The following exercises are of 3 kinds:

(E) = easy ; (M) = medium or (H) = hard

the basic exam contains one easy exercise and a medium exercise. The maximum grade is 24/30

The advanced exam contains one medium and one hard exercise. The maximum grade is 30L.

The total time for a written exam is 1h.

Linear Systems

(E) With $x^{(0)} = (0, 0, 0)$, compute two iterations of the Jacobi method to solve the system $Ax = b$, where

$$A = \begin{bmatrix} 4 & 2 & 0 \\ 0 & 2 & 1 \\ 1 & -1 & 4 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}.$$

(E) With $x^{(0)} = (0, 0, 0)$, compute two iterations of the Gauss-Seidel method to solve the system $Ax = b$, where

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \qquad b = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

(E) Compute the LU factorization of the matrix

$$A = \begin{bmatrix} 2 & 3 & 4 \\ -4 & -5 & -7 \\ 6 & 8 & 13 \end{bmatrix}$$

(E) Let $b \in \mathbb{R}$. Consider the matrix

$$B = \begin{bmatrix} 2 & b \\ b & 4 \end{bmatrix}$$

1. For which values of b is B positive definite?
2. What is the condition number of B?

(M) Write the pseudocode of forward substitution method used to solve linear systems where the matrix is lower triangular and describe its computational cost

(M) Describe the concept of preconditioner for iterative method. In particular, what are the features of a good preconditioner?

(M) Splitting iterative methods: definition and condition for convergence.

(M) Write the pseudocode of the Gauss-Seidel method.

(H) Write the pseudocode of the Gaussian elimination method (with pivoting).

(H) Write the pseudocode of the LU factorization method, without pivoting, and describe its computing cost.

(H) Consider the linear system $Ax = b$, and its perturbed version $A\tilde{x} = \tilde{b}$. State and prove an upper bound on the relarive error $\|x - \tilde{x}\|_2 \,/\, \|x\|_2$.

(H) Describe a method to compute the inverse of a given nonsingular matrix and discuss its computational cost.

(H) Prove the Theorem of convergence of the Jacobi methods for diagonally dominant matrices.

Eigenvalues

(E) With starting vector $x^{(0)} = (1, 0)$, compute two iterations of the power method on the matrix

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

(H) Describe the Pagerank algorithm explaining the meaning of the various terms and expressions (a discussion of the computational efficiency of the method is not required)

(H) Write the pseudocode of the Pagerank algorithm, possibly in a version which is computationally efficient for large matrices.

Nonlinear Systems

(E) With initial guess x_0 = Fourier point, apply one Newton iteration to the equation

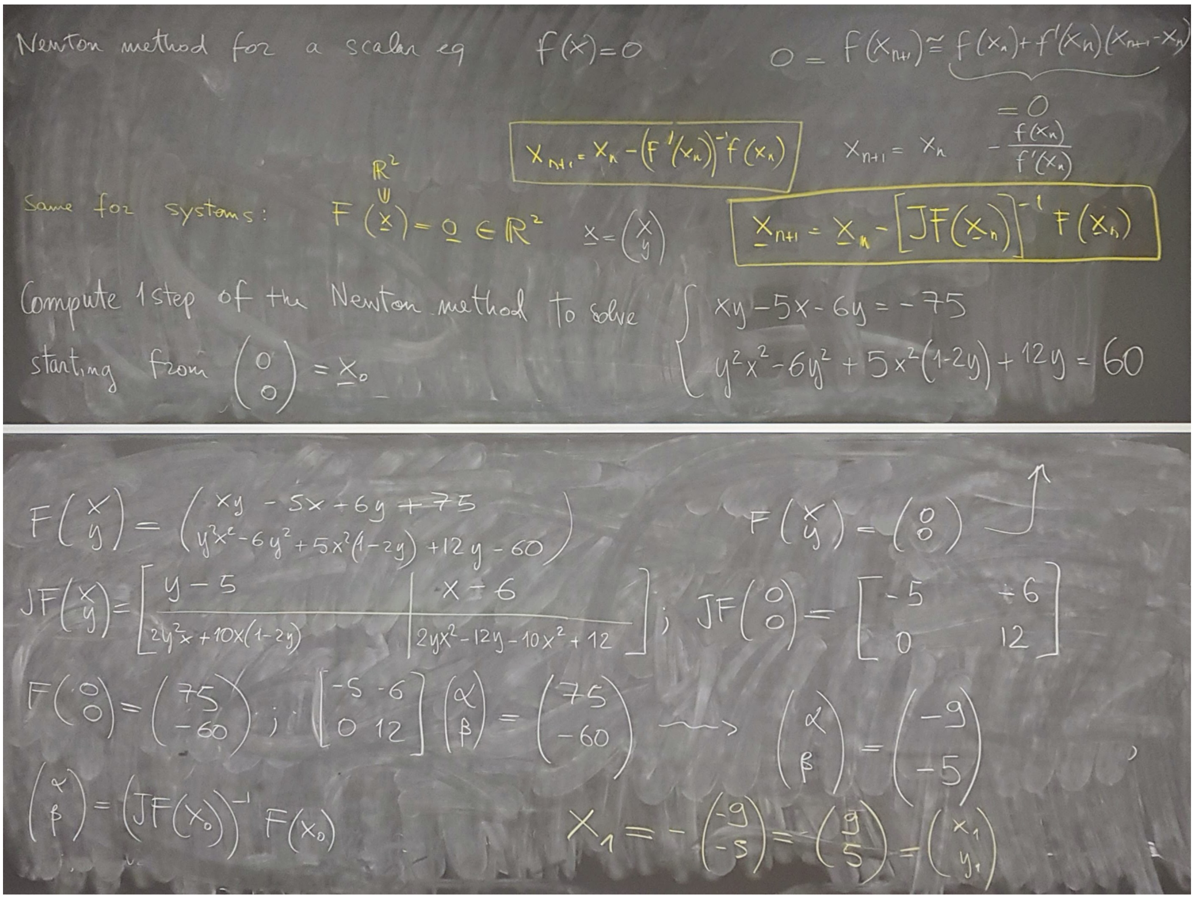
$$(x^2 - 6x + 4)(x + 2) = 0 \qquad x \in [0, 1]\ .$$

(M) Write the pseudocode of the bisection method.

