Exercises NMES - 15.1.2025

Exercise 1

Write the LU factorization, without pivoting, of:

$$\begin{array}{c} (r_1) \\ (r_2) \\ (r_3) \end{array} \begin{bmatrix} 2 & 4 & 4 \\ 1 & 5 & 7 \\ 3 & 12 & 18 \end{bmatrix}$$

showing the intermediate computations.

Solution: The steps are the same as for the Gaussian elimination method. Let $L = I_n$ and compute the entries in the first column of L while eliminating the elements in the first column of A below the diagonal: $l_{2,1} = \frac{a_{2,1}}{a_{1,1}} = \frac{1}{2}$ and $l_{3,1} = \frac{a_{3,1}}{a_{1,1}} = \frac{3}{2}$. Now replace the row (r_2) and (r_3) with

$$(r_2) - l_{2,1} \cdot (r_1) = (1,5,7) - \frac{1}{2} \cdot (2,4,4) = (1,5,7) - (1,2,2) = (0,3,5)$$
$$(r_3) - l_{3,1} \cdot (r_1) = (3,12,18) - \frac{3}{2} \cdot (2,4,4) = (3,12,18) - (3,6,6) = (0,6,12)$$

respectively. Now the matrix A becomes

$$\begin{array}{c} (r_1) := (r_1) \\ (r_2) := (r_2) - l_{2,1} \cdot (r_1) \\ (r_3) := (r_3) - l_{3,1} \cdot (r_1) \end{array} \begin{bmatrix} 2 & 4 & 4 \\ 0 & 3 & 5 \\ 0 & 6 & 12 \end{bmatrix}$$

and L is

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & 0 & 1 \end{bmatrix}.$$

Now repeat the computations above to eliminate the portion of the second row of A below the diagonal. So $l_{3,2} = \frac{a_{3,2}}{a_{2,2}} = 2$; replace now the third row (r_3) with $(r_3) - l_{3,2}(r_2)$.

$$(r_3) - l_{3,2} \cdot (r_2) = (0,6,12) - 2 \cdot (0,3,5) = (0,6,12) - (0,6,10) = (0,0,2).$$

So the matrix A has now become

$$\begin{array}{c} (r_1) := (r_1) \\ (r_2) := (r_2) \\ (r_3) := (r_3) - l_{3,2} \cdot (r_2) \end{array} \begin{bmatrix} 2 & 4 & 4 \\ 0 & 3 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

and L is

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & 2 & 1 \end{bmatrix}.$$

Exercise 2

Compute the linear regression $r(x) = c_0 + c_1 x$ for the set of points

$$(-3,0),\ (-2,0),\ (-1,0),\ (1,1),\ (2,2),\ (3,4).$$

Solution: To compute the solution recall the least square linear problem

$$\begin{bmatrix} m & \sum_{i=1}^m x_i \\ \sum_{i=1}^m x_i & \sum_{i=1}^m (x_i)^2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^m y_i \\ \sum_{i=1}^m y_i x_i \end{bmatrix}.$$

which in this case is

$$\begin{bmatrix} 6 & 0 \\ 0 & 28 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 17 \end{bmatrix},$$

gives the solution which is

$$c_0 = \frac{7}{6}, \qquad c_1 = \frac{17}{28}.$$

Exercise 3

Starting from $x^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, compute 2 iterations of the Jacobi method applied to the system $Ax = b$, where

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Solution: Recall that for iterative methods, we can split $A = M - N$, so that the general step of an iterative method is

$$x^{(k+1)} = x^{(k)} + M^{-1} r^{(k)}$$

In the case of Jacobi method, $M = \texttt{diag}(A)$

$$M = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Before starting, compute the residual $r^{(0)}$

$$r^{(0)} = b - Ax^{(0)} = b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

