Polynomial approximation of PDEs with stochastic coefficients

Partial differential equations with stochastic coefficients conveniently model problems in which the data of a given PDE (coefficients, forcing terms, boundary conditions) are affected by uncertainty, due e.g. to measurement errors, limited data availability or intrinsic variability of the described system.

Assuming that the uncertainty in the problem can be described by a set of N random variables $y_1, \ldots, y_N$, the solution u of the PDE at hand can be seen as an N-variate random function, $u = u(y_1, \ldots, y_N)$, for which one may wish to compute mean and variance, or the probability that it exceeds a given threshold; such analysis is usually referred to as "Uncertainty Quantification". This could be achieved with a straightforward Monte Carlo method, that may however be very demanding in terms of computational costs. Methods based on polynomial approximations of u have thus been introduced, aiming at exploiting the possible degree of regularity of u with respect to $y_1, \ldots, y_N$ to alleviate the computational burden. Such polynomial approximations can be obtained e.g. with Galerkin projections or collocation methods (e.g. the sparse grids collocation method) over the support of the random variables. Although effective for problems with a moderate number of random dimensions, these methods suffer from a degradation of their performance as N increases ("curse of dimensionality"). Minimizing the impact of the "curse of dimensionality" is therefore a key point for the application of polynomial methods to high-dimensional problems.

In this talk we will discuss these methods and explore possible strategies to determine efficient polynomial approximations of u (the so-called "best M-terms" approximation of u). In particular, we will consider a "knapsack approach", in which we estimate the cost and error contribution of each possible component of the polynomial approximation, and then we choose the components with the highest error/cost ratio. We will present theoretical convergence results obtained for some specific problems as well as numerical results showing the efficiency of the proposed approach.