Parabolic Quasi-Variational Inequality with Gradient Constraints

Nobuyuki Kenmochi
School of Education (Mathematics), Bukkyo University
96 Kitahananobo-cho, Murasakino, Kita-ku, Kyoto, 603-8301 Japan
(E-mail: kenmochi@bukkyo-u.ac.jp)

Abstract. We know many mechanical phenomena whose dynamics is described by a class of quasi-variational inequalities of the parabolic type. Our system consists of a second-order parabolic variational inequality with gradient constraint depending on the temperature, coupled with the heat equation:

$$u_t - \nu\Delta u + \partial I_{K_\theta(t)}(u) + g(u) \ni f \quad \text{in } Q,$$

$$\theta_t - \kappa\Delta\theta + h(x,t,u) = 0 \quad \text{in } Q,$$

$$\frac{\partial u}{\partial n} = 0, \quad \theta = 0 \ \text{ on } \Sigma,$$

$$u(\cdot,0) = u_0, \quad \theta(\cdot,0) = \theta_0 \quad \text{in } \Omega.$$

Here,Ω is a bounded smooth domain in $\mathbf{R}^N$ with $1 \le N \le 3$, $\Gamma := \partial\Omega$, $Q := \Omega \times (0,T)$ with $0 < T < \infty$ and $\Sigma := \Gamma \times (0,T)$; ν and κ are positive constants, $K_\theta(\cdot)$ is the closed convex set in $H^1(\Omega)$ defined by

$$K_\theta(t) := \{v \in H^1(\Omega);\ |\nabla v| \le \psi(\theta(\cdot,t)) \text{ a.e. on } \Omega\}$$

for a given positive and smooth continuous function $\psi(\cdot)$ on $\mathbf{R}$. Further, $I_{K_\theta}(\cdot)$ is the indicator function of $K_\theta(\cdot)$ in $L^2(\Omega)$ and $\partial I_{K_\theta(\cdot)}$ is its subdifferential. Also, $h(x,t,u)$ is a smooth function on $\overline{\Omega} \times \mathbf{R} \times \mathbf{R}$, $g(\cdot)$ is a smooth function on $\mathbf{R}$, and f is a function given in $L^2(Q)$ and $u_0 \in H^1(\Omega)$ with $|\nabla u_0| \le \psi(\theta_0)$ as well as $\theta_0 \in H^2(\Omega) \cap H^1_0(\Omega)$ given as data.

Since the temperature is unknown in our problem and should be determined as a part of the solution, the constraint function is unknown as well. In this sense, our problem includes the quasi-variational structure, and in the mathematical treatment one of main difficulties comes from it. Our approach to the problem is based on the abstruct theory of quasi-variational inequalities with non-local constraint which evolved in a paper in Banach Center Pub.(Vol.86,pp.175-194,2009) by Kano-Kenmochi-Murase. In this talk, we prove the existence of weak solutions in higher space dimensions and the existence of strong solutions in one space dimension.