First iteration
Firstly, we have to compute $M^{-1}r^{(0)}$, which amounts to solving the linear system
$$Mu = r^{(0)}.$$
Since M is diagonal, it is easy to solve the linear system, obtaining the solution u
$$u = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix},$$
so $x^{(1)} = x^{(0)} + u = (1/2,\ 0,\ 1/2)^T$. Then compute the residual $r^{(1)} = b - Ax^{(1)}$
$$r^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{3}{2} \\ 0 \\ \frac{3}{2} \end{bmatrix} = - \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}$$

Second iteration
As before, compute the solution of the linear system $Mu = r^{(1)}$, that yields
$$x^{(2)} = x^{(1)} + u = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix} - \begin{bmatrix} \frac{1}{4} \\ 0 \\ \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ 0 \\ \frac{1}{4} \end{bmatrix}.$$

Exercise 4

Apply two bisection iterations to solve the equation
$$x^3 + 3x - 2 = 0 \qquad in\,[0, 1].$$
Solution: Let $F(x) = x^3 + 3x - 2$. We are looking the solution in the interval $[0, 1]$, so first check that $F(0)$ and $F(1)$ have different sign. Indeed
$$F(0) = -2, \qquad F(1) = 2$$
.
First iteration.
We start with $a = 0$ and $b = 1$. Compute the midpoint $m = \frac{a+b}{2}$, and evaluate $F(m)$.
$$m = \frac{1}{2}, \qquad F(m) = \frac{1}{8} + \frac{3}{2} - 2 = -\frac{3}{8}.$$
Therefore, since $F(m) < 0$, replace update a, $a := m$.

Second iteration.
Compute the midpoint $m = \frac{a+b}{2}$ and evaluate $F(m)$.

$$m = \frac{3}{4}, \qquad F(m) = \frac{27}{64} + \frac{9}{4} - 2 = \frac{43}{64}.$$

Since, $F(m) > 0$, then for the next iteration we update b, $b := m$.

Exercise 5

With initial guess $\underline{x}^{(0)} = [1, 1]^T$ apply one Newton iteration to find an approximate solution of the system

$$\underline{F}(\underline{x}) = \begin{bmatrix} x_1^2 - 2x_1 + x_2 + 7 \\ 2x_1 - x_2 + 2 \end{bmatrix}.$$

Solution: Recall that the Newton's iteration for multivariate functions is given by

$$\underline{x}^{(k+1)} = \underline{x}^{(k)} - \left(J_F(\underline{x}^{(k)})\right)^{-1} \cdot \underline{F}(\underline{x}^{(k)}),$$

where J_F is the Jacobian matrix of $\underline{F}$. We have

$$J_F(\underline{x}) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 - 2 & 1 \\ 2 & -1 \end{bmatrix}.$$

For $k = 0$, that is the first iteration, we have

$$\underline{F}(\underline{x}^{(0)}) = \begin{bmatrix} 1^2 - 2 + 1 + 7 \\ 2 - 1 + 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}, \quad \text{and} \quad J_F(\underline{x}^{(0)}) = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix},$$

and we have to compute $\underline{u} = [u_1, u_2]^T$ solution of $J_F(\underline{x}^{(0)}) \cdot \underline{u} = \underline{F}(\underline{x}^{(0)})$. The solution of this linear system is

$$\begin{cases} u_2 = 7 \\ 2u_1 - u_2 = 3 \end{cases} \quad \rightarrow \begin{cases} u_2 = 7 \\ u_1 = \dfrac{3+7}{2} = 5 \end{cases}$$

and the first newton iteration step gives

$$\underline{x}^{(1)} = \underline{x}^{(0)} - \underline{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 5 \\ 7 \end{bmatrix} = - \begin{bmatrix} 4 \\ 6 \end{bmatrix}.$$

Exercise 6

Given the function $f(x) = \cos(2\pi\, x)$ compute its Lagrange interpolant of degree 2 through the points $x_1 = 0$, $x_2 = 1/2$, $x_3 = 1$.

