf{B}$ in a fundamental domain for the B-field.

5. Ricci-flat metrics

We will continue with the setting of Theorem 4.5. In other words, we have a fixed Jacobian elliptic fibration $j : J \to \mathbf{P}^1$. Our goal is to show that there exists an ϵ_0 such that for any $f : X \to \mathbf{P}^1$ with Jacobian $j : J \to \mathbf{P}^1$, and any $\epsilon < \epsilon_0$, and any metric ω_ϵ given by Theorem 4.5, there exists a function u_ϵ such that $\omega_\epsilon + i\partial\bar{\partial}u_\epsilon$ is a Ricci-flat metric, and furthermore that u_ϵ is very small in the $C^{k,\alpha}$ sense. Of course, that such a u_ϵ exists is Yau's proof of the Calabi conjecture. Here we apply standard techniques, following [20], to obtain control of u_ϵ. As mentioned in the introduction, the only subtle difference is that as $\epsilon \to 0$, $Diam(X) \to \infty$, and this requires us to be a bit more careful in estimating constants. However, we follow [20] closely.

More precisely, we wish to solve the equations

$$
\begin{gathered}
(\omega_\epsilon + i\partial\bar{\partial}u_\epsilon)^2 = e^{F_\epsilon}\omega_\epsilon^2 \\
\int_X u_\epsilon \omega_\epsilon^2 = 0.
\end{gathered}
\tag{5.1}
$$

Here $F_\epsilon = \log(\frac{\Omega\wedge\bar{\Omega}/2}{\omega_\epsilon^2})$. By [35], we know such a u_ϵ exists.

We begin with some standard lemmas. For convenience, we will assume $Vol(X) = 1$. This can be achieved since we are holding the volume of X constant anyway, so we just scale the original Ω so that $\int_J (\operatorname{Re}\Omega)^2 = 1$.

Lemma 5.1. *Let X, ω_ϵ be as in Theorem 4.5. Assume $Vol(X) = 1$. Then there exists a function $I(\epsilon)$ depending only on ϵ and J with $I(\epsilon) \geq C\epsilon^5$, C depending only on J, such that:*

(1) *For any function f on X such that $\int_X f\omega_\epsilon^2 = 0$,*

$$\|df\|_2^2 \geq I(\epsilon)\|f\|_4^2.$$

(2) *For any function f on X,*

$$\|df\|_2^2 \geq I(\epsilon)(\|f\|_4^2 - \|f\|_2^2).$$

Proof. These are the standard Sobolev inequalities, but we just need to be careful about the constants. We have, by [23], Lemmas 1 and 2, for a function f such that $\int_X f\omega_\epsilon^2 = 0$,

$$\|df\|_2^2 \geq C_2 \|f\|_4^2$$

while for an arbitrary function, we have

$$\|df\|_2^2 \geq D(4)C_2(\|f\|_4^2 - \|f\|_2^2).$$

Here, we are using Li's notation for the constants $C_0, C_1, C_2, D(n)$ and the fact that the volume is 1 and the dimension is 4. Again by [23], $D(4)$ is an absolute constant, $C_2 = D(4)C_0^{1/2}$, and $2C_1 \geq C_0 \geq C_1$, where C_1 is the constant in the isoperimetric inequality

$$C_1(\min\{V(M_1), V(M_2)\})^3 \leq V(N)^4$$

where V denotes volume, and N is any codimension one submanifold of X dividing it into M_1 and M_2. In [9], Croke calls this constant $\Phi(M)$.

Theorem 13 from [9] says that

$$C_1 \geq C_4 \left(\int_0^{Diam(X)} ((\sqrt{1/K}) \sinh(\sqrt{K} r))^3 dr \right)^{-5},$$

where C_4 again is an absolute constant, and $Ric(X) \geq -3K$, where $3K \leq D_3 e^{-D_4/\epsilon}$ by Theorem 4.5, (4). Now the integral is bounded above by

$$Diam(X)(\sqrt{1/K} \sinh(\sqrt{K} Diam(X)))^3.$$

Now by Theorem 4.5, (5), $\sqrt{K} Diam(X) \to 0$ as $\epsilon \to 0$, so for sufficiently small ϵ, using the first term of the Taylor series expansion of sinh, this is bounded by

$$C_5 Diam(X)^4 \leq C_6 \epsilon^{-2}$$

so $C_1 \geq C_7 \epsilon^{10}$, hence $C_0 \geq C_8 \epsilon^{10}$ and we can take

$$I(\epsilon) = \min(D(4), 1)C_2 \geq C_9 \epsilon^5. \qquad \text{q.e.d.}$$

Lemma 5.2. (The C^0 estimate.) *Let u_ϵ be the solution to Equations (5.1). There exists a constant C depending only on J, such that for all $\epsilon < \epsilon_0$, (ϵ_0, D_2 as in Theorem 4.5)*

$$\|u_\epsilon\|_\infty \leq C\epsilon^{-5} e^{-D_2/\epsilon}.$$

Proof. The starting point is the inequality (23) of [20]:

$$\int_X |d|u_\epsilon|^{p/2}|^2 \leq Ap \int_X |F_\epsilon||u_\epsilon|^{p-1}.$$

All integrals are with measure ω_ϵ^2. See also the expanded derivation of this inequality in [24]. One can check the constant A is independent of p and ϵ.