(M) Write the pseudocode of the Newton method.

(M) With initial guess $x^{(0)} = (0, 0)$ apply one Newton iteration to the nonlinear system

$$\begin{cases} xy - 5x - 6y = -75 \\ y^2x^2 - 6y^2 + 5x^2\,(1 - 2y) + 12y = 60 \end{cases}$$



(H) Give the statement of Newton convergence theorem for nonlinear equations. Prove that the order of convergence of the methods is 2.

Least Squares

(E) Compute the linear regression $r(x) = c_0 + c_1 x$ for the set of points

$$(-3, 1),\ (-2, 0),\ (-1, 3),\ (1, 1),\ (2, 4),\ (3, 3).$$

(M) Least square method for data fitting: give the definition and describe its use for computing the linear regression line.

(H) Prove that if $A \in \mathbb{R}^{m\times n}$, with $m > n$, has rank n, the system $A^T Ax = A^T b$ admits a unique solution.

Interpolation and Quadrature

Given $f(x) = -2x^2 + x^3 - 4$ determine its Lagrange interpolant of deg 2 w.r.t. $x_1 = -1$, $x_2 = 0$, $x_3 = 2$.

$-1 \quad 0 \quad 2$

$$L_1(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} = \frac{(x\ )(x-2)}{(-1)(-3)} = \frac{1}{3}(x)(x-2) = \frac{1}{3}x^2 - \frac{2}{3}x$$

$$L_2(x) = \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} = \frac{(x+1)(x-2)}{(1)(-2)} = -\frac{1}{2}(x+1)(x-2) = -\frac{1}{2}(x^2-x-2) = -\frac{1}{2}x^2 + \frac{1}{2}x + 1$$

$$L_3(x) = \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} = \frac{(x+1)(x)}{(3)(2)} = \frac{1}{6}(x^2+x) = \frac{1}{6}x^2 + \frac{1}{6}x$$

$$(\Pi f)(x) = f(x_1)L_1(x) + f(x_2)L_2(x) + f(x_3)L_3(x) \qquad \begin{cases} f(x_1) = -2-1-4 = -7 \\ f(x_2) = -4 \\ f(x_3) = -8+8-4 = -4 \end{cases}$$

$$= \underbrace{\left(-\frac{7}{3} + 2 - \frac{2}{3}\right)}_{-1} x^2 + \underbrace{\left(+\frac{14}{3} - 2 - \frac{2}{3}\right)}_{2} x - 4$$

(E) Given the function $f(x) = -2x^2 + x - x$, use the composite midpoint rule to approximate the value of $\int_{-1}^{2} f(x)\,dx$. Use three subintervals

(E) Define the Simpson rule on one interval; use it to compute $\int_{-1}^{1} x^2\,dx$; check its exactness

(M) Write the pseudo code of a composite Simpson rule

(H) State and prove the theorem of existence and uniqueness of the Lagrange interpolant of a given function

(E) Given $f(x) = x^4 - 2x^3$ compute its Lagrange interpolant of degree 3

(E) Given $f(x) = x^3 - 2x^2 + x - 4$, use the composite midpoint rule on $[-3,3]$, splitting the interval in 3 subintervals (uniform subdivision), in order to approximate

$$\int_{-3}^{3} f(x)\,dx$$

(H) Give a description of the composite midpoint quadrature rule, state and prove an error bound.

(H) What is the order of precision (or exactness) of the midpoint rule? State and prove it.

Ordinary differential equations

(E)

$$\begin{cases} y'(t) = t\,y(t) \\ y(1) = 1 = y_0 \end{cases}$$

Compute 1 step of Heun method with $\Delta t = \frac{1}{2}$

$f(t,y) = t \cdot y$

H.M.
$$\begin{cases} y^*_{n+1} = y_n + \Delta t\, f(t_n, y_n) \\ y_{n+1} = y_n + \underbrace{\frac{\Delta t}{2}\Big(f(t_n, y_n) + f(t_{n+1}, y^*_{n+1})\Big)}_{\simeq \int_{t_n}^{t_{n+1}} f(t, y(t))\,dt} \end{cases}$$

$t_0 = 1 \qquad t_1 = t_0 + \Delta t = \frac{3}{2}$

$y^*_1 = y_0 + \Delta t\, f(t_0, y_0) = 1 + \frac{1}{2} = \frac{3}{2}$

$y_1 = y_0 + \frac{\Delta t}{2}\Big(f(t_0, y_0) + \underbrace{f(t_1, y^*_1)}_{\frac{3}{2}\cdot\frac{3}{2}}\Big) = 1 + \frac{1}{4}\left(1 + \frac{9}{4}\right) = 1 + \frac{13}{16} = \frac{29}{16}$

(M) Write the (pseudo) code for the Heun scheme

(H) Derive the explicit Euler, implicit Euler, Crank-Nicolson methods from suitable quadrature rules, explaining each step