Solution 1: Recall that the Lagrange interpolant of f of degree k, over the points $x_1, x_2, \dots, x_{k+1}$, is

$$\Pi_k(f) := \sum_{i=1}^{k+1} f(x_i) L_i(x), \quad \text{where} \quad L_i(x) := \prod_{\substack{j=1 \\ j\neq i}}^{k+1} \frac{(x - x_j)}{(x_i - x_j)}.$$

In order to compute $\Pi_2(f)$ on the points x_1, x_2, x_3, we have

$$\begin{aligned}
L_1(x) &= \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} = \frac{(x - \frac{1}{2})(x - 1)}{(-\frac{1}{2})(-1)} = (2x - 1)(x - 1), \\
L_2(x) &= \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)} = \frac{(x - 0)(x - 1)}{(\frac{1}{2})(-\frac{1}{2})} = -4x(x - 1), \\
L_3(x) &= \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)} = \frac{(x - 0)(x - \frac{1}{2})}{(1)(\frac{1}{2})} = x(2x - 1),
\end{aligned}$$

and also

$$\begin{aligned}
f(x_1) &= \cos(2\pi x_1) = \cos(0) = 1, \\
f(x_2) &= \cos(2\pi x_2) = \cos(\pi) = -1, \\
f(x_3) &= \cos(2\pi x_3) = \cos(2\pi) = 1.
\end{aligned}$$

Finally

$$\begin{aligned}
\Pi_2(f) &= f(x_1)L_1(x) + f(x_2)L_2(x) + f(x_3)L_3(x) \\
&= 1 \cdot (2x - 1)(x - 1) + (-1) \cdot (-4x)(x - 1) + 1 \cdot x(2x - 1) \\
&= 2x^2 - 2x - x + 1 + 4x^2 - 4x + 2x^2 - x \\
&= 8x^2 - 8x + 1.
\end{aligned}$$

Solution 2: An alternative solution consists in the following observation. $\Pi_2(f)$ is the degree two polynomial $c_1 + c_2 x + c_3 x^2$, where the coefficients c_1, c_2, c_3 are solution of the following linear system

$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \end{bmatrix}$$

For this exercise we have

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{4} \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

In particular $c_1 = 1$ and it remains to solve

$$\begin{cases} 2c_2 + c_3 = -8 \\ c_2 + c_3 = 0 \end{cases} \rightarrow \begin{cases} 2c_2 + c_3 = -8 \\ c_3 = -c_2 \end{cases} \rightarrow \begin{cases} c_2 = -8 \\ c_3 = 8 \end{cases}$$

Again we found that $\Pi_2(f) = 1 - 8x + 8x^2$.

Exercise 7

Describe the Crank-Nicolson scheme for the solution of an ODE and explain its relation with the trapezoidal quadrature rule. Then, compute one step of the Crank-Nicolson scheme for the problem

$$\begin{cases} y'(t) = 2t\,(1 - y(t)) \\ y(0) = 3 \end{cases}$$

selecting $\Delta t = 1$.

Solution: Let $t_n = t_0 + n\Delta t$ and $y_n = y(t_n)$. We are given $t_0 = 0$ and $\Delta t = 1$, and have to compute one step of Crank-Nicolson, i.e. compute y_1, for $t_1 = 1$.

For the Crank-Nicolson scheme, we approximate the derivative using a finite difference and the ODE field with the average of the fields at two consecutive timesteps

$$\frac{y_n - y_{n-1}}{\Delta t} = \frac{1}{2}(f(t_n, y_n) + f(t_{n-1}, y_{n-1})).$$

In our case, $f(t, y) = 2t(1 - y)$ and $\Delta = 1$, therefore

$$y_n - y_{n-1} = \frac{1}{2}(2t_n(1 - y_n) + 2t_{n-1}(1 - y_{n-1})).$$

In order to compute y_1, substitute t_0, t_1, y_0 in the equation above and solve for y_1.

$$\begin{aligned} y_1 - y_0 &= \frac{1}{2}(2t_1(1 - y_1) + 2t_0(1 - y_0)) \\ y_1 - 3 &= \frac{1}{2}(2 \cdot 1 \cdot (1 - y_1) + 2 \cdot 0 \cdot (1 - 3)), \end{aligned}$$

to obtain $y_1 = 2$.