We apply this first with $p = 2$. The left-hand side is $\|du_\epsilon\|_2^2 \geq A_1\epsilon^5\|u_\epsilon\|_4^2$ by Lemma 5.1, (1), so by applying Hölder's inequality to the right-hand side, we get

$$\left(\int_X |u_\epsilon|^4\right)^{1/2} \leq A_2\epsilon^{-5}\left(\int_X |F_\epsilon|^{4/3}\right)^{3/4}\left(\int_X |u_\epsilon|^4\right)^{1/4}$$

or

$$\begin{aligned} \|u_\epsilon\|_4 &\leq A_3\epsilon^{-5}\left(\int_X |F_\epsilon|^{4/3}\right)^{3/4} \\ &\leq C_1\epsilon^{-5}e^{-D_2/\epsilon}. \end{aligned} \tag{5.2}$$

Now for arbitrary p, using Lemma 5.1, (2)

$$\begin{aligned} \|u_\epsilon\|_{2p}^p = \left(\int_X |u_\epsilon^{2p}|\right)^{1/2} &= \|\,|u_\epsilon|^{p/2}\|_4^2 \\ &\leq A_4\epsilon^{-5}\|\,d|u_\epsilon|^{p/2}\|_2^2 + \|\,|u_\epsilon|^{p/2}\|_2^2 \\ &\leq A_5p\epsilon^{-5}\left(\int_X |F_\epsilon|\,|u_\epsilon|^{p-1}\right) + \|\,|u_\epsilon|^{p/2}\|_2^2. \end{aligned}$$

Applying Hölder's inequality to the first term, we have, with $q = p$, $q' = 1/(1-1/p) = p/(p-1)$,

$$\begin{aligned} \int_X |F_\epsilon|\,|u_\epsilon|^{p-1} &\leq \|F_\epsilon\|_p\,\|\,|u_\epsilon|^{p-1}\|_{p/(p-1)} \\ &= \|F_\epsilon\|_p\,\|u_\epsilon\|_p^{p-1}, \end{aligned}$$

so

$$\begin{aligned} \|u_\epsilon\|_{2p}^p &\leq A_5p\epsilon^{-5}\|F_\epsilon\|_p\,\|u_\epsilon\|_p^{p-1} + \|u_\epsilon\|_p^p \\ &= \left(A_5p\epsilon^{-5}\|F_\epsilon\|_p + \|u_\epsilon\|_p\right)\,\|u_\epsilon\|_p^{p-1}. \end{aligned} \tag{5.3}$$

Now we claim that if we set $p_n = 2^{n+1}$, there exists constants C_n such that

$$\|u_\epsilon\|_{p_n} \le C_n \epsilon^{-5} e^{-D_2/\epsilon}$$

for all $\epsilon < \epsilon_0$. This holds for $n = 1$ by (5.2). Suppose it holds for a given n. Then by (5.3),

$$\begin{aligned}\|u_\epsilon\|_{p_{n+1}}^{p_n} &\le (A_5 p_n \epsilon^{-5} D_1 e^{-D_2/\epsilon} + C_n \epsilon^{-5} e^{-D_2/\epsilon})(C_n \epsilon^{-5} e^{-D_2/\epsilon})^{p_n - 1} \\ &\le \begin{cases} (A_5 D_1 2^{n+1} + 1)(C_n \epsilon^{-5} e^{-D_2/\epsilon})^{p_n} & \text{if } C_n \ge 1; \\ (A_5 D_1 2^{n+1} + 1)(\epsilon^{-5} e^{-D_2/\epsilon})^{p_n} & \text{if } C_n \le 1. \end{cases}\end{aligned}$$

Thus we can take

$$C_{n+1} \le \begin{cases} (A_5 D_1 2^{n+1} + 1)^{2^{-(n+1)}} C_n & \text{if } C_n \ge 1; \\ (A_5 D_1 2^{n+1} + 1)^{2^{-(n+1)}} & \text{if } C_n \le 1. \end{cases}$$

It then follows as in [20], page 299, that $C_n \le A_6$ for some constant A_6 independent of n and ϵ. Thus we conclude that

$$\|u_\epsilon\|_\infty \le A_6 \epsilon^{-5} e^{-D_2/\epsilon}$$

for all $\epsilon < \epsilon_0$. q.e.d.

Lemma 5.3. (The C^2 estimate.) *Let u_ϵ be the solution to Equations (5.1). There are constants C and ϵ_0 depending only on J (possibly smaller than the ϵ_0 of Theorem 4.5) such that for all $\epsilon < \epsilon_0$,*

$$C^{-1}\omega_\epsilon \le \tilde{\omega}_\epsilon \le C\omega_\epsilon$$

where $\tilde{\omega}_\epsilon = \omega_\epsilon + i\partial\bar{\partial}u_\epsilon$.

Proof. Let $R_\epsilon = \sup_{i\neq j} |R_{i\bar{i}j\bar{j}}|$, where $R_{i\bar{i}j\bar{j}}$ is the holomorphic bisectional curvature of the metric ω_ϵ, and the supremum is over all points of X and unitary bases at each point. Since the holomorphic bisectional curvature determines the curvature, ([6], pg. 76) and $\sup\|R\| \to \infty$ as $\epsilon \to 0$ by Theorem 4.5 (6), we must have $R_\epsilon > 1$ for small ϵ. So if $c_\epsilon = 2R_\epsilon$, then $c_\epsilon + \inf R_{i\bar{i}j\bar{j}} \ge R_\epsilon > 1$. Here the infinum is as before over all unitary frames and points on X. Then [35], (2.22), reads

$$\begin{aligned} e^{c_\epsilon u_\epsilon}\Delta'(e^{-c_\epsilon u_\epsilon}(2+\Delta u_\epsilon)) \ge{} & (\Delta F_\epsilon - 4\inf_{i\neq j} R_{i\bar{i}j\bar{j}}(x)) \\ & - 2c_\epsilon(2+\Delta u_\epsilon) \\ & + (c_\epsilon + \inf_{i\neq j} R_{i\bar{i}j\bar{j}}(x))\, e^{-F_\epsilon}\,(2+\Delta u_\epsilon)^2 \end{aligned}$$

where the infina are now only at the given point (but still over all unitary bases). Here Δ' is the Laplacian with respect to the metric $\omega_\epsilon + i\partial\bar\partial u_\epsilon$, and Δ is the Laplacian with respect to ω_ϵ. Let

$$k(x) = -\inf_{i\neq j} R_{i\bar i j\bar j}(x)/R_\epsilon,$$

so that $|k(x)| \leq 1$.

