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ORIGINAL ARTICLE

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An existence result for a model of complete damage in elastic materials with reversible evolution

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Abstract In this paper, we consider a model describing evolution of damage in elastic materials, in which stiffness completely degenerates once the material is fully damaged. The model is written by using a phase transition approach, with respect to the damage parameter. In particular, a source of damage is represented by a quadratic form involving deformations, which vanishes in the case of complete damage. Hence, an internal constraint is ensured by a maximal monotone operator. The evolution of damage is considered "reversible", in the sense that the material may repair itself. We can prove an existence result for a suitable weak formulation of the problem, rewritten in terms of a new variable (an internal stress). Some numerical simulations are presented in agreement with the mathematical analysis of the system.

Keywords Complete damage · Phase transition · Non-smooth PDE system · Existence result for weak solutions

1 Introduction

Damage models may be introduced as an inelastic response in materials due to breaking of cohesive bonds in the microscopic structure (see e.g. [21]). Among different approaches, in this paper we are interested in finding some macroscopic description of the phenomenon, the ultimate goal being the possibility of introducing an effective predictive model for computational engineering tools.

The literature dealing with the phenomenon of damage is very rich and covers different research fields. Indeed, the study of damage offers a non-trivial interplay between non-smooth mechanics, analysis of nonlinear partial differential equations (PDE), calculus of variations and computational mechanics.

In this paper, we mainly focus on (volume) damage process in continuum materials. However, let us briefly recall that this kind of phenomenon is related (from both the physical and analytical points of view) to other

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problems arising in the mechanics of materials, as fractures in brittle materials (see e.g. [8,12]), the contact with adhesion [4], surface damage and delamination [22], inelastic and plastic behaviour [9,13].

Our approach is based on a macroscopic model, proposed by Frémond (see e.g. [14,15] for a first local existence result in 1D) in the spirit of phase transition models. Hence, we refer to [5,6,16,17,23], and references therein, for some analytical results on this type of model.

In the last years, the formulation has been extended to the case of glued materials in [11] and successfully applied to the problem of debonding of reinforced structural element [3,25]. A similar approach, even if it is derived via variational techniques can be found in [8]. In fact, the energy formulation leads to equation of motion similar to the one adopted here. Different material behaviours can be obtained by specific choice of the energy functional [12,18].

In particular, we are restricting ourselves to isotropic damage, in the small-strain regime, and we do not account for thermal effects. The main idea is that the PDE system describing the evolution of the phenomenon may be recovered using a variational principle, which is mainly based on a generalized version of the principle of virtual powers. Indeed, it is assumed that both the thermomechanical equilibrium of the mechanical system and its evolution are deduced from a balance between dissipative and non-dissipative (internal and external) forces. By our specific choice of energy and dissipations functionals, the problem results of rate-dependent type. We recall that different approaches are used in the literature concerning a static description of the phenomenon (mainly by use of minimizing technique), rate-independent evolution (by introducing a suitable notion of solutions, as the energetic solutions, and solving the problem in terms of a stability inequality and energy balance), see e.g. [7,20,21].

Since the complete derivation of the model proposed by Frémond is well known, we do not detail its derivation here, but rather give the main steps. The state variables (in terms of which the equilibrium is defined) are deformations, in a small-strain regime, represented by ∇u. We are assuming that the displacement u is a scalar quantity to simplify some technicalities in the proofs of the analytical results. a phase damage parameter χ, as well as its gradient $\nabla\chi$. The damage parameter represents a *macroscopic measure* of the state of damage of the material, as it can be interpreted as a local proportion of unbroken microscopic bonds in the material for volume unit. Indeed, $\chi = 0$ means that the material is completely damaged, $\chi = 1$ that it is undamaged, and $\chi \in (0, 1)$ that the intermediate situation is present. The choice (see [14]) of considering the gradient $\nabla\chi$ a state variable can be justified by the assumption that there is some local interaction at a microscopic level between damage and non-damaged zones. Besides it physical justification, we point out that this term plays a crucial role in the mathematical treatment of the resulting nonlinear system as it provides sufficient spatial regularity for χ allowing us to apply some useful compactness results.

From a mathematic point of view, the resulting PDE system presents interesting features. As far as we known, an existence result for the complete original Frémond model (global in time) is not known in a three-dimensional setting. This is mainly due to the fact that the system degenerates once the material is completely damaged, so that the validity of the equations is no longer assured. The main difficulties come from the coexistence of degenerating terms, higher-order nonlinearities and the presence of non-/smooth multivalued operators. More precisely (see the discussion in the next section), the modelling approach proposed by Fremond proposes a system of the type

$$-\operatorname{div}(\chi^2\nabla u) = 0 \tag{1.1}$$

$$\partial_t\chi - \Delta\chi + \partial I_{[0,1]}(\chi) \ni w - \chi|\nabla u|^2 \tag{1.2}$$

thus implying that the elastic properties of the material degenerate when $\chi \searrow 0$, i.e. when complete damage appears. Consequently, the above model ceases to describe the behaviour of the material when the sample experiences a complete damage. The main mathematical problem consists then in finding a proper notion of weak solution capable of describing the complete damage. In this paper, we propose a notion of solution able to describe the complete damage phenomenon. This type of weak solution originates from an approximation scheme. Thus, as a side effect, we also prove an existence result. The main feature of the new notion of solution consists in the use of a new internal variable, which we call internal stress, in place of the deformation strain (see also [21] for some related result). As anticipated, this new variable originates from a (standard) approximation procedure that consists in removing the degeneracy of the momentum balance equation. More precisely, we replace the momentum equation (1.1) with

$$-\operatorname{div}((\chi_\varepsilon^2 + \varepsilon)\nabla u_\varepsilon) = 0.$$

Then, the main problems are related to the analysis of the limit when $\varepsilon \searrow 0$. Let us point out that this approximation and the following passage to the limit procedure has been already used to deal with this kind of damage systems, e.g. in [7]. In particular, since the deformation gradient (at the approximation level) is always weighted by powers of the damage variable χ, it comes out that the possibility of identifying the weak limit of $\chi_\varepsilon^2 \nabla u_\varepsilon$ is related to the occurrence of regions of $\Omega \times (0, T)$ where the sample is completely damaged. The analysis suggests that a good descriptor of the behaviour of the material should be $\zeta := \overline{\chi^{1/2}\nabla u}$, namely the weak limit in $L^2(\Omega \times (0, T))$ of $\chi_\varepsilon^{1/2}\nabla u_\varepsilon$. We call ζ internal stress. Moreover, since the right-hand side of (1.2) (at the approximated level) contains the term $(\chi_\varepsilon^{1/2}\nabla u_\varepsilon)^2$ it turns to be bounded only in $L^1(\Omega \times (0, T))$. Consequently, we deduce from properties of weak limits that there exists a bounded non-negative measure μ on $\Omega \times (0, T)$ such that

$$(\chi_\varepsilon^{1/2}|\nabla u_\varepsilon|)^2 \xrightarrow{\varepsilon \searrow 0} \zeta^2 + \mu \quad \text{in } \mathcal{D}'(\Omega \times (0, T)).$$

The measure μ is called defect measure, and it is related to the lack of strong convergence of $\chi_\varepsilon^{1/2}\nabla u_\varepsilon$. It is important to note that the emergence of the measure μ is not only a mathematical issue, but rather its presence is justified from the physical point of view. In fact, the lack of strong convergence for the sequence of internal stress is related to the presence of regions where complete damage manifests, i.e. $\chi = 0$. Consequently, we propose that the occurrence of the measure μ and its positivity should be an indicator of the emergence of damage. For a slightly modified system, we are able to rigorously justify this conjecture and prove that if there exists some $(\bar{x}, \bar{t})$ for which $\chi(\bar{x}, \bar{t}) > 0$, then there should be an open neighbour B of $(\bar{x}, \bar{t})$ with $\mu(B) = 0$. It remains an open problem to obtain the same result for our original system. In the last part of the paper, we present some numerical simulations which suggest the validity of this proposal for our system. Let us finally point out that we are able to deal with a reversible evolution of χ, i.e. we do not impose any constraint on the sign of χ_t. This is mainly due to technical mathematical reasons. Indeed, adding a constraint on χ_t would lead to a doubly nonlinear character of the damage evolution equation, which we are not able to tackle to the lack of regularity of the left-hand side when passing to the limit in the equation. Actually, it should be pointed out that there are technologically relevant materials (for instance rubbers and polymers) which display such a healing property.

1.1 The model

Let us consider an elastic body, located in a smooth bounded domain in $\mathbb{R}^3$, and look for its damage evolution during a finite time interval $(0, T)$.

