Existence and uniqueness results for Fokker–Planck equations in Hilbert spaces

G. Da Prato, Scuola Normale Superiore, Pisa

Let H be a separable Hilbert space. We are given a linear self-adjoint negative operator $A : D(A) \subset H \to H$, a family of nonlinear operators $b(t) : D(b(t)) \subset H \to H,\ t \in [0,T]$, and a linear bounded operator $B \in L(H)$.

We are concerned with the stochastic differential equation

$$dX = (AX + b(t,X))dt + BdW(t), \quad X(0) = x \in H, \quad t \in [0,T].$$

Let $\mathscr{L}(t),\ t \in [0,T]$, be the corresponding Kolmogorov operators

$$\mathscr{L}(t) = \frac{1}{2}\mathrm{Tr}\ [BB^*D^2\varphi] + \langle Ax + b(t,x), D\varphi\rangle.$$

Given a Borel probability measure ζ, we want to find a family of Borel probability measures $\mu_t,\ t \in [0,T]$, such that $\mu(0) = \zeta$ and the following Fokker–Planck equation is fulfilled

$$\frac{d}{dt}\int_H \varphi(x)\,\mu_t(dx) = \int_H \mathscr{L}\varphi(x)\,\mu_t dx, \tag{1}$$

for all φ belonging to a suitable space of test function $\mathscr{E}$.

We shall discuss some recent existence and uniqueness results for (1) proved in collaboration with V. Bogachev and M. Röckner. When B has a bounded inverse a very general uniqueness result can be proved. Some situations will be also discussed when B is degenerate, possibly $B = 0$, where Kolmogorov operators reduce to transport operators.