Now suppose $e^{-c_\epsilon u_\epsilon}(2+\Delta u_\epsilon)$ assumes its maximum at $x \in X$. Then by the maximum principal, the Laplacian must be non-positive there, so at the point x

$$\begin{aligned}
0 \geq & e^{c_\epsilon u_\epsilon}\Delta'(e^{-c_\epsilon u_\epsilon}(2+\Delta u_\epsilon))\\
\geq & (\Delta F_\epsilon + 4k(x)R_\epsilon) - 2c_\epsilon(2+\Delta u_\epsilon) + (c_\epsilon - k(x)R_\epsilon)e^{-F_\epsilon}(2+\Delta u_\epsilon)^2\\
= & (\Delta F_\epsilon + 4k(x)R_\epsilon) - 4R_\epsilon(2+\Delta u_\epsilon) + (2-k(x))R_\epsilon e^{-F_\epsilon}(2+\Delta u_\epsilon)^2\\
= & e^{-F_\epsilon}(2-k(x))R_\epsilon\Bigg[\left((2+\Delta u_\epsilon) - \frac{2e^{F_\epsilon}}{2-k(x)}\right)^2 - \left(\frac{2e^{F_\epsilon}}{2-k(x)}\right)^2\\
& \qquad\qquad + \frac{e^{F_\epsilon}(\Delta F_\epsilon + 4R_\epsilon k(x))}{(2-k(x))R_\epsilon}\Bigg]
\end{aligned}$$

and since $|k(x)| \leq 1$, we get

$$\left|(2+\Delta u_\epsilon) - \frac{2e^{F_\epsilon}}{2-k(x)}\right| \leq \left|\left(\frac{2e^{F_\epsilon}}{2-k(x)}\right)^2 - \frac{e^{F_\epsilon}(\Delta F_\epsilon + 4R_\epsilon k(x))}{(2-k(x))R_\epsilon}\right|^{1/2}.$$

If we are outside of the region where the gluing is taking place, then $F_\epsilon = 0$, so we get

$$\left|(2+\Delta u_\epsilon) - \frac{2}{2-k(x)}\right| \leq \left|\left(\frac{2}{2-k(x)}\right)^2 - \frac{4k(x)}{2-k(x)}\right|^{1/2},$$

or

$$\begin{aligned}
2+\Delta u_\epsilon \leq & \frac{2}{2-k(x)} + \left|\left(\frac{2}{2-k(x)}\right)^2 - \frac{4k(x)}{2-k(x)}\right|^{1/2}\\
= & 2.
\end{aligned}$$

In the gluing region, by Theorem 4.5 (6), there is a constant C_1

$$|k(x)| \leq C_1\epsilon/R_\epsilon.$$

Also in the gluing region, we can use the bounds of Theorem 4.5, (3) on F_ϵ and ΔF_ϵ, to get, for a constant C_2 bounding e^{F_ϵ},

$$\begin{aligned} 2+\Delta u_\epsilon &\leq \frac{2e^{F_\epsilon}}{2-k(x)} + \left| \left(\frac{2e^{F_\epsilon}}{2-k(x)}\right)^2 - \frac{e^{F_\epsilon}(\Delta F_\epsilon + 4R_\epsilon k(x)))}{(2-k(x))R_\epsilon} \right|^{1/2} \\ &\leq \frac{2C_2}{2-C_1\epsilon/R_\epsilon} + \left| \left(\frac{2C_2}{2-C_1\epsilon/R_\epsilon}\right)^2 + \frac{C_2(D_1 e^{-D_2/\epsilon} + 4C_1\epsilon)}{2-C_1\epsilon/R_\epsilon} \right|^{1/2}. \end{aligned}$$

Now as $\epsilon \to 0$, $\epsilon/R_\epsilon \to 0$ by Theorem 4.5, (6). So what we get is

$$(2+\Delta u_\epsilon)(x) \leq C_3$$

for sufficiently small ϵ, and C_3 independent of ϵ.

Now

$$e^{-c_\epsilon u_\epsilon(y)}(2+\Delta u_\epsilon)(y) \leq e^{-c_\epsilon u_\epsilon(x)}(2+\Delta u_\epsilon)(x)$$

for all points y, so

$$\begin{aligned} 2+\Delta u_\epsilon &\leq e^{c_\epsilon(u_\epsilon(y)-u_\epsilon(x))}C_3 \\ &\leq e^{c_\epsilon(\sup u_\epsilon - \inf u_\epsilon)}C_3 \\ &\leq e^{R_\epsilon C_4 \epsilon^{-5} e^{-D_2/\epsilon}} C_3. \end{aligned}$$

By Theorem 4.5, (6),

$$R_\epsilon \epsilon^{-5} e^{-D_2/\epsilon} \to 0,$$

so we get

$$2+\Delta u_\epsilon \leq C_5$$

for sufficiently small ϵ.

