Exercises NMES - 13.1.2025

Exercise 1

Apply the Gaussian elimination method, without pivoting, to solve the linear system $Ax = b$, where

$$\begin{matrix}(r_1)\\(r_2)\\(r_3)\end{matrix}\begin{bmatrix}2 & 4 & 10\\2 & 6 & 20\\1 & 4 & 18\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-6\\12\\18\end{bmatrix}$$

showing the intermediate computations.

Solution: First we eliminate the first column of A under the diagonal term $a_{1,1}$. Compute $l_{2,1} = \frac{a_{2,1}}{a_{1,1}} = 1$ and $l_{3,1} = \frac{a_{3,1}}{a_{1,1}} = \frac{1}{2}$. Then perform

$$\begin{matrix}(r_1)\\(r_2)=(r_2)-l_{2,1}\cdot(r_1)\\(r_3)=(r_3)-l_{3,1}\cdot(r_1)\end{matrix}\begin{bmatrix}2 & 4 & 10\\0 & 2 & 10\\0 & 2 & 13\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-6\\18\\21\end{bmatrix}$$

In the second step of the Gaussian elimination we eliminate the second column of A under the diagonal term $a_{2,2}$. Compute $l_{3,2} = \frac{a_{3,2}}{a_{2,2}} = 1$. Then perform

$$\begin{matrix}(r_1)\\(r_2)\\(r_3)=(r_3)-l_{3,2}\cdot(r_2)\end{matrix}\begin{bmatrix}2 & 4 & 10\\0 & 2 & 10\\0 & 0 & 3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-6\\18\\3\end{bmatrix}$$

Finally compute the solution by *back-substitution* method:

$$3x_3 = 3; \to x_3 = 1;$$
$$2x_2 + 10x_3 = 18; \to 2x_2 = 18 - 10; \to x_2 = 4;$$
$$2x_1 + 4x_2 + 10x_3 = -6; \to 2X_1 = -6 - 10 - 16; \to x_1 = -16;$$

Finally the solution of the linear system is $x = \begin{bmatrix}-16\\4\\1\end{bmatrix}$.

Exercise 2

Compute the quadratic least-square approximation $r(x) = c_0 + c_1 x + c_2 x^2$ for the set of points

$$(-2, \frac{5}{2}),\ (-1, 0),\ (0, -1),\ (1, 0),\ (2, \frac{5}{2}).$$

Solution: Here we want to minimize the quadratic function

$$F(c_0, c_1, c_2) = \sum_{i=1}^{m} \left(F(x_i) - c_0 - c_1 \cdot x_i - c_2 \cdot x_i^2)\right)^2.$$

The problem is equivalent to solve

$$(LS2) \begin{bmatrix} m & \sum_{i=1}^m x_i & \sum_{i=1}^m (x_i)^2 \\ \sum_{i=1}^m x_i & \sum_{i=1}^m (x_i)^2 & \sum_{i=1}^m (x_i)^3 \\ \sum_{i=1}^m (x_i)^2 & \sum_{i=1}^m (x_i)^3 & \sum_{i=1}^m (x_i)^4 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^m y_i \\ \sum_{i=1}^m x_i y_i \\ \sum_{i=1}^m (x_i)^2 y_i \end{bmatrix}.$$

For the given set of points, it holds

$$(LS2) \begin{bmatrix} 5 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 34 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 20 \end{bmatrix}.$$

We get immediately $c_1 = 0$. And then

$$\begin{cases} 5c_0 + 10c_2 = 4 \\ 10c_0 + 34c_2 = 20 \end{cases} \rightarrow \begin{cases} 5c_0 + 10c_2 = 4 \\ 14c_2 = 12 \end{cases} \rightarrow \begin{cases} c_0 + = \frac{4}{5} - \frac{60}{35} = -\frac{32}{35} \\ c_2 = \frac{6}{7} \end{cases}$$

Therefore the solution is $r(x) = -\frac{32}{35} + \frac{6}{7}x^2$.

Exercise 3

Starting from $x^{(0)} = (0, 0, 0)^T$, compute 2 iterations of the Gauss-Seidel method applied to the system $Ax = b$, where

$$A = \begin{bmatrix} 1 & 4 & 1 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Solution: Recall that for iterative methods, we can split $A = M - N$, so that the general step of an iterative method is

$$x^{(k+1)} = x^{(k)} + M^{-1} r^{(k)}$$

In the case of Gauss-Seidel method, $M =$ `tril(A)`

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}.$$

Before starting, compute the residual $r^{(0)}$

$$r^{(0)} = b - Ax^{(0)} = b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

First iteration
Firstly, we have to compute $M^{-1}r^{(0)}$, which amounts to solving the linear system

$$Mu = r^{(0)}.$$

Since M is lower triangular, it is easier to solve the linear system by forward-substitution, obtaining the solution u

$$u = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix},$$

so $x^{(1)} = x^{(0)} + u = (1, -1, 3)^T$. Then compute the residual $r^{(1)} = b - Ax^{(1)}$

$$r^{(1)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & 4 & 1 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Second iteration As before, compute the solution of the linear system

$$Mu = r^{(1)}$$

using forward substitution. This yields

$$x^{(2)} = x^{(1)} + u = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 7 \end{bmatrix}.$$

Exercise 4

Starting from $v^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, compute 2 iterations of the power method on the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

returning the eigenvector and eigenvalue approximation and showing the intermediate steps.

Solution: In the first iteration of the power method we compute:

$$
\begin{aligned}
w_1 &= Av_0 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} =; \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\
\|w_1\| &= \sqrt{2^2 + 1^2} = \sqrt{5}; \\
v_1 &= \frac{w_1}{\|w_1\|} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}; \\
\lambda_1 &= v_1^T A v_1 = \frac{1}{5} \begin{bmatrix} 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \frac{14}{5}.
\end{aligned}
$$

In the second iteration of the power method we compute:

$$
\begin{aligned}
w_2 &= Av_1 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \cdot \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 5 \\ 4 \end{bmatrix}; \\
\|w_2\| &= \frac{1}{\sqrt{5}} \sqrt{5^2 + 4^2} = \sqrt{\frac{41}{5}}; \\
v_2 &= \frac{w_2}{\|w_2\|} = \frac{1}{\sqrt{41}} \begin{bmatrix} 5 \\ 4 \end{bmatrix}; \\
\lambda_2 &= v_2^T A v_2 = \frac{1}{41} \begin{bmatrix} 5 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 5 & 4 \end{bmatrix} \cdot \begin{bmatrix} 14 \\ 13 \end{bmatrix} = \frac{122}{41}.
\end{aligned}
$$

Exercise 5

With initial guess $x^{(0)} = 1$ apply one Newton iteration to find an approximate solution of the equation

$$(2x - 1)\left(3x^2 - 2x + 1\right) = 0$$

Solution: We have to solve the equation $F(x) = 0$ where

