Iterative Solution of Elliptic Equations with a Small Parameter

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Several applications in science and engineering are modelled by partial differential equations involving a small parameter defining a certain central characteristic of the problem. Generic examples are reaction dominated diffusion-reaction equation where the diffusivity parameter is usually very small and equations governing the deformation of thin structures, e.g. beams, plates and shells. All of the above examples share the characteristic that they can be written in the form

$$t^2 L_1 u + L_0 u = f \tag{1}$$

where t^2 is a small, positive parameter and L_1 and L_0 are positive semi-definite linear partial differential operators, such that the operator $t^2 L_1 + L_0$ is coercive.

Equations of the type (1) are discretized using finite-difference or finite-element methods giving rise to a finite-dimensional matrix equation

$$t^2 A_1 x + A_0 x = b \tag{2}$$

where the matrices A_i are assumed to reflect the properties of the operators L_i and the vectors x and b describe the unknown u and the load f with respect to some finite-dimensional basis.

To obtain an approximate solution of (1) one must solve the linear system of equations (2), and in principle, as the parameter t^2 varies one must solve a similar equation several times. However, since it seems likely that the equations do not differ very much from each other one can ask if it were possible to combine the solution procedures in some way. Our aim is to give an affirmative answer and show a methodology how equations of type (2) can be solved for several values of the parameter t^2. The key ingredient of the method will be computing the Cholesky decomposition of the matrix A_0 and using this as a preconditioner to conjugate gradient method.

However, as our examples show this may not be sufficient to obtain an effective method and therefore we use deflated version of the conjugate gradient method to suppress the extremal eigenvalues hindering the convergence of the iteration. Novelty of the method is that solution of the systems and collection of the eigenvectors can be interleaved also in the deflated version of the algorithm.