The balance equations of the system are the classical momentum balance and a new balance equation of micro-forces responsible for damage phenomenon. They are written in Ω as follows

$$-\operatorname{div} \sigma = f, \tag{1.3}$$

$$B - \operatorname{div} \mathbf{H} = 0, \tag{1.4}$$

where σ is the Cauchy stress and B, $\mathbf{H}$ are new interior stresses related to the damage of the material. Indeed, the equations are recovered by a generalization of the principle of virtual powers proposed by M. Frémond, in which internal micro-forces and micro-velocities are included (see [14] for a detailed derivation of these balance equations).They are combined with suitable boundary conditions. In particular, assuming that no exterior surface forces are applied at a microscopic level, we have

$$\mathbf{H} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega, \tag{1.5}$$

while we restrict ourselves to experiments with some fixed Dirichlet value for the displacement on the boundary, say

$$u = u_\Gamma \quad \text{on } \partial\Omega. \tag{1.6}$$

The involved physical quantities σ, B, $\mathbf{H}$ are recovered in terms of the free energy functional (for non-dissipative parts) and the pseudo-potential of dissipation (for dissipative contributions). We first introduce the free energy functional

$$\Psi(\nabla u, \chi, \nabla, \chi) = \frac{1}{2}K(\chi)|\nabla u|^2 + \frac{1}{2}|\nabla\chi|^2 + w(1-\chi) + I_{[0,1]}(\chi), \tag{1.7}$$

where $K(\chi) \geq 0$ is the stiffness of the material, the term $w(1-\chi)$ represents the cohesion of the material where $w > 0$, and $I_{[0,1]}(\chi)$ is the indicator function of the interval $[0, 1]$ (forcing χ to assume physical admissible values in the range [0, 1]). The choice of the stiffness provides stress–strain relation. In particular, as during the damage evolution a material loses its stiffness, it is required that $K(\chi)$ vanishes as $\chi \searrow 0$. In this paper, we let (see following remark for a justification of this choice)

$$K(\chi) = \chi^2.$$

Hence, we introduce the pseudo-potential of dissipation Φ, which is a non-negative convex function w.r.t. dissipative variables, vanishing for zero dissipation. In particular, we assume that it depends just on the dissipative variable χ_t and let

$$\Phi(\chi_t) = \frac{1}{2}|\chi_t|^2. \tag{1.8}$$

In this paper, in the pseudo-potential we have not introduced any constraint on the "direction" of the evolution of χ, i.e. we are not forcing any sign of χ_t as we are considering a reversible damage process. Even if this assumption can be a limitation in the analysis of classical structural materials characterized by irreversible damage process, bio-materials and smart polymers that exhibit healing properties can be modelled with the proposed model.

Now, the state quantities in (1.3) and (1.4) are specified by constitutive relations in terms of Ψ and Φ

$$\begin{aligned} \sigma &= \frac{\partial \Psi}{\partial \nabla u} \\ B &= \frac{\partial \Psi}{\partial \chi} + \frac{\partial \Phi}{\partial \chi_t} \\ \mathbf{H} &= \frac{\partial \Psi}{\partial \nabla \chi}. \end{aligned} \tag{1.9}$$

Thus, combining (1.9) in (1.3), (1.4), (1.5) and (1.6), we get the PDE system in Ω

$$-\operatorname{div}(\chi^2 \nabla u) = f, \tag{1.10}$$

$$\chi_t - \Delta\chi + \partial I_{[0,1]}(\chi) \ni w - \chi|\nabla u|^2, \tag{1.11}$$

$$\partial_n \chi = 0, \quad u = u_\Gamma \quad \text{on } \partial\Omega. \tag{1.12}$$

Hence, we assume suitable initial conditions (for χ). The notation $\partial I_{[0,1]}$ stands for the sub-differential (in the sense of convex analysis, cf. [2]) of the indicator function of the interval [0, 1]. It is defined for $\chi \in [0, 1]$ and $\partial I_{[0,1]}(\chi) = 0$ if $\chi \in (0, 1)$, $\partial I_{[0,1]}(0) = (-\infty, 0]$ and $\partial I_{[0,1]}(1) = [0, +\infty)$. In the following, we will use the notation $\beta = \partial I_{[0,1]}$ and by $\widehat{\beta} = I_{[0,1]}$.

Let us briefly comment on the above system (1.10)–(1.11). Note that, as χ may reach the value 0, equation (1.10) may degenerate and ∇u is not controlled. This is a problem, in particular because the gradient ∇u gives a (quadratic) contribution as source of damage in (1.11). In addition, note that on the right-hand side of (1.11) the quadratic mechanical contribution $|\nabla u|^2$ is multiplied by χ, so that one could expect that it vanishes once the material is damaged, i.e. $\chi = 0$. However, this is not a priori true, due to the fact that we cannot make an identification, separately, of χ and ∇u. In addition, once a possible L^2 estimate could be proved for ∇u, the right-hand side of (1.4) is characterized by the presence of a L^1 source. Thus, to solve the PDE system, we have to introduce a suitable notion of solution given in terms of a new internal variable (corresponding to an internal stress) ζ and the damage variable χ. In particular, we do not recover at the end information on the function u and its boundary condition. However, the system is rewritten in a consistent formulation, for which we are able to prove some energy stability estimate.

The paper is organized as follows. In Sect. 2, we introduce the notation, the assumptions and the weak formulation of the problem. Hence, after making precise the notion of solution we refer to, we state the main existence result theorem. In Sect. 3, we write an approximation of our model, letting the stiffness coercive, i.e. bounded from below by some $\varepsilon > 0$. After proving some a priori estimates on the approximated solutions, not depending on ε, by compactness and semicontinuity arguments, we pass to the limit as $\varepsilon \searrow 0$. In Sect. 5, we show some computational results, which are in accordance with the analytical result.

2 The existence result: statement and results

In this section, we make precise of the notation we use. We are considering a smooth bounded domain $\Omega \subset \mathbb{R}^3$. We fix a final time $T > 0$ of the evolution of our phenomenon. Hence, we let

$$H := L^2(\Omega), \quad V := H^1(\Omega), \quad V_0 := H^1_0(\Omega)$$

so that (V, H, V') forms an Hilbert triplet, where H and V are endowed with their usual scalar products and norms (and H identified with its dual). For the sake of simplicity, the same symbol will be used both for a space and for any power of it. We note that the norms $\|v\|_{V_0}$ and $\|\nabla v\|_H$ are equivalent for $v \in V_0$, thanks to the Poincaré inequality. Hence, we use the notation $\langle \cdot, \cdot \rangle$ for the duality pairing between V' and V and $\langle \cdot, \cdot \rangle_0$ for the duality in V_0, V_0' (the same notations are used for powers of functional spaces). Note in particular that, in the following, we will intend the operators $-\Delta$ and $-\mathrm{div}$ in some duality pairings. More precisely, we have

$$\langle -\Delta u, v \rangle = \int_\Omega \nabla u \cdot \nabla v, \quad \langle -\mathrm{div}\, \mathbf{a}, w \rangle_0 = \int_\Omega \mathbf{a} \cdot \nabla w,$$

for any $u, v \in V$, $w \in V_0$, $\mathbf{a} \in L^2(\Omega)^N$ (here $N = 3$). We prescribe

$$\chi_0 \in V, \quad \chi_0 \in [0, 1] \quad \text{a.e.} \tag{2.1}$$

and

$$u_\Gamma \in H^1(0, T; H^{1/2}(\Gamma)). \tag{2.2}$$

We introduce the harmonic extension $\tilde{u}_\Gamma$ of u_Γ. It gives, by well-known elliptic results that (cf. (2.2))

$$\tilde{u}_\Gamma \in H^1(0, T; V). \tag{2.3}$$

Then, we introduce the (closed) convex subset of $H^1(\Omega)$

$$V_\Gamma := \left\{ v \in H^1(\Omega) : \ v - \tilde{u}_\Gamma \ \in V_0 \right\}. \tag{2.4}$$

Note that V_Γ is actually independent of the choice of the particular extension $\tilde{u}_\Gamma$ (here we used the harmonic extension) and depends only on the boundary condition u_Γ.

To introduce the notion of solution for our problem, we first make precise an approximated version, depending on a regularizing parameter $\varepsilon > 0$. Once it is proved that there exists a corresponding solution, we will show that it converges in suitable way to the solution of the limit problem, so that the final solution is defined through an approximation procedure.

Definition 2.1 A triplet of functions $(u_\varepsilon, \chi_\varepsilon, \xi_\varepsilon)$ is solution for the approximated damage problem, for $\varepsilon > 0$ fixed, if

$$u_\varepsilon \in L^\infty(0, T; V_\Gamma), \tag{2.5}$$

$$\chi_\varepsilon \in H^1(0, T; H) \cap L^\infty(0, T; V), \quad \chi \in [0, 1] \quad \text{a.e. in } Q, \tag{2.6}$$

$$\xi_\varepsilon \in L^2(0, T; H) \tag{2.7}$$

with

$$\partial_n \chi_\varepsilon, = 0 \quad \text{on} \quad \partial\Omega \tag{2.8}$$

and solving, a.e. in $(0, T)$, the system

$$-\mathrm{div}\left((\varepsilon + \chi_\varepsilon^2)\nabla u_\varepsilon\right) = 0 \quad \text{in } V_\Gamma' \tag{2.9}$$

$$\partial_t \chi_\varepsilon - \Delta \chi_\varepsilon + \xi_\varepsilon = w - \chi_\varepsilon |\nabla u_\varepsilon|^2 \quad \text{a.e. in} \ \ Q. \tag{2.10}$$

$$\xi_\varepsilon \in \beta(\chi_\varepsilon) \quad \text{a.e. in}\ \Omega \tag{2.11}$$

with

$$\chi_\varepsilon(0) = \chi_0. \tag{2.12}$$

Now, we are in the position of introducing the notion of solution for our problem

Definition 2.2 A couple of functions (χ, ζ) with

$$\chi \in H^1(0,T;H) \cap L^\infty(0,T;V) \tag{2.13}$$
$$\zeta \in L^2(0,T;H) \tag{2.14}$$

is a solution to the *complete damage system* if there exist a subsequence ε_n and a triplet of solutions for the approximated damage problem in the sense by Definition 2.1 $(u_\varepsilon, \chi_\varepsilon, \xi_\varepsilon)$ such that for $n \nearrow +\infty$

$$\chi_{\varepsilon_n} \rightharpoonup \chi \quad \text{in } H^1(0,T;H) \cap L^2(0,T;V), \tag{2.15}$$
$$\chi_{\varepsilon_n}^{1/2} \nabla u_{\varepsilon_n} \rightharpoonup \zeta \quad \text{in } L^2(0,T;H) \tag{2.16}$$

with $\chi(0) = \chi_0$ a.e. in Ω and the following equations are satisfied in the sense of distributions

$$-\operatorname{div}(\chi^{3/2}\zeta) = 0, \tag{2.17}$$
$$\chi_t - \Delta\chi + \xi = w - (\zeta^2 + \mu) \tag{2.18}$$

with μ a positive Radon measure and $\xi \in L^2(0,T;H)$ with $\xi \in \beta(\chi)$ a.e. in Q. Moreover, (χ, ζ) should verify the following energy inequality

$$\begin{aligned} &\frac{1}{2}\int_\Omega \chi(t)\zeta^2(t) + \frac{1}{2}\int_\Omega |\nabla\chi(t)|^2 + \int_\Omega \widehat{\beta}(\chi(t)) - \int_\Omega w\chi(t) \\ &\quad \le \frac{1}{2}\int_\Omega \chi(s)\zeta^2(s) + \frac{1}{2}\int_\Omega |\nabla\chi(s)|^2 + \int_\Omega \widehat{\beta}(\chi(s)) - \int_\Omega w\chi(s) + \int_s^t \int_\Omega \chi^{3/2}\zeta\nabla\partial_t\tilde{u}_\Gamma \end{aligned} \tag{2.19}$$

for a.e. (s,t).

Note that in order to simplify the notation, in the rest of the paper we will not relabel the subsequences.

Remark 2.1 Due to (2.13)–(2.14), equation (2.17) is actually solved in $L^2(0,T;V')$. So, we will show that we can obtain it in $L^2(0,T;V'_\Gamma)$ due to (2.16) and passing to the limit in (2.9).

Remark 2.2 The energy inequality (2.19) means that the energy of the system, written in terms of the new internal stress ζ decreases along the evolution. Note that in the case that also the applied volume external forces are not zero in (2.17), the energy inequality (2.19) has to be modified by adding on the right-hand side the actual power of the external forces in the interval (s,t), i.e. $\int_s^t \int_\Omega f\partial_t(u - \tilde{u}_\Gamma)$. This type of inequality provide a notion of "energetic solution" which is comparable to the notion introduced for rate-independent evolutions (see [7]). Finally, let us point out that the variable ζ plays the role of an internal stress and it accounts for the product of the (possibly degenerating) phase parameter and the deformations.

For this notion of solution, we are able to prove the following existence result

Theorem 2.3 *Under the assumptions* (2.1)–(2.3)*, there exists a solution for the complete damage system in the sense of Definition* 2.2*.*

The proof of the theorem is presented in Sects. 3 and 4.

3 The approximated problem

In this section, we detail the solution of the approximated problem using the notion of solutions introduced by Definition 2.1. To this aim, we make use of the Schaefer fixed point theorem. In the following, for the sake of simplicity we make use of the same symbol c for possibly different positive constants depending just on Ω, T, and the data of the problem, but not on ε.

The following theorem holds true.

Theorem 3.1 *Let* (2.1), (2.2) *hold. Then, there exists a solution* $(u_\varepsilon, \chi_\varepsilon, \xi_\varepsilon)$ *to* (2.9)–(2.11), (2.8), (2.12), *fulfilling* (2.5)–(2.7)

$$u_\varepsilon \in L^\infty(0,T;W^{1,p}(\Omega)), \quad p > 2, \tag{3.1}$$

$$\chi_\varepsilon \in L^\infty(Q) \cap L^2(0,T;W^{2,p/2}(\Omega)). \tag{3.2}$$

In addition, it is proved the following energy inequality (for a.e. s, t*, including* $s = 0$*)*

$$\begin{aligned}\frac{1}{2}\int_\Omega(\varepsilon+\chi_\varepsilon^2(t))|\nabla u_\varepsilon(t)|^2 + \int_s^t \|\partial_t\chi_\varepsilon\|_H^2 + \frac{1}{2}\|\nabla\chi_\varepsilon(t)\|_H^2 - \int_\Omega w\chi_\varepsilon(t) + \int_\Omega \widehat{\beta}(\chi_\varepsilon(t))\\ \le \int_\Omega w\chi_\varepsilon(s) + \frac{1}{2}\int_\Omega(\varepsilon+\chi_\varepsilon^2(s))|\nabla u_\varepsilon(s)|^2\\ + \frac{1}{2}\|\nabla\chi(s)\|_H^2 + \int_\Omega\widehat{\beta}(\chi_\varepsilon(s)) + \int_s^t\int_\Omega(\varepsilon+\chi_\varepsilon^2)\nabla u_\varepsilon\nabla\partial_t\tilde{u}_\Gamma.\end{aligned} \tag{3.3}$$

Remark 3.2 Note that Theorem 3.1 provides an existence result for a damage system in which stiffness does not completely degenerates once the material is completely damaged.

Remark 3.3 As far as the energy inequality (3.3), we point out that it implies (see (1.7) and (1.8))

$$\int_\Omega \Psi_\varepsilon(t) \le \int_\Omega \Psi_\varepsilon(s) + \int_s^t\int_\Omega(\varepsilon+\chi_\varepsilon^2)\nabla u_\varepsilon\nabla\partial_t\tilde{u}_\Gamma, \tag{3.4}$$

where (cf. (1.7))

$$\Psi_\varepsilon(\nabla u,\chi,\nabla\chi) = \frac{1}{2}(\varepsilon+\chi^2)|\nabla u|^2 + \frac{1}{2}|\nabla\chi|^2 + w(1-\chi) + \widehat{\beta}(\chi). \tag{3.5}$$

Note that in (3.3) the term $\int_s^t \|\partial_t\chi_\varepsilon\|_H^2 \ge 0$ is a dissipative one.

To prove Theorem 3.1, we apply the Schaefer fixed point theorem to a suitable operator whose fixed points will provide a solution to our original problem. We introduce the correct space in which we are looking for the fixed point

$$Y := \{z \in L^2(0,T;H),\ z \in [0,1] \text{ a.e. in } Q\}. \tag{3.6}$$

The operator $\mathcal{T} : Y \to Y$ will be constructed as a composition of the two operators $\mathcal{T}_1$ and $\mathcal{T}_2$ giving, respectively, the solution of (2.9) once χ_ε is fixed and the solution of (2.10) once u_ε is fixed. The construction of $\mathcal{T}$ and the proof of the existence result are as follows.

Step 1: construction of $\mathcal{T}_1$

First of all, we deal with (2.9). Let $\bar{\chi} \in Y$ fixed in place of χ in (2.9) and look for a (unique) solution $u_\varepsilon = \mathcal{T}_1(\chi_\varepsilon)$ to (2.8) and

$$\int_\Omega(\varepsilon+\bar{\chi}^2)\nabla u_\varepsilon\cdot\nabla v = 0, \quad \forall v \in V_0. \tag{3.7}$$

Applying the Lax–Milgram theorem, it is a standard matter to obtain for a.a. $t \in [0,T]$ the existence and uniqueness of a solution to (3.7) $u(t) \in V$, with $(u-\tilde{u}_\Gamma)(t) \in V_0$.

Hence, we can take as test function in (3.7) $v = u - \tilde{u}_\Gamma$ and use the Poincaré inequality (cf. also (3.6)) to infer that

$$\int_\Omega(\varepsilon+\bar{\chi}^2)|\nabla(u_\varepsilon-\tilde{u}_\Gamma)|^2 = -\int_\Omega(\varepsilon+\bar{\chi}^2)\nabla\tilde{u}_\Gamma\nabla(u_\varepsilon-\tilde{u}_\Gamma) \le \delta\|u_\varepsilon-\tilde{u}_\Gamma\|_{V_0}^2 + c_\delta\int_\Omega|\nabla\tilde{u}_\Gamma|^2, \tag{3.8}$$

where δ has to be taken sufficiently small and c_δ (coming by the Young inequality) does not depend on ε. It results (once more using Poincaré inequality)

$$\|u_\varepsilon-\tilde{u}_\Gamma\|_{L^\infty(0,T;V_0)} \le c, \tag{3.9}$$

and consequently (due to the regularity of $\tilde{u}_\Gamma$)

$$\|u_\varepsilon\|_{L^\infty(0,T;V)} \le c. \tag{3.10}$$

Note, in particular, that the constant c in (3.10) does not depend on the choice of $\bar{\chi}$ (as $\bar{\chi} \in [0,1]$ a.e.). Hence, using results introduced, for example, in [27] leads to improve regularity of u_ε (exploiting the fact that now $\varepsilon > 0$), i.e. for a.e. t,

$$\nabla u_\varepsilon(t) \in L^p(\Omega), \quad \text{for some } p > 2. \tag{3.11}$$

Note that p in (3.11) is actually less than 4.

The next step is studying the continuity of $\mathcal{T}_1$. To this aim, let us take $\bar{\chi}_n, \bar{\chi} \in Y$ such that the following strong convergence holds.

$$\bar{\chi}_n \to \bar{\chi} \quad \text{in } Y$$

(where Y is endowed with the topology norm induced by $L^2(0,T;H)$). Setting $u_n = \mathcal{T}_1(\bar{\chi}_n)$ and $u = \mathcal{T}_1(\bar{\chi})$ we show that $u_n \to u$ in $L^2(0,T;V)$ (from now on, in this part of the section, we do not specify the dependence on ε which is fixed). First, we aim to pass to the limit in (3.7) written for $\bar{\chi}_n$. Note that (3.10) holds for u_n independently of n. Thus, by weak star compactness results, we extract a subsequence such that

$$u_n \rightharpoonup^* u \quad \text{in } L^\infty(0,T;V). \tag{3.12}$$

Then due to the fact that $\bar{\chi}_n$ is uniformly bounded (see (3.6)) and (3.10) holds, we can easily deduce that

$$\|(\varepsilon + \bar{\chi}_n^2)\nabla u_n\|_{L^\infty(0,T;H)} \le c,$$

and thus (for some subsequence), the following convergence holds

$$(\varepsilon + \bar{\chi}_n^2)\nabla u_n \rightharpoonup^* \zeta \quad \text{in } L^\infty(0,T;H). \tag{3.13}$$

To identify ζ, we argue as follows. First, we observe that $\bar{\chi}_n$ converges a.e., and the same holds for $\bar{\chi}_n^2$. Thus, it is clear that thanks to the Lebesgue-dominated convergence theorem, up to a subsequence,

$$\bar{\chi}_n^2 \to \bar{\chi}^2 \quad \text{in } L^s(Q), \quad \text{for any } s < +\infty, \tag{3.14}$$

so that (cf. (3.12)) ζ in (3.13) can be identified with $\zeta = (\varepsilon + \bar{\chi}^2)\nabla u$.

At this point, we are in the position to pass to the limit in (3.7) and, by uniqueness of the solution for the limit equation, identify $u = \mathcal{T}_1(\bar{\chi})$. This allows us to extend the above convergence to the whole sequences.

Now, it remains to prove that u_n strongly converges. To this aim, let us use a contracting argument: we take the difference of (3.7) written for $(u_n, \bar{\chi}_n)$ and $(u, \bar{\chi})$ and fix $v = u_n - u$ (now $v \in V_0$). We have

$$\int_\Omega (\varepsilon + \bar{\chi}_n^2)|\nabla(u_n - u)|^2 = -\int_\Omega (\bar{\chi}_n^2 - \bar{\chi}^2)\nabla u \nabla(u_n - u). \tag{3.15}$$

Thus, owing to (3.14) and (3.11) (note that $|\nabla u|^2$ belongs to $L^{p/2}(\Omega)$, $p/2 > 1$), exploiting the Young inequality, we can deduce

$$\int_\Omega (\frac{\varepsilon}{2} + \bar{\chi}_n^2)|\nabla(u_n - u)|^2 \le \frac{2}{\varepsilon}\int_\Omega |\bar{\chi}_n^2 - \bar{\chi}^2|^2|\nabla u|^2 \to 0. \tag{3.16}$$

Thus, at the end (once more using Poincaré's inequality)

$$u_n \to u \quad \text{in } L^\infty(0,T;V). \tag{3.17}$$

Step 2: construction of $\mathcal{T}_2$

The second step consists in fixing $u_\varepsilon = \mathcal{T}_1(\bar{\chi})$ on the right-hand side of (2.10). We denote with $\mathcal{T}_2$ the operator that assign to u_ε the solution of (2.10), namely we set

$$\chi_\varepsilon = \mathcal{T}_2(\mathcal{T}_1(\bar{\chi}))$$

Once u_ε is fixed with regularity above (3.10)–(3.11), the operator is well-defined thanks to standard results in the theory of parabolic equations with maximal monotone operators. More precisely, the existence of χ_ε follows by parabolic theory combined with a fixed point argument; namely, one first solves (2.10) for a fixed χ_ε (and u_ε) on the right-hand side, and subsequently, one uses a fixed point theorem. We will skip this argument and refer, for example, to [2]. In particular, let us point out that, once we have solved (2.10), we observe that the right-hand side belongs to $L^\infty(0, T; L^{p/2}(\Omega))$. Thus, the corresponding equation makes sense, for example, in the dual of $L^2(0, T; L^{p^*}(\Omega)) \cap L^2(0, T; V)$, where $\frac{1}{p^*} + \frac{2}{p} = 1$ and the corresponding solution χ belongs to Y. Uniqueness of the solution mainly follows from monotonicity and contracting arguments.

Now, let us test (2.10) (where u_ε is fixed) by χ_ε and integrate over $(0, t)$. We obtain, integrating by parts in time and exploiting the positivity of the solution χ_ε,

$$\frac{1}{2}\|\chi_\varepsilon(t)\|_H^2 - \frac{1}{2}\|\chi_0\|_H^2 + \int_0^t \|\nabla\chi_\varepsilon\|_H^2 + \int_0^t\int_\Omega \chi_\varepsilon^2|\nabla u_\varepsilon|^2 \le \int_0^t\int_\Omega w\chi_\varepsilon. \tag{3.18}$$

Here, we have used the monotonicity of β and (2.1), so that

$$\int_0^t\int_\Omega \xi_\varepsilon\chi_\varepsilon \ge 0.$$

Hence, just applying the Gronwall lemma we get

$$\|\chi_\varepsilon\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \le c, \tag{3.19}$$

and the constant c actually does not depend on u_ε.

Then, we can (formally) test (2.10) by $\xi_\varepsilon \in \beta(\chi_\varepsilon)$ and integrate over $(0, t)$

$$\int_0^t\int_\Omega |\xi_\varepsilon|^2 + \int_0^t\int_\Omega \chi_\varepsilon\xi_\varepsilon|\nabla u_\varepsilon|^2 \le \int_0^t\int_\Omega w\xi_\varepsilon. \tag{3.20}$$

In particular, here we have used the fact that a chain rule for maximal monotone graphs and the definition of β (see also (2.1)) lead to

$$\int_0^t\int_\Omega \xi_\varepsilon\partial_t\chi_\varepsilon = \int_\Omega \widehat{\beta}(\chi_\varepsilon(t)) - \widehat{\beta}(\chi_0) \ge 0.$$

and analogously, by monotonicity of β and the chain rule, we have also

$$\int_0^t\int_\Omega (-\Delta\chi_\varepsilon)\xi_\varepsilon \ge 0.$$

Hence, by definition of β (as $\xi_\varepsilon \in \beta(\chi_\varepsilon)$ a.e.) we can deduce that,

$$\int_0^t\int_\Omega \chi_\varepsilon\xi_\varepsilon|\nabla u_\varepsilon|^2 \ge 0. \tag{3.21}$$

Eventually, due to the Young inequality applied to (3.20), it follows

$$\|\xi_\varepsilon\|_{L^2(0,T;H)}^2 \le c. \tag{3.22}$$

Now, we have deduced that the equation may be rewritten (weakly) as

$$\partial_t\chi_\varepsilon - \Delta\chi_\varepsilon = -\xi_\varepsilon + w - \chi_\varepsilon|\nabla u_\varepsilon|^2, \tag{3.23}$$

and the right-hand side is bounded, at least, in $L^2(0, T; L^{p/2}(\Omega))$. Thus, by parabolic arguments, we have in addition that

$$\|\chi\|_{W^{1,p/2}(0,T;L^{p/2}(\Omega))\cap L^{p/2}(0,T;W^{2,p/2}(\Omega))} \le c. \tag{3.24}$$

Remark 3.4 Note that in the case of a double nonlinear structure of the equation, e.g. in the case of an irreversible constraint for the evolution of the damage parameter (i.e. for the sign of its time derivative), we could not perform the above estimate, and thus, we could not pass to the limit in the equation. Here we are actually able to identify a.e. the internal reaction ξ ensuring the physical constraint on the damage parameter. Some weak results for irreversible evolution can be found, for example, in [20,24].

Step 3: construction of $\mathcal{T}$

Now, we have constructed the operator $\mathcal{T} : Y \to Y$, defined by

$$\mathcal{T}(\bar{\chi}) := \mathcal{T}_2(\mathcal{T}_1(\bar{\chi}))$$

which turns out to be well-defined and compact thanks to (3.19) and (3.22) and the compactness results in [26]. To prove that it is continuous (with respect to the norm of Y induced by $L^2(0, T; H)$), we take a sequence in Y $\bar{\chi}_n \to \bar{\chi}$ in $L^2(0, T; H)$ and show that $\chi_n = \mathcal{T}(\bar{\chi}_n) \to \chi = \mathcal{T}(\bar{\chi})$ with respect to the same norm. Here, we do not explicitly write the dependence on ε of the solutions. Owing to the continuity of $\mathcal{T}_1$ (we have proved before), we have that

$$\mathcal{T}_1(\bar{\chi}_n) \to \mathcal{T}_1(\bar{\chi})$$

in $L^\infty(0, T; V)$. Hence, thanks to (3.19) and (3.22) and weak–strong compactness, we deduce (at least for subsequence)

$$\chi_n \rightharpoonup^* \chi \quad \text{in } W^{1,p/2}(0, T; L^{p/2}(\Omega)) \cap L^{p/2}(0, T; W^{2,p/2}(\Omega)) \cap L^2(0, T; V) \cap L^\infty(Q), \tag{3.25}$$
$$\chi_n \to \chi \quad \text{in } L^2(0, T; H). \tag{3.26}$$

Actually, due to (3.25)–(3.26) and the Lebesgue theorem we also have that

$$\chi_n \to \chi \quad \text{in } L^s(Q),\ s < +\infty. \tag{3.27}$$

Finally (see (3.22))

$$\xi_n \rightharpoonup \xi \quad \text{in } L^2(0, T; H). \tag{3.28}$$

Note that, combining (3.28) with (3.26), we can identify $\xi \in \beta(\chi)$.Thus, we are in the position of passing to the limit in (2.10) written for n as $n \to +\infty$), exploiting in particular (3.25). Note that the term $\chi_n|\nabla u_n|^2$ is bounded in $L^\infty(0, T; L^{p/2}(\Omega))$ and that it converges a.e. to $\chi|\nabla u|^2$ due to the strong convergence of χ_n and u_n, so that we can identify its weak limit

$$\chi_n|\nabla u_n|^2 \rightharpoonup^* \chi|\nabla u|^2 \quad \text{in } L^\infty(0, T; L^{p/2}(\Omega)). \tag{3.29}$$

By uniqueness of the solution (once u is fixed), we can identify $\chi = \mathcal{T}(\bar{\chi})$, which concludes our proof.

Step 4: conclusion of the proof

Now, we can conclude the proof of the theorem. We use the Schaefer's fixed point theorem [10, Ch. 9,Theorem 4]. Since ε is fixed, also in this part of the section, we remove the ε-dependence from u_ε and χ_ε. Thus, assume that for some $\lambda \in [0, 1]$ we have $\chi_\lambda \in Y$ such that $\chi_\lambda = \lambda\mathcal{T}(\chi_\lambda)$. Let $u_\lambda = \mathcal{T}_1(\chi_\lambda)$. By construction, χ_λ, u_λ solve

$$\begin{cases} \partial_t \chi_\lambda - \Delta\chi_\lambda + \xi_\lambda + \chi_\lambda|\nabla u_\lambda|^2 = \lambda w, & \text{in } (L^2(0, T; L^{p^*}(\Omega)) \cap L^2(0, T; V))' \\ \xi_\lambda \in \beta(\chi_\lambda) \text{ in } \ L^2(0, T; H). \end{cases} \tag{3.30}$$

First of all note that $\chi_\lambda \in L^\infty(Q)$ (actually much more is true). Thus, testing the equation with χ_λ and using the monotonicity of β (see Step 2 above, namely estimate (3.19)), we have the following estimate which turns to be independent of λ

$$\|\chi_\lambda\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \le c$$

with $c > 0$ independent of λ (and of u_λ). The estimate above gives that the set

$$\mathcal{X} := \bigcup_{\lambda\in[0,1]} \{\chi_\lambda : \quad \chi_\lambda = \lambda\mathcal{T}(\chi_\lambda)\}$$

is bounded in Y. Thus, the Schaefer's theorem gives the existence of a fixed point χ for $\mathcal{T}$ and thus the existence of a solution to the approximate problem (2.9)–(2.10) $(\mathcal{T}_1(\chi), \chi)$.

4 A priori estimates and passage to the limit

In this section, we aim to pass to the limit in (2.9)–(2.10) as $\varepsilon \searrow 0$. To this aim, we perform some further estimates on the solutions, which do not depend on the parameter ε. As a result, we will prove the existence Theorem 2.3. Let us point out that some of the estimates below are formally performed. Indeed, we should proceed by a further regularization of the equations to get sufficient regularity for the test functions, approximated the maximal monotone graph by its Moreau–Yosida approximation, and then passing to the limit. However, the procedure is analogous of that we are going to detail, and thus for the sake of clarity, we directly deal with the limit system and proceed formally.

4.1 First a priori estimate

We prove inequality (3.3). To this aim, we test (2.9) by $\partial_t(u_\varepsilon - \tilde{u}_\Gamma)$, (2.10) by $\partial_t\chi_\varepsilon$, add the resulting equations and integrate over $(0, t)$. Note that we are allowed to use these test functions by virtue of (2.5), (2.6), (2.8). Combining some terms and integrating by parts in time, we get (recall that in our model w is a constant)

$$\begin{aligned} &\varepsilon\int_0^t\int_\Omega |\nabla\partial_t u_\varepsilon|^2 + \int_0^t\int_\Omega \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}((\varepsilon+\chi_\varepsilon^2)|\nabla u_\varepsilon|^2) + \int_0^t\int_\Omega(\varepsilon+\chi_\varepsilon^2)\nabla u_\varepsilon\nabla\partial_t\tilde{u}_\Gamma + \int_0^t\int_\Omega \|\partial_t\chi_\varepsilon\|_H^2 \\ &\quad + \int_\Omega\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\nabla\chi_\varepsilon\|_H^2 - \int_0^t\int_\Omega w\partial_t\chi_\varepsilon \le 0, \end{aligned} \tag{4.1}$$

as, by applying the chain rule for sub-differentials, it follows

$$\int_0^t\int_\Omega \beta(\chi_\varepsilon)\partial_t\chi_\varepsilon = \int_\Omega I_{[0,1]}(\chi_\varepsilon(t)) - \int_\Omega I_{[0,1]}(\chi_0) = 0, \tag{4.2}$$

as $\chi_0 \in [0, 1]$ a.e. Thus, (3.3) easily follows. Now, exploiting the Hölder and the Young inequalities, we can get from (4.1) the following bound

$$\begin{aligned} &\varepsilon\textstyle\int_0^t \|\nabla\partial_t u_\varepsilon\|_H^2 + \frac{\varepsilon}{2}\|\nabla u_\varepsilon(t)\|_H^2 + \frac{1}{2}\|\chi_\varepsilon(t)\nabla u_\varepsilon(t)\|_H^2 + \int_0^t\|\partial_t\chi_\varepsilon\|_H^2 + \frac{1}{2}\|\nabla\chi_\varepsilon(t)\|_H^2 \\ &\quad \le c_0 + \textstyle\frac{1}{2}\int_0^t\|\partial_t\chi_\varepsilon\|_H^2 + c\int_0^t(\varepsilon^{1/2}\|\nabla u_\varepsilon\|_H + \|\chi_\varepsilon\nabla u_\varepsilon\|_H)\|\nabla\partial_t\tilde{u}_\Gamma\|_H \end{aligned} \tag{4.3}$$

where c_0 depends in particular on the initial data and on w.

Thus, using Gronwall's lemma (and Poincaré's inequality), by virtue of (2.3), we may conclude that (see for a comparison (3.24))

$$\|\chi_\varepsilon\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \le c \tag{4.4}$$

$$\varepsilon^{1/2}\|u_\varepsilon\|_{H^1(0,T;V)} \le c, \tag{4.5}$$

$$\|\chi_\varepsilon\nabla u_\varepsilon\|_{L^\infty(0,T;H)} \le c. \tag{4.6}$$

Note that here the constant c does not depend on ε.

4.2 Second a priori estimate

We can proceed testing (2.10) by $\xi_\varepsilon \in \beta(\chi_\varepsilon)$ (cf. (2.7)) and integrating in time. We first observe that by monotonicity of β we get

$$\int_0^t \langle -\Delta\chi_\varepsilon, \xi_\varepsilon\rangle \geq 0. \tag{4.7}$$

Thus, also recalling (4.2), we have

$$\int_0^t \int_\Omega |\xi_\varepsilon|^2 \leq \int_0^t \int_\Omega w|\xi_\varepsilon| - \int_0^t \int_\Omega \chi_\varepsilon |\nabla u_\varepsilon|^2 \xi_\varepsilon, \tag{4.8}$$

so that we can eventually deduce

$$\|\xi_\varepsilon\|_{L^2(0,T;H)} \leq c, \tag{4.9}$$

applying the Young inequality and observing that (cf. (3.21))

$$-\int_0^t \int_\Omega \chi_\varepsilon |\nabla u_\varepsilon|^2 \xi_\varepsilon \leq 0. \tag{4.10}$$

4.3 Third a priori estimate

Finally, if we test (2.10) by the constant function 1 and integrate in time, we get

$$\int_0^t \int_\Omega \chi_\varepsilon |\nabla u_\varepsilon|^2 = -\int_\Omega \chi_\varepsilon(t) + \int_\Omega \chi_\varepsilon(0) - \int_0^t \int_\Omega \xi_\varepsilon + \int_0^t \int_\Omega w \leq c, \tag{4.11}$$

where the boundedness of the right-hand side follows from (4.4) and (4.9). Thus, we can deduce

$$\|\chi_\varepsilon^{1/2}\nabla u_\varepsilon\|_{L^2(0,T;H)} \leq c. \tag{4.12}$$

4.4 Passage to the limit

Using compactness arguments, by virtue of (4.4) we infer that there exists $\chi \in H^1(0,T;H) \cap L^\infty(0,T;V)$ and a subsequence of ε (not relabelled) such that

$$\chi_\varepsilon \rightharpoonup *\chi \quad \text{in } H^1(0,T;H) \cap L^\infty(0,T;V), \tag{4.13}$$

and by strong compactness (see [26])

$$\chi_\varepsilon \to \chi \quad \text{in } C^0([0,T];H^{1-\sigma}(\Omega)),\ \sigma > 0. \tag{4.14}$$

As a consequence, after recalling that $\|\chi_\varepsilon\|_{L^\infty(q)} \leq c$ (as $\chi_\varepsilon \in [0,1]$ a.e.), applying the generalized Lebesgue theorem, one may deduce

$$\chi_\varepsilon \xrightarrow{\varepsilon \searrow 0} \chi \quad \text{in } L^p(Q) \ \text{ for any } \ p < +\infty. \tag{4.15}$$

Hence, due to (4.9), we have

$$\xi_\varepsilon \rightharpoonup \xi \quad \text{in } L^2(0,T;H), \tag{4.16}$$

for some ξ, which we are able to identify as an element of the sub-differential. Indeed, combining (4.15) and (4.16), it follows that $\xi \in \beta(\chi)$. Then, by (4.12) there exists some $\zeta \in L^2(0,T;H)$ such that

$$\chi_\varepsilon^{1/2}\nabla u_\varepsilon \rightharpoonup \zeta \quad \text{in } L^2(0,T;H). \tag{4.17}$$

Now, we are in the position of passing to the limit (weakly) in (2.9), after observing that (4.15) and (4.17) lead to

$$\chi_\varepsilon^2 \nabla u_\varepsilon \rightharpoonup \chi^{3/2}\zeta \quad \text{in } L^2(0,T;H). \tag{4.18}$$

In particular, after observing that

$$\varepsilon\nabla u_\varepsilon \to 0 \quad \text{in } L^\infty(0,T;H), \tag{4.19}$$

due to (4.5), we get the weak formulation (actually in V_0) of (2.17). Now, we deal with the passage to the limit (in V') for (2.10). First, we note that on the left-hand side, we just get the weak limit exploiting (4.13) and (4.16). As far as the quadratic term on the right-hand side, due to the fact that we have just a weak convergence in $L^2(0,T;H)$ of $\chi_\varepsilon^{1/2}\nabla u_\varepsilon$ (see (4.17)), we can deduce that there exists some positive measure μ such that, in the sense of measures

$$\chi_\varepsilon|\nabla u_\varepsilon|^2 \to \zeta^2 + \mu. \tag{4.20}$$

Thus, passing to the limit as $\varepsilon \searrow 0$, we get equation (2.18) verified in the sense of distributions. Indeed, we recall that if f_ε weakly converges to f in $L^2(Q)$, then there exists a positive Radon measure μ (named the defect measure) such that

$$f_\varepsilon^2 \to f^2 + \mu \quad \text{in the sense of Radon measures.} \tag{4.21}$$

This amounts to say that

$$\langle f_\varepsilon^2, \phi\rangle = \langle f^2, \phi\rangle + \langle \mu, \phi\rangle, \quad \forall \phi \in C_c(Q).$$

Now, to prove (4.21) recall that being the sequence $g_\varepsilon := |f_\varepsilon - f|^2$ (here $f_\varepsilon = \chi_\varepsilon^{1/2}\nabla u_\varepsilon$) bounded in $L^1(Q)$, we have that there exists a positive Radon measure μ with finite mass for which

$$\lim_{\varepsilon\searrow 0}\int_Q |f_\varepsilon(x,t) - f(x,t)|^2\phi(x,t)\mathrm{d}x\,\mathrm{d}t = \langle \mu, \phi\rangle, \quad \forall \phi \in C_c(Q).$$

Thus, expanding the square in the left, we have

$$\lim_{\varepsilon\searrow 0}\int_Q \Big(|f_\varepsilon(x,t)|^2\phi(x,t) - 2f_\varepsilon(x,t)f(x,t)\phi(x,t)\Big)dx\,dt = -\int_Q |f(x,t)|^2\phi(x,t)dx\,dt + \langle \mu, \phi\rangle,$$

which, recalling that f_ε weakly converges to f in $L^2(Q)$, gives (4.21). We can observe that μ is a positive defect measure which expresses the lack of strong convergence in $L^2(Q)$ of the sequence $v_\varepsilon = \chi_\varepsilon^{1/2}\nabla u_\varepsilon$. In particular, there holds that $\mu(A) = 0$ if and only if v_ε strongly converges to ζ in $L^2(A)$ for $A \subseteq Q$. We will discuss in the next section some properties of this measure μ.

4.5 Energy inequality

We apply weak semicontinuity of norms to deduce from (4.13), (4.15), (4.16), (4.17) that for almost all s, t

$$\|\chi^{1/2}(t)\zeta(t)\|_H \le \liminf_{\varepsilon\searrow 0}\|\chi_\varepsilon(t)\nabla u_\varepsilon(t)\|_H, \tag{4.22}$$

$$\|\nabla\chi(t)\|_H \le \liminf_{\varepsilon\searrow 0}\|\nabla\chi_\varepsilon(t)\|_H, \tag{4.23}$$

$$\|\partial_t\chi\|_{L^2(s,t;H)} \le \liminf_{\varepsilon\searrow 0}\|\partial_t\chi_\varepsilon\|_{L^2(s,t;H)}. \tag{4.24}$$

Hence, we apply weak–strong convergence result (see (4.14) and (4.17)) to deal with the limit of (4.1) as $\varepsilon \searrow 0$, where the integration is taken on (s,t). Applying (4.22)–(4.24) to (3.3), it is a standard matter to get that (2.19) follows. We are thus in the position of concluding the proof of Theorem 2.3.

5 The defect measure

It is a challenging and interesting open problem to locate the regions where the measure μ possibly concentrates and, moreover, to characterize them, in a rigorous way, in terms of regions where complete damage appears, i.e. $\chi = 0$. Indeed, in the numerical simulations we are going to present in the following section, it seems that the measure μ is concentrated exactly in those regions where the material experience a complete damage.

In the following Proposition, we try to highlight some relationships between the regions in which χ vanishes, i.e. the damage points, and regions with positive measure μ. However, this Proposition requires some extra assumptions on the momentum u, on the damage function χ and for their approximating sequences. It is not clear whether these (extra) properties are guaranteed or not by the model. Essentially. we need that $\chi_\varepsilon \xrightarrow{\varepsilon\searrow 0} \chi$ strongly in $C^0(Q)$ and some equicontinuity with respect to time for the sequence u_ε. Thus, we prove the Proposition for a slightly modified system for which the above requirements are satisfied. It is a an open problem to prove the same result for the original system.

5.1 The modified system

To ensure equicontinuity for the sequence u_ε, we add a viscosity term to the momentum balance equation. Thus, it will turn out that that the sequence u_ε verifies

$$\|\partial_t u_\varepsilon\|_{L^2(0,T;H)} \leq C, \tag{5.1}$$

for some constant C independent of ε.

Moreover, we require the further assumption for the approximating sequence χ_ε

$$\|\chi_\varepsilon\|_{L^\infty(0,T;W^{1,p}(\Omega))} \leq c, \quad p > 3, \tag{5.2}$$

so that

$$\chi_\varepsilon \to \chi \quad \text{in } C^0(Q), \tag{5.3}$$

and $\chi \in L^\infty(0,T;W^{1,p}(\Omega))$.

The requirement that χ_ε (and hence χ) satisfies (5.2) (and (5.3)) can be accomplished by replacing the Laplacian in (1.11) with a p-laplacian with p sufficiently large, i.e. $p > 3$ in three dimensions. For definitiveness, we replace the original system (2.9)–(2.11) with

$$\partial_t u_\varepsilon - \operatorname{div}\left((\varepsilon + \chi_\varepsilon^2)\nabla u_\varepsilon\right) = 0 \tag{5.4}$$

$$\partial_t \chi_\varepsilon - \Delta_p \chi_\varepsilon + \xi_\varepsilon = w - \chi_\varepsilon|\nabla u_\varepsilon|^2, \tag{5.5}$$

$$\xi_\varepsilon \in \beta(\chi_\varepsilon) \quad \text{a.e. in } \Omega \tag{5.6}$$

coupled with initial conditions for u_ε and for χ_ε and with the same boundary conditions of the original system. The existence of a weak solution for the system above follows from a standard argument (see, for instance, the proof of Theorem 3.1 and [5]). Moreover, the sequences u_ε and χ_ε satisfy the same estimates as in (4.4)–(4.6), (4.9), (4.12) and (5.1). Thus, the analogous of Theorem 2.3 holds with the sole exception that in the notion of solution we have to include the function $u \in H^1(0,T;H)$ and that we add $\partial_t u$ in (2.17).

Proposition 5.1 *Suppose that (5.2) (and thus (5.3)) holds and that there exists $\bar t > 0$ and $\bar x$ such that $\chi(\bar x, \bar t) > 0$, then there exists some $\delta_1, \delta_2 > 0$ and open neighbours of $\bar x$ and of $\bar t$ such that $\mu(B_{\delta_1}(\bar x) \times I_{\delta_2}(\bar t)) = 0$.*

Proof By the assumptions of theorem, there exist $c > 0$ and $\tilde\delta_1, \tilde\delta_2 > 0$ such that

$$\min\{\chi_\varepsilon, \chi\} \geq c \quad \text{in } \bar B_{\tilde\delta_1}(\bar x) \times I_{\tilde\delta_2}(\bar t). \tag{5.7}$$

As a consequence, there holds that (recall (4.6)) uniformly for $t \in I_{\tilde\delta_2}(\bar t)$

$$\int_{B_{\tilde\delta_1}(\bar x)} |\nabla u_\varepsilon(x,t)|^2 dx = \int_{B_{\tilde\delta_1}(\bar x)} \frac{1}{\chi_\varepsilon(x,t)} \chi_\varepsilon(x,t)|\nabla u_\varepsilon(x,t)|^2 dx \leq \frac{1}{c}\int_{B_{\tilde\delta_1}} \chi_\varepsilon(x,t)|\nabla u_\varepsilon(x,t)|^2 dx. \tag{5.8}$$

Thus, we get $\nabla u_\varepsilon \in L^\infty(I_{\tilde\delta_2}(\bar t); L^2(B_{\tilde\delta_1}))$ (see (4.12)). Now we show that actually $u_\varepsilon \in L^2(I_{\tilde\delta_1}(\bar t); W^{2,q}(B_{\tilde\delta_{1/2}}))$, for some suitable q we will chose later on. To this end, consider the equation

$$\partial_t u_\varepsilon - \operatorname{div}\left((\varepsilon + \chi_\varepsilon^2)\nabla u_\varepsilon\right) = 0 \quad \text{in } B_{\tilde\delta_1} \times I_{\tilde\delta_2}(\bar t). \tag{5.9}$$

Let ψ denote a positive, smooth cut-off function (independent of time) such that $\psi \equiv 1$ in $B_{\tilde\delta_{1/2}}(\bar x)$ and $\psi = 0$ on $\Omega \setminus B_{\tilde\delta_1}(\bar x)$.

Then, set $w_\varepsilon := \psi u_\varepsilon$. We have that

$$\operatorname{div}\left((\varepsilon + \chi_\varepsilon^2)\nabla w_\varepsilon\right) = \operatorname{div}\left((\varepsilon + \chi_\varepsilon^2)\psi\nabla u_\varepsilon\right) + \operatorname{div}\left((\varepsilon + \chi_\varepsilon^2)u_\varepsilon\nabla\psi\right)$$

Hence, simple computations show that w_ε solves

$$\begin{cases} \operatorname{div}\left((\varepsilon + \chi_\varepsilon^2)\nabla w_\varepsilon\right) = F_\varepsilon, & \text{in } B_\delta \\ w_\varepsilon = 0 \quad \text{on } \partial B_\delta, \end{cases} \tag{5.10}$$

where

$$F_\varepsilon := -\psi \partial_t u_\varepsilon + 2(\varepsilon + \chi_\varepsilon^2)\nabla u_\varepsilon \cdot \nabla \psi + 2u_\varepsilon \chi_\varepsilon \nabla \chi_\varepsilon \cdot \nabla \psi + 2\psi \chi_\varepsilon \nabla u_\varepsilon \cdot \nabla \chi_\varepsilon + (\varepsilon + \chi_\varepsilon^2) u_\varepsilon \Delta \psi.$$

Now, recall that χ_ε is bounded (uniformly w.r.t. ε) in $L^\infty(0, T; W^{1,p}) \cap L^\infty(Q)$ $(p > 3)$ and it is bounded from below in $B_{\tilde{\delta}_1} \times I_{\tilde{\delta}_2}(\bar{t})$. Moreover, we have that $\partial_t u_\varepsilon$ is uniformly bounded in $L^2(0, T; H)$. Thus, by the Hölder inequality, we have that F_ε is bounded in $L^2(I_{\tilde{\delta}_2}(\bar{t}); L^q(B_\delta))$, where $q > 1$. Note that the value of q exclusively depends on p. In particular, we can choose p so large in such a way that $q = 2 - \rho$ for any $\rho > 0$. Therefore, we have that the standard elliptic regularity (recall that w_ε solves a linear strongly elliptic problem) gives that the function w_ε is bounded in $L^2(I_{\tilde{\delta}_2}(\bar{t}); W^{2,q}(B_{\tilde{\delta}_1}))$, which means that, since $w_\varepsilon \equiv u_\varepsilon$ in $B_{\tilde{\delta}_1/2}$, u_ε is bounded, uniformly w.r.t ε in $L^2(I_{\tilde{\delta}_2}(\bar{t}); W^{2,q}(B_{\tilde{\delta}_1/2}))$. Now, choosing, e.g. $q > 6/5$, by Sobolev's injection and by the Aubin–Lions compactness Lemma, we have that the estimate above implies, up to the extraction of a not relabelled subsequence of ε, $\nabla u_\varepsilon \xrightarrow{\varepsilon \searrow 0} \nabla u$ strongly in $L^2(I_{\tilde{\delta}_2}(\bar{t}); L^2(B_{\tilde{\delta}_1/2}))$, at least. Consequently, we obtain that $\chi_\varepsilon^{1/2} \nabla u_\varepsilon \xrightarrow{\varepsilon \searrow 0} \zeta = \chi^{1/2} \nabla u$ strongly in $L^2(I_{\tilde{\delta}_2}(\bar{t}); L^2(B_{\tilde{\delta}_1/2}))$ and hence $\mu_{\bar{t}}(I_{\tilde{\delta}_2}(\bar{t}); B_{\tilde{\delta}_1/2}) = 0$. Thus, the result follows with the choice $\delta_1 = \tilde{\delta}_1/2$ and $\delta_2 = \tilde{\delta}_2$.

6 Numerical simulations

Now, we present numerical simulations based on the damage models presented in Sect. 3. In particular, two finite element evolutive tests will be illustrated underlying the obtained analytic evidences. Firstly, the paradigmatic case of a one- dimensional bar subjected to traction is proposed, and secondly, a bi-dimensional problem with non-homogenous solution is investigated.

In the numerical simulations, a time discretization of the quasi-static evolution has been considered: we introduce a discrete set of loading parameters $0 = t_0 \leq\leq t_N = t_{\max}$. The solution of system of equations of motion for each time step is consolidated for this kind of problem [8,11,12,19]. The coupled damage-mechanics model is solved in a semi-coupled fashion. At each time step, it is achieved by alternating solution of the equations of system until convergence. Moreover, the constraint $\partial I_{[0,1]}(\chi)$ is solved numerically by projection within the iterative process. In this algorithm, the spatial discretization is obtained using the Galerkin finite element method. The model has been implemented in a program based upon the Open Source package deal.II [1].

In order to limit spurious behaviour due to space and time discretization, specific choices have been adopted. First of all, a simplified version of the model is investigated. In particular, the term $\partial_t \chi$ has been neglected; this assumption avoids the spurious effect of the time discretization of the evolution of the damage variable χ. The effect of this viscous term is to give finite velocity of the damage evolution and does not introduce relevant information to our analysis. Its physical effect is studied at length in [14]. Moreover, the material has been assumed homogenous and with isotropic behaviour. At the beginning of the loading process at time $t = 0$, the material is completely undamaged so $\chi = \chi_0 = 1$. The adopted mesh for the domain is quasi-uniform, and linear shape functions have been chosen to approximate the displacement and the damage fields.

The applied loads are always monotonic and do not change in sign. In particular, all the analyses have been carried out under displacement control. An explicit linear relationship between the time and the imposed displacement is introduced. In the simulations, this avoids local snap-back at the material level.

Finally, the parameter ε has been set equal to 10^{-16} in such a way it only prevents from singularity at the solution stage without introducing any appreciable residual stiffness within the material. Anyway, up to value $\varepsilon = 10^{-6}$ the solution is unaffected by the choice of ε. Higher values of ε as adopted in [8] do not influence the damage path but introduce non-negligible residual stiffness in the material that may lead to unrealistic equilibrium path.

6.1 1D bar in tension

The solution of the evolution problem for the traction of a one- dimensional bar made of homogeneous material of length L, so that $x \in [0, L]$, is described.

The left extremity of the bar $x = 0$ is kept fixed by imposing $u = 0$, while at $x = L$ an incremental displacement $u = t$ is applied. The condition $\chi = 1$ is imposed at the constrained borders. This condition

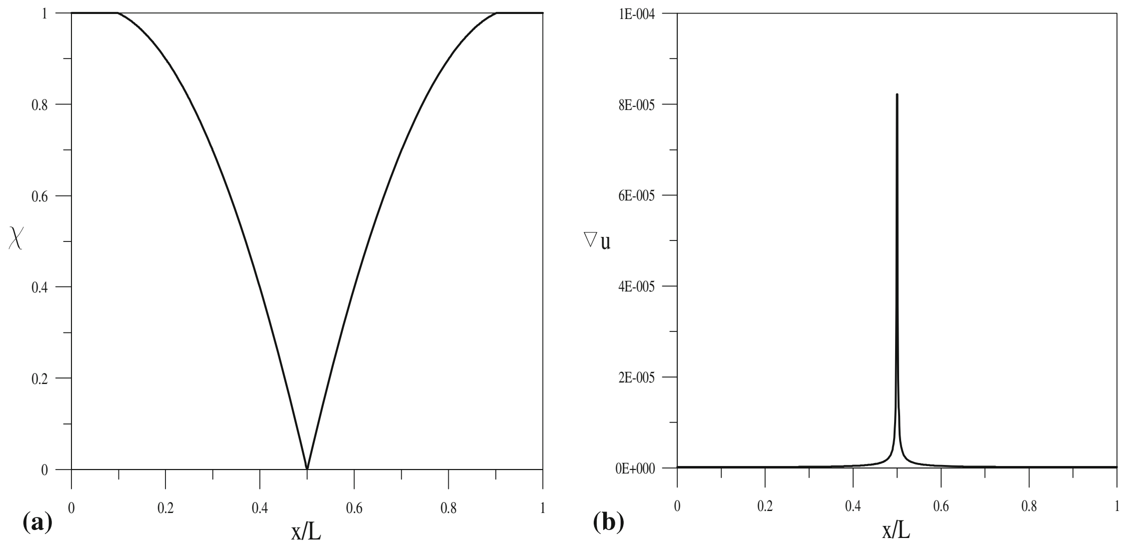

Fig. 1 1D example: maps of the field χ and value of ∇u along the bar after rupture

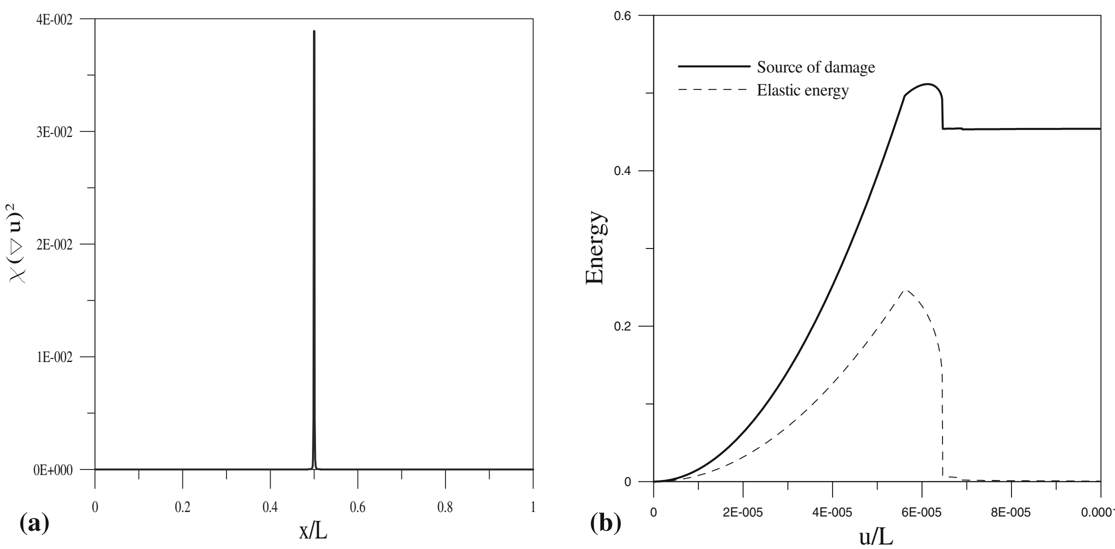

Fig. 2 1D example: **a** maps of source of damage $\chi\,(\nabla u)^2$ along the bar and **b** evolution of elastic energy and source of damage as a function of the imposed displacement

forbids the development of fractures exactly at the boundary, although they are free to appear at a small distance. This effect may well interpret the confining effects offered in a real experimental set-up by fractional contactor gluing of the supports. This aspect is debated at length in [12].

Figure 1 reports the value of χ and of the displacement gradient ∇u along the bar in stress-free condition once that rupture has occurred. Three zones can be distinguished: an unbroken zones where $\chi = 1$ and the gradient of the displacement is zero, two transition zones where χ varies between 1 and 0 and the displacement gradually increases and a central core where the material is disaggregated $\chi \approx 0$ and characterized by high deformation value. Moreover, in Fig. 2a is plotted the value of the source of damage $\chi\,(\nabla u)^2$ along the bar after rupture that is not null only in the central core. Figure 2b plots the elastic energy and the source of damage integrated along the bar as a function of the imposed displacement that are

$$\frac{1}{2}\int_{\Omega} \chi^2\,(\nabla u)^2\,dx\ ,\ \int_{\Omega} \chi\,(\nabla u)^2\,dx, \tag{6.1}$$

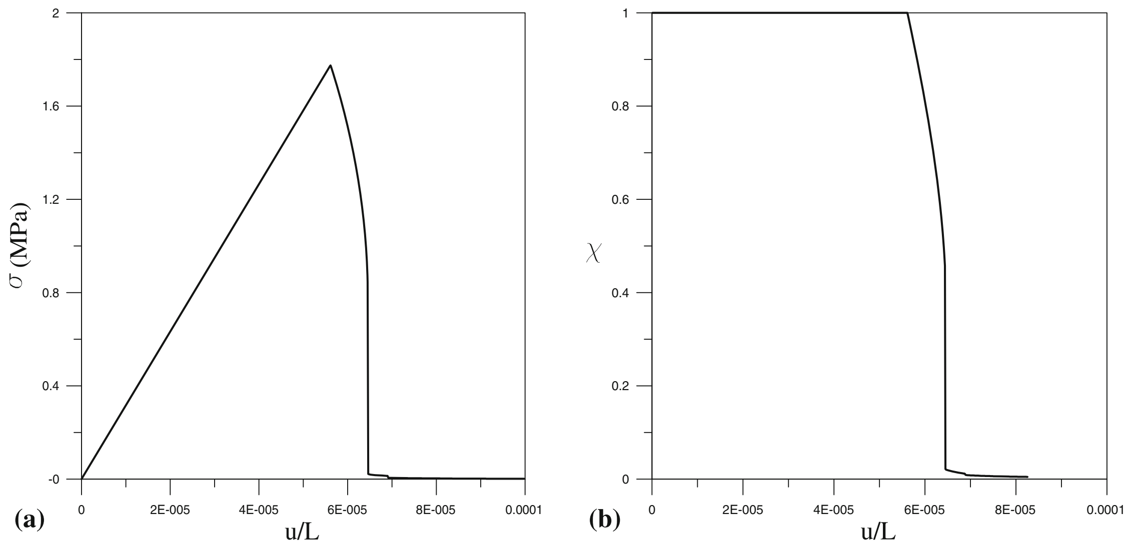

Fig. 3 1D example: **a** stress along the bar as a function of the imposed displacement and **b** evolution of damage variable χ calculated at $x = 0.5L$ as a function of the imposed displacement

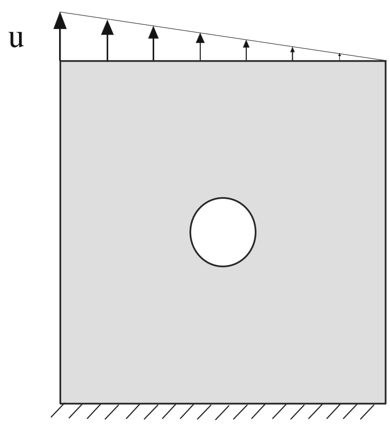

Fig. 4 2D example: geometry and boundary conditions

being $\Omega = [0, L]$. The elastic energy starts growing quadratically. Once that the damaging mechanism begins the elastic energy decreases and assumes null value at rupture. Differently, the source of damage remains constant at rupture, thus revealing a conservation of the energy.

Moreover, the global response of the bar in term of stress–displacement is plotted in Fig. 3a. The bar reveals linear elastic behaviour since the source of damage is smaller than the damage threshold w. After, the response is typical of quasi-brittle material characterized by softening of mechanical properties.

Finally, in Fig. 3b the evolution of damage χ at a point in the middle of the bar $x = 0.5L$ is plotted as a function of the imposed displacement. The material moves from an undamaged to a nearly completed damage state rather rapidly. After the damage variable asymptotically tends to zero as the displacement grows; for example, at $u/L = 8 * 10^{-5}$ the damage is $\chi \approx 5 * 10^{-3}$. At this stage, the constraint $\partial I_{[0,1]}(\chi)$ does not act.

6.2 2D square plate with a hole

Figure 4 depicts the second case that we considered. It represents a two-dimensional problem of a body in plane strain condition. This is composed of a squared linear elastic matrix of side L with a central circular hole of radius R. The solid is homogeneous, and the material is characterized by isotropic constitutive behaviour.

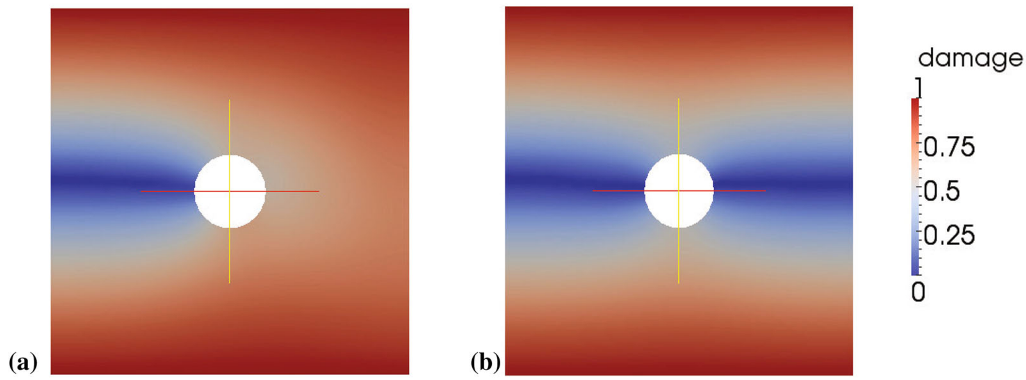

Fig. 5 2D example: maps of the field χ for $u_y(-L/2) = 0.175$ mm and $(-L/2) = 0.4$ mm

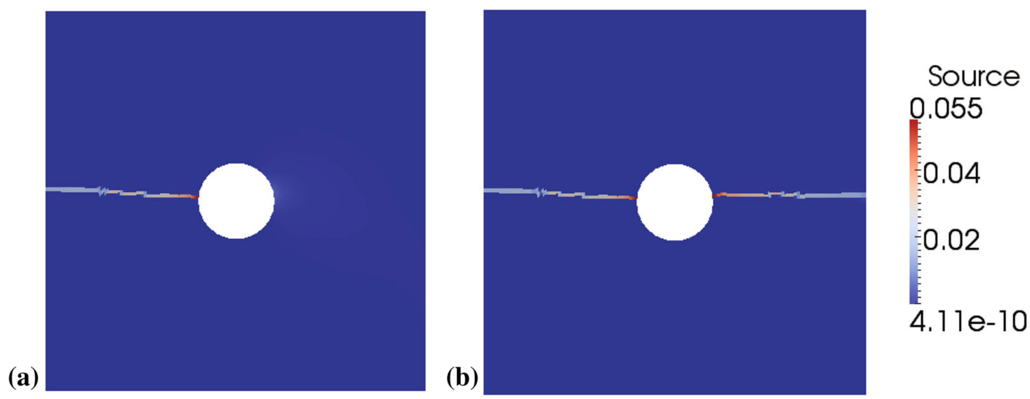

Fig. 6 2D example: maps of the source of damage $\chi\,(\nabla u)^2$ for $u_y(-L/2) = 0.175$ mm and $u_y(-L/2) = 0.4$ mm

The lower portion of the boundary is keep fixed by imposing null values to the displacement components $u_x = 0$, $u_y = 0$, whereas the upper side of the square presents an assigned vertical displacement u_y which varies linearly along the border that reads

$$u_y(x) = t\left(1 - \frac{2x}{L}\right), \tag{6.2}$$

while the horizontal displacement is left free to vary. The two vertical sides are traction free.

Figure 5 plots the evolution of the damage field χ predicted by the model for different values of the displacement field u_y. It clearly appears that, due to the non-homogeneous state of stress within the body, two damage zones initiate near the hole as a consequence of the stress concentration and subsequently propagate up to the vertical borders for different values of the imposed displacement: firstly in the left portion of the body Fig. 5a and then at the right side Fig. 5b.

As for the previous example, in Fig. 6 the values of the source of damage $\chi\,(\nabla u)^2$ in the domain is reported for the two damage states previously illustrated revealing non-null values only in the zones of almost totally damaged materials. This occurs also in the variational approach [8]. Locally, these zones have a width equivalent to the finite element size. It should be underlined the fact that once rupture has occurred in the left portion of the square, here the value of the source of damage remains constant, while it increases in the undamaged right side. Figure 7 plots the elastic energy and the source of damage as reported in (6.1) but now integrated in the square domain as a function of the imposed displacement. The source of damage is an increasing function of the displacement field that reaches an asymptotic value as the damage zones separates into two distinct portions the solid. Contextually, the elastic energy tends to a null value after two softening paths representative of the propagation of the damage inside the domain along the cracked zones.

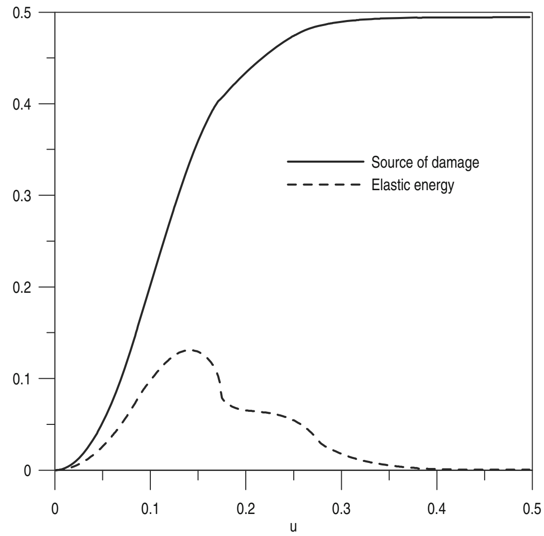

Fig. 7 2D example: evolution of elastic energy and source of damage as a function of the imposed displacement

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