Now working in a choice of coordinates z_1, z_2 at a point so that $\partial_{z_1}, \partial_{z_2}$ are unitary at the point with respect to ω_ϵ and which also diagonalizes $\tilde{\omega}_\epsilon = \omega_\epsilon + i\partial\bar{\partial}u_\epsilon$, then

$$(\tilde{\omega}_\epsilon)_{i\bar{j}} = \delta_{ij}(1+(u_\epsilon)_{i\bar{i}}),$$

and each $1+u_{i\bar{i}}$ is positive, so $1+(u_\epsilon)_{i\bar{i}} \leq C_5$, so $\tilde{\omega}_\epsilon \leq C_5\omega_\epsilon$. Also,

$$\tilde{\omega}_\epsilon^2 = \prod (1+(u_\epsilon)_{i\bar{i}})\,\omega_\epsilon^2 = e^{F_\epsilon}\omega_\epsilon^2.$$

Since $1+(u_\epsilon)_{i\bar{i}}$ is bounded above by C_5, it must be bounded below by something close to C_5^{-1}, so changing C_5 slightly if necessary, we get

$$C_5^{-1}\omega_\epsilon \leq \tilde{\omega}_\epsilon \leq C_5\omega_\epsilon. \qquad \text{q.e.d.}$$

We note here that for some purposes, Lemma 5.3 is already sufficient. For example, if we wish to know that the fibres collapse to points as $\epsilon \to 0$, Lemma 5.3 along with Proposition 3.5 tells us the diameter of each fibre under the Ricci-flat metric goes to zero as $\epsilon \to 0$. However, if we wish to get a clearer picture of the asymptotic behaviour of the metric, we need stronger results.

Lemma 5.4. (The $C^{2,\alpha}$ estimate.) *Let u_ϵ be the solution to Equations (5.1). If $U \subseteq B$ is a simply connected open set with $\overline{U} \subseteq B_0 = B \setminus \Delta$, then there exists constants α and ϵ_0 and a polynomial P, depending on J and U, such that*

$$\|u_\epsilon\|_{C^{2,\alpha}} \leq P(\epsilon^{-1})$$

in $f^{-1}(U)$ for all $\epsilon < \epsilon_0$ and $\mathbf{B}$ in a fundamental domain for the B-field (see Remark 4.6). Here the $C^{2,\alpha}$ norm is on $f^{-1}(U)$ as defined in Lemma 4.1, and so α, ϵ_0 and P also depend on the choice of holomorphic coordinate y and fixed bounded domain T', as specified in the proof below.

Proof. We need to apply the basic result of [11], Theorem 17.14. However, we must be careful about the constants. Let $\pi : \mathcal{T}^*_B \to B$ be the projection, and let $T' \subseteq \pi^{-1}(U) \subseteq \mathcal{T}^*_B$ be a fixed bounded domain which contains a fundamental domain of each fibre of f over U. We will be computing norms in the domain T'. To do so, we choose a holomorphic coordinate y in the base, yielding holomorphic canonical coordinates x, y on $\mathcal{T}^*_U$. Now take a bigger open set $T(\epsilon)$ containing T'. This open set will also be bounded, but will depend on ϵ. We choose it as follows. First let $V \subseteq B_0$ be an open set with $\overline{U} \subseteq V$, $\overline{V} \subseteq B_0$, and the holomorphic coordinate y extending to V. Let $T \subseteq \pi^{-1}(V)$ be a domain containing T' and containing a fundamental domain of each fibre over V. Let

$$T(\epsilon) = \{(x, y) \in \mathcal{T}^*_V \mid \text{there exists } (\tilde{x}, y) \in T \text{ with } |x - \tilde{x}| < \epsilon^{-1/2}\}.$$

The point of this choice is as follows. Consider the change of variable $y' = \epsilon^{-1/2} y$, $x' = \epsilon^{1/2} x$. Then using x', y' to identify $\mathcal{T}^*_U$ with a subset of $\mathbf{C}^2$, we get $Dist(\partial\overline{T(\epsilon)}, T') \geq 1$, for sufficiently small ϵ, in the euclidean distance in $\mathbf{C}^2$.

Pulling back ω_ϵ to $T(\epsilon)$, we can write

$$\omega_\epsilon = i\partial\bar{\partial}(\varphi_1 + \varphi_2)$$

where $i\partial\bar{\partial}\varphi_1$ is a semi-flat metric and $i\partial\bar{\partial}\varphi_2$ is the correction to this metric resulting from the gluing process. By applying Lemma 4.1, we can choose ϵ sufficiently small so that the $C^{2,\alpha}$ norm of φ_2 on T' is as small as we like (and the C^2 norm of φ_2 on $T(\epsilon)$). On the other hand, φ_1 can be taken to be a translation of the Kähler potential given for the standard semi-flat metric in Example 2.2. Since we have chosen $\mathbf{B}$ in a fundamental domain, we can then bound the $C^{2,\alpha}$ norm of φ_1 on T' independently of $\mathbf{B}$ as a polynomial in ϵ^{-1}. The same is true of the C^2 norm of φ_1 on $T'(\epsilon)$.

Now the equation that u_ϵ satisfies is

$$(i\partial\bar{\partial}\psi_\epsilon)^2 = \Omega \wedge \bar{\Omega}/2$$

where $\psi_\epsilon = \varphi_1 + \varphi_2 + u_\epsilon$. Thus a $C^{2,\alpha}$ bound on ψ_ϵ polynomial in ϵ^{-1} yields a $C^{2,\alpha}$ bound on u_ϵ polynomial in ϵ^{-1}. Now changing coordinates between x, y and x', y' also only affects the $C^{2,\alpha}$ norm of a function by a factor polynomial in ϵ^{-1}, so we can work with respect to the coordinates x', y'. Now in these coordinates,

$$i\partial\bar{\partial}(\varphi_1) = \frac{i}{2}(W_0(dx' + \epsilon b dy') \wedge \overline{(dx' + \epsilon b dy')} + W_0^{-1} dy' \wedge d\bar{y}').$$

By looking at the explicit form of b for the semi-flat metric, we see ϵb in fact goes to zero as $\epsilon \to 0$ on $T(\epsilon)$. Thus the eigenvalues of $i\partial\bar{\partial}\varphi_1$ on $T(\epsilon)$, i.e., the eigenvalues of the matrix

$$\begin{pmatrix} W_0 & \epsilon b W_0 \\ \epsilon \bar{b} W_0 & W_0^{-1} + \epsilon^2 |b|^2 \end{pmatrix},$$

can be bounded below and above by some constants λ and Λ independently of ϵ. Since φ_2 is small, the same is true of ω_ϵ on $T(\epsilon)$. Finally, by Lemma 5.3, the eigenvalues of $i\partial\bar{\partial}\psi_\epsilon$ are bounded below and above by $C^{-1}\lambda$ and $C\Lambda$, independently of ϵ. Furthermore, Lemmas 5.2 and 5.3 imply the C^2 norm of ψ_ϵ on $T(\epsilon)$ is bounded by a polynomial in ϵ^{-1}. We can now apply [11], Theorem 17.14 to the domains $T' \subseteq T(\epsilon)$, to obtain the desired result. q.e.d.

We shall now follow the standard method of continuity from [35], and, for $t \in [0, 1]$, look at the solution $u_{\epsilon,t}$ to the equation

$$(\omega_\epsilon + i\partial\bar{\partial}u_{\epsilon,t})^2 = (1 + t(e^{F_\epsilon} - 1))\omega_\epsilon^2, \tag{5.4}$$

$$\int_X u_{\epsilon,t}\ \omega_\epsilon^2 = 0. \tag{5.5}$$

We set $\omega_{\epsilon,t} = \omega_\epsilon + i\partial\bar{\partial}u_{\epsilon,t}$, the Kähler form of a metric on the given complex manifold X. For $t = 0$, we just get back our original (glued) metric, whilst $t = 1$ is the case we have just looked at, yielding the Ricci flat metric with Kähler form $\tilde{\omega}_\epsilon$. Since $\log(1 + t(e^{F_\epsilon} - 1))$ has the same properties as F_ϵ for $t \in [0, 1]$, all the above estimates of Lemmas 5.2–5.4 work equally well for $u_{\epsilon,t}$. In particular,

$$C^{-1}\omega_\epsilon \leq \omega_{\epsilon,t} \leq C\omega_\epsilon$$

for some constant C independent of $t \in [0, 1]$ and ϵ, and

$$\|u_{\epsilon,t}\|_{C^{2,\alpha}} \leq P(\epsilon^{-1}),$$

with the polynomial P independent of $t \in [0, 1]$ and ϵ.

Moreover, the Ricci form of the metric $\omega_{\epsilon,t}$ is given by

$$\frac{i}{2\pi}\partial\bar{\partial}(F_\epsilon - \log(1 + t(e^{F_\epsilon} - 1))),$$

and so the Ricci curvature $Ric_{\omega_{\epsilon,t}}$ has a similar lower bound (independent of t) as Ric_{ω_ϵ}.

Lemma 5.5. *Let $G_{\epsilon,t}(x, y)$ denote Green's function for the Laplacian $\Delta_{\epsilon,t}$ associated to the metric $\omega_{\epsilon,t}$, normalised so that*

$$\int_X G_{\epsilon,t}(x, y)\,\omega^2_{\epsilon,t}(x) = 0.$$

Then, for ϵ sufficiently small and any $t \in [0, 1]$,

$$G_{\epsilon,t}(x, y) \geq -A\epsilon^{-11},$$

for some constant A independent of ϵ and t.

Proof. For ease of notation, we drop the suffices ϵ, t. We follow the proof of Lemma 3.3 from [24], which is due to Peter Li. The volume of X is 1, and we set $K(x, y, s) = H(x, y, s) - 1$, where H is the heat kernel on X. As in [24], we need to find a lower bound for the integral of $K(x, y, s)$ over $1 \leq s \leq \infty$, of the same form as that claimed for $G(x, y)$. Lu observes that

$$K(x, y, s) \geq -K^{1/2}(x, x, s)K^{1/2}(y, y, s),$$

and that furthermore, for any $x \in X$,

$$K(x, x, s) \leq K(x, x, 1)e^{-\lambda(s-1)},$$

for all $s \geq 1$, where λ denotes the first (positive) eigenvalue of the Laplacian. If now we can suitably bound λ from below, and $K(x,x,1)$ from above, we'll be able to integrate the resulting function which bounds $K^{1/2}(x,x,s)K^{1/2}(y,y,s)$ from above, obtaining a lower bound for $\int_1^\infty K(x,y,s)ds$.

The bound from below for λ comes from Theorem 4 on page 116 of [31]. Since the metric is within a fixed constant factor of our original metric, all the quantities in the given formula are known, and so using Theorem 4.5, and we deduce that

$$\lambda \geq A_1 Diam(X)^{-2} \geq A_2\epsilon,$$

for appropriate absolute constants A_1, A_2. The proof of Lemma 5.1 may be applied to the metric $\omega_{\epsilon,t}$ to obtain a similar bound on the Sobolev constant, and then the bound from above for $K(x,x,1)$ is implied by Equation (3.12) of [36], where the argument given there has been run for the function $K(x,y,s) = H(x,y,s) - 1$ (so in particular $\int_X K(x,z,s)\,\omega_{\epsilon,t}^2(z) = 0$). For an appropriate constant A_3 independent of t, we have

$$K(x,x,1) \leq A_3\epsilon^{-10}.$$

Thus for all $s \geq 1$

$$K(x,x,s) \leq A_3\epsilon^{-10}e^{-A_2\epsilon(s-1)},$$

which then implies that

$$K(x,y,s) \geq -A_4\epsilon^{-10}e^{-A_2\epsilon(s-1)},$$

for some constant A_4 independent of ϵ and t. On integrating, we obtain the claimed bound in the form stated (a rather more involved argument in fact gives a bound $K(x,x,1) \leq A_3'\epsilon^{-3}$, and hence $G_{\epsilon,t}(x,y) \geq -A'\epsilon^{-4}$, but this extra accuracy is not required). q.e.d.

We are now ready for our main theorem.

Theorem 5.6. *For any simply connected open set $U \subseteq B_0$ with $\overline{U} \subseteq B_0$, and any $k \geq 2$, $0 < \alpha < 1$, there exists constants C, C', and ϵ_0 such that for all choices of $\mathbf{B}$ in a fundamental domain for the B-field and any $\epsilon < \epsilon_0$ giving ω_ϵ as in Theorem 4.5, and u_ϵ satisfying the equations*

$$(\omega_\epsilon + i\partial\bar{\partial}u_\epsilon)^2 = e^{F_\epsilon}\omega_\epsilon^2$$
$$\int_X u_\epsilon\omega_\epsilon^2 = 0$$

with $F_\epsilon = \log(\frac{\Omega\wedge\bar\Omega/2}{\omega_\epsilon^2})$, *we have*

$$\|u_\epsilon\|_{C^{k,\alpha}} \le Ce^{-C'/\epsilon}.$$

Here, the norm is as in Lemma 4.1 on the region $f^{-1}(U)$, *and the constants* C, C' *are independent of* ϵ.

Proof. This is now completely standard. Following [20] and [24], we differentiate (5.4) with respect to t, getting

$$\Delta_{\epsilon,t}\frac{du_{\epsilon,t}}{dt} = \frac{e^{F_\epsilon}-1}{1+t(e^{F_\epsilon}-1)}.$$

The right hand side is very small, which along with the estimate of Lemma 5.5, allows us to bound $du_{\epsilon,t}/dt$. Indeed, by Green's formula and (5.5), we have

$$\frac{du_{\epsilon,t}(x)}{dt} = -\int_X \left(\Delta_{\epsilon,t}\frac{du_{\epsilon,t}}{dt}\right)\tilde G_{\epsilon,t}(x,y)\omega^2_{\epsilon,t}(y).$$

Here $\tilde G_{\epsilon,t}$ is the Green's function for the Laplacian for $\omega_{\epsilon,t}$, normalized so that $\inf_X \tilde G_{\epsilon,t} = 0$. Lemma 5.5 tells us that $\int_X \tilde G_{\epsilon,t}(x,y)\omega^2_{\epsilon,t}(y) \le A\epsilon^{-11}$ for some constant A independent of ϵ and t, so bounds on F_ϵ imply

$$\|du_{\epsilon,t}/dt\|_{C^0} \le C_1 e^{-C_2/\epsilon} \tag{5.6}$$

for some constants C_1 and C_2 independent of t and ϵ, for sufficiently small ϵ.

We can now apply the interior Schauder estimates (see [11] Theorem 6.2) to obtain

$$\|du_{\epsilon,t}/dt\|_{C^{2,\alpha}} \le C_3 e^{-C_4/\epsilon} \tag{5.7}$$

for sufficiently small ϵ. This holds for α as given by Lemma 5.4. We note that a certain amount of care must be taken in applying these estimates: first, we need to use the estimate of (5.6) and Lemma 5.4 on a larger open set U' with $\overline{U} \subseteq U' \subseteq B_0$. Second we note that by Lemma 5.4, the $C^{0,\alpha}$ estimates for the coefficients of the second order operator $\Delta_{\epsilon,t}$ depend only polynomially on ϵ^{-1}, and the same is true, much as in the proof of Lemma 5.4, for the constants λ and Λ needed in applying [11], Theorem 6.2. The constant arising in the Schauder estimate can be verified to depend only polynomially on λ and Λ. Taking these things into account, one obtains (5.7).

Now integrating (5.7) with respect to t we obtain

$$\|u_\epsilon\|_{C^{2,\alpha}} \leq C_3 e^{-C_4/\epsilon}.$$

Using Schauder estimates again repeatedly in the standard way (see [35], Formula (4.5) and following text), one can then find for each k, constants C and C' such that

$$\|u_\epsilon\|_{C^{k,\alpha}} \leq C e^{-C'/\epsilon}.$$

To get this inequality for any α, one uses the interpolation inequalities.
q.e.d.

Remark 5.7. The construction of the Ooguri–Vafa metric in §3 clearly works also for singular fibres of type I_n, simply by quotienting at the appropriate stage by $\epsilon n\mathbf{Z}$ instead of $\epsilon\mathbf{Z}$, and the above proofs go through unchanged in this case. Thus all the results of this section remain valid for elliptic K3 surfaces with semi-stable fibres.

6. Gromov–Hausdorff convergence

We now return to the notion of convergence alluded to in the introduction. We wish to show that with the proper normalization, the results of §5 imply that in the large complex structure limit, K3 surfaces in fact converge to 2-spheres. To make this precise, we first recall the notion of Gromov–Hausdorff distance. The definition given below can be easily seen to be equivalent to a definition in terms of ϵ-dense subsets, c.f. [30] pg. 276.

Definition 6.1. Let (X, d_X), (Y, d_Y) be two compact metric spaces. Suppose there exists maps $f : X \to Y$ and $g : Y \to X$ (not necessarily continuous) such that for all $x_1, x_2 \in X$,

$$|d_X(x_1, x_2) - d_Y(f(x_1), f(x_2))| < \epsilon$$

and for all $x \in X$,

$$d_X(x, g \circ f(x)) < \epsilon,$$

and the two symmetric properties for Y hold. Then we say the Gromov–Hausdorff distance between X and Y is at most ϵ. The Gromov–Hausdorff distance $d_{GH}(X, Y)$ is the infinum of all such ϵ.

The Gromov–Hausdorff distance defines a topology on the set of compact metric spaces, and hence a notion of convergence. It follows from results of Gromov (see e.g. [30], pg. 281, Cor. 1.11) that the class of compact Ricci-flat manifolds with diameter $\leq D$ is precompact. Thus in particular, if we have a sequence of Calabi–Yau n-folds whose complex structure converges to a large complex structure limit point (or any other boundary point for that matter) and whose metrics have diameter bounded above, then there is a convergent subsequence, and then the basic question is: what is the limit? The conjecture which motivated the work of this paper is the following:

Conjecture 6.2. *Let $\overline{\mathcal{M}}$ be a compactified moduli space of complex deformations of a simply-connected Calabi–Yau n-fold X with holonomy group $SU(n)$, and let $p \in \overline{\mathcal{M}}$ be a large complex structure limit point (see [27] for the precise Hodge-theoretic definition of this notion). Let (X_i, g_i) be a sequence of Calabi–Yau manifolds with Ricci-flat Kähler metric which are complex deformations of X, with the sequence $[X_i] \in \overline{\mathcal{M}}$ converging suitably to p, and $C_1 \geq Diam(X_i) \geq C_2 > 0$ for all i. Then a subsequence of (X_i, g_i) converges to a metric space (X_∞, d_∞), where X_∞ is homeomorphic to S^n. Furthermore, d_∞ is induced by a Riemannian metric on $X_\infty \setminus \Delta$, with $\Delta \subseteq X_\infty$ a set of codimension 2.*

A similar conjecture was also made by Kontsevich, Soibelman and Todorov (see [22], [25]).

Remark 6.3. Conjecture 6.2 is obvious in the elliptic curve case (ignoring the fact that elliptic curves are not simply-connected), no matter how the sequence of points approaches the large complex structure limit point. However, in the K3 case, more care must be taken. In this paper, we have considered limits mirror to points approaching the large Kähler limit along a ray in the Kähler cone. However, if a sequence of points approaching the large Kähler limit approaches the boundary of the projectivized Kähler cone, we might expect further degeneracies in the Gromov-Hausdorff limits. For example, a product of two elliptic curves $E_1 \times E_2 = \mathbf{R}^4/\mathbf{Z}^4$, with a metric

$$\begin{pmatrix} \epsilon^{-3} & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & \epsilon \end{pmatrix}$$

has a special Lagrangian fibration given by projection on the the first and third factors, and has fibres of area ϵ. When we normalise the metrics to have diameter one, the sequence of Riemannian manifolds converges to an S^1 as $\epsilon \to 0$. As pointed out to us by N.C. Leung, this construction descends to the corresponding Kummer surfaces. The limit of the Kummer surfaces is then a closed interval.

Thus we expect that the correct restriction on sequences of points in the complex moduli space in Conjecture 6.2 should correspond in the mirror to Kähler classes staying within a proper subcone of the Kähler cone. We can now prove the conjecture for the limits of K3 surfaces considered in this paper, where the Kähler class tends to ∞ along a ray, which we have seen reduces to the following result.

Theorem 6.4. *Let $j : J \to B$ be an elliptically fibred* K3 *surface with a section and singular fibres all of type I_1, and let $f_i : X_i \to B$ be a sequence of elliptically fibred* K3 *surfaces with jacobian j. Let ω_i correspond to a Ricci-flat Kähler metric on X_i with ω_i^2 independent of i, and with $\int_{f_i^{-1}(b)} \omega_i = \epsilon_i \to 0$ as $i \to \infty$. Then the sequence of Riemannian manifolds $(X_i, \epsilon_i\omega_i)$ converges in the Gromov–Hausdorff sense to B, the metric on B being induced from the* (*singular*) *Riemannian metric given, in local coordinates, by $W_0^{-1} dy \otimes d\bar{y}$, with W_0 as defined in* §4.

Proof. As usual, after choosing a topological zero-section of each X_i, we can identify X_i with J as a manifold. We may then view the ω_i as corresponding to a sequence of Riemannian metrics g_i on J, and prove that the sequence $J_i = (J, \epsilon_i g_i)$ converges in the Gromov–Hausdorff sense to B (with the given metric).

Using Remark 4.6, we can choose the class $\mathbf{B}_i$ determining ω_i in a fundamental domain for the B-field by making, for each i, a judicious choice of zero-section σ_0.

Consider now B along with the metric $W_0^{-1} dy \otimes d\bar{y}$. Near each singular fibre one can find a coordinate y so that $\tau_1 = 1$ and $\tau_2 = \frac{1}{2\pi i} \log y + h(y)$, for some holomorphic function h, and from this one can see that each point of $\Delta \subseteq B$ is at finite distance under this metric, and thus B becomes a compact metric space using geodesic distance.

Now we need to show that for each $\delta > 0$, $d_{GH}(J_i, B) < \delta$ for sufficiently large i. We will apply Definition 6.1 to the maps $f_i = j : J \to B$ and $\sigma_0 : B \to J$.

Choose, using Corollary 3.7 and Lemma 5.3, for each point $p_j \in \Delta$, a small disc D_j around p_j with the property that:

(1) $Diam(D_j) < \delta/100$.

(2) $Diam(f^{-1}(D_j)) < \delta/100$ for sufficiently small ϵ_i.

Let $U = B \setminus \bigcup D_j$. Now let $x_1, x_2 \in J$. Let γ be a path joining x_1 and x_2 such that, for a given i,

$$l_{\epsilon_i g_i}(\gamma) < d_{\epsilon_i g_i}(x_1, x_2) + \delta/100.$$

Here l denotes length, and the subscript denotes the metric being used. At the risk of increasing the length of γ by $24\delta/100$, we can assume that γ enters and leaves each $f^{-1}(D_j)$ at most once, and write $\gamma = \gamma_1 + \gamma_2$, with $\gamma_1 \subseteq f^{-1}(U)$ and $\gamma_2 \subseteq f^{-1}(\bigcup D_j)$, with $l_{\epsilon_i g_i}(\gamma_2) \leq 24\delta/100$. Now if $f^{-1}(U)$ carried a semi-flat metric, then $f^{-1}(U) \to U$ would in fact be a Riemannian submersion, and distances decrease under submersions. On the other hand, if ϵ_i is sufficiently small, it follows from Theorem 5.6 that the metric $\epsilon_i g_i$ is close to a semi-flat metric in the C^0 sense. Thus for sufficiently large i, depending on δ,

$$l_B(f(\gamma_1)) \leq l_{SF}(\gamma_1) \leq l_{\epsilon_i g_i}(\gamma_1) + C(\epsilon_i),$$

where l_{SF} denotes length with respect to the suitably normalized semi-flat metric close to $\epsilon_i g_i$, and $C(\epsilon_i)$ is a constant depending on ϵ_i (and δ) but independent of the path. Furthermore, $C(\epsilon_i) \to 0$ as $\epsilon_i \to 0$. Thus, possibly replacing $f(\gamma_2)$ with a shorter path, we see that

$$\begin{aligned} d_B(f(x_1), f(x_2)) &\leq l_B(f(\gamma_1)) + 24\delta/100 \\ &\leq l_{\epsilon_i g_i}(\gamma_1) + C(\epsilon_i) + 24\delta/100. \end{aligned}$$

Thus for sufficiently small ϵ_i, we always have

$$d_B(f(x_1), f(x_2)) < d_{\epsilon_i g_i}(x_1, x_2) + \delta.$$

Next, let $y_1, y_2 \in B$, and let γ be a path joining y_1 and y_2 with

$$l_B(\gamma) < d_B(y_1, y_2) + \delta/100.$$

As before, we can assume that γ enters and leaves each D_i once, and write $\gamma = \gamma_1 + \gamma_2$. Consider now the metric on $\sigma_0(B)$; locally, this takes the form $\epsilon_i(W^{-1} + W|b|^2)dy \otimes d\bar{y}$ for some W and b. Again, the metric on $f^{-1}(U)$ is close to a semi-flat metric, hence this metric is close, in the C^0 sense, to $(W_0^{-1} + \epsilon_i^2 W_0 |b_{SF}|^2)dy \otimes d\bar{y}$. Now the point of choosing $\mathbf{B}_i$ to be in a fundamental domain for the B-field is that $|b_{SF}|^2$ can then be uniformly bounded, independent of i. Thus for small ϵ_i, this metric is close to the given metric on B. Thus there exists a constant $C(\epsilon_i)$ with $C(\epsilon_i) \to 0$ as $\epsilon_i \to 0$ such that

$$l_{\epsilon_i g_i}(\sigma_0(\gamma_1)) \leq l_B(\gamma_1) + C(\epsilon_i).$$

Therefore $d_{\epsilon_i g_i}(\sigma_0(y_1), \sigma_0(y_2)) \leq l_B(\gamma) + C(\epsilon_i) + 24\delta/100$ so for sufficiently small ϵ_i,

$$d_{\epsilon_i g_i}(\sigma_0(y_1), \sigma_0(y_2)) \leq d_B(y_1, y_2) + \delta.$$

Thus for sufficiently small ϵ,

$$|d_B(y_1, y_2) - d_{\epsilon_i g_i}(\sigma_0(y_1), \sigma_0(y_2))| < \delta$$

for all $y_1, y_2 \in B$.

If $x_1, x_2 \in J$, similar arguments show that

$$d_{\epsilon_i g_i}(x_1, x_2) < d_B(f(x_1), f(x_2)) + \delta$$

by joining x_1 and x_2 by a path which first connects x_1 to $\sigma_0(B)$ inside a fibre or inside $f^{-1}(D_j)$ for some j, then follows a geodesic inside $\sigma_0(B)$ to the fibre containing x_2, and then connects up to x_2 inside this fibre. The inequality follows for sufficiently small ϵ_i since the diameter with respect to $\epsilon_i g_i$ of any fibre $f^{-1}(y)$ for $y \in U$, for small ϵ_i, is bounded by $C\epsilon_i$, where C depends only on the periods over U.

This shows

$$|d_{\epsilon_i g_i}(x_1, x_2) - d_B(f(x_1), f(x_2))| < \delta$$

for sufficiently small ϵ_i. Finally, similar methods show

$$|d_{\epsilon_i g_i}(x_1, x_2) - d_{\epsilon_i g_i}(\sigma_0(f(x_1)), \sigma_0(f(x_2)))| < \delta$$

for all $x_1, x_2 \in X$, and ϵ_i sufficiently small. q.e.d.

Remark 6.5. The metric on the base B is McLean's metric (see [26], [19], [14]) on the base of the special Lagrangian T^2-fibration obtained by hyperkähler rotation. In higher dimensions we also expect this metric to appear in the limit, showing a residual effect of the conjectural special Lagrangian fibration. This metric would then be singular along some subset of the limit, corresponding to the limit of the discriminant loci of the conjectural special Lagrangian fibrations. We hope this will be codimension 2. See [16] for further speculation along these lines.

Conversely, we hope that one approach to understanding the existence of special Lagrangian fibrations would be to prove Conjecture 6.2, which gives us insight into the behaviour of Ricci-flat metrics near large complex structure limits. However, it is clear that any approach to prove Conjecture 6.2 in higher dimensions must be substantially different to the one given here for K3 surfaces, where we have made use of the existence of special Lagrangian fibrations as well as the hyperkähler trick to reduce to a question of Kähler degenerations.

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