Some exercises to close the a.a. 20-21

1) Use the GEM (no pivoting) to solve the linear system

$$\begin{array}{c} * (-7) \\ * (-3) \\ \end{array} \begin{bmatrix} 1 & 7 & 5 \\ 7 & 50 & 39 \\ 3 & 23 & 25 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{array}{c} \\ * (-2) \\ \end{array} \begin{bmatrix} 1 & 7 & 5 \\ 0 & 1 & 4 \\ 0 & 2 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 7 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$2x_3 = 1 \quad \Rightarrow \quad x_3 = 1/2$$

$$x_2 + 4x_3 = 0 \quad \Rightarrow \quad x_2 = -2$$

$$x_1 + 7x_2 + 5x_3 = 0 \quad \Rightarrow \quad x_1 = +14 - \frac{5}{2} = 23/2$$

2) compute the LU factorization (no pivoting ✓) of the matrix:

$$A = \begin{bmatrix} 1 & 7 & 5 \\ 7 & 50 & 39 \\ 3 & 23 & 25 \end{bmatrix}$$

$$L * U = A$$

$$U = \begin{bmatrix} 1 & 7 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 7 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

3) Starting from $x^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, write 2 its. of the Jacobi method to solve

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 4 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underline{x}^{(n+1)} = - \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 4 \\ 1 & 1 & 0 \end{bmatrix} \underline{x}^{(n)} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

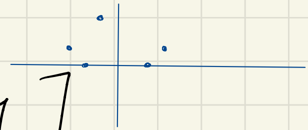
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underline{x}^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \underline{x}^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underline{x}^{(2)} = - \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix}$$

$$\underline{x}^{(2)} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

Compute the regression line for the points: $(-3,1)$, $(-2,0)$, $(-1,3)$, $(2,0)$, $(3,1)$

$$p(x) = a_1 + a_2 x$$



$$\underbrace{\begin{bmatrix} 1 & -3 \\ 1 & -2 \\ 1 & -1 \\ 1 & +2 \\ 1 & +3 \end{bmatrix}}_{A} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

$$A^T A \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = A^T \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -3 & -2 & -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 1 & -2 \\ 1 & -1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ -1 & 27 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -1 \\ -1 & 27 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix} \ \dots$$

Compute two iterations of the power method (for eigenvalue / eigenvector approximation) on the matrix

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

with starting vector $x^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$w^{(1)} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$x^{(1)} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$W^{(2)} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 5 \\ -4 \end{bmatrix}$$

$$x^{(2)} = \frac{W^{(2)}}{\| W^{(2)} \|} = \frac{\cancel{\frac{1}{\sqrt{5}}} \begin{bmatrix} 5 \\ -4 \end{bmatrix}}{\cancel{\frac{1}{\sqrt{5}}} \sqrt{41}} = \frac{\begin{bmatrix} 5 \\ -4 \end{bmatrix}}{\sqrt{41}}$$

$$\left(\underline{x}^{(2)}\right)^T A \, \underline{x}^{(2)} =$$

$$= \frac{1}{41} \cdot \left(\begin{bmatrix} 5 & -4 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ -4 \end{bmatrix} \right)$$

$$= \frac{1}{41} \cdot \left(\begin{bmatrix} 5 & -4 \end{bmatrix} \begin{bmatrix} 14 \\ -13 \end{bmatrix} \right)$$

$$= \frac{70 + 52}{41} = \frac{122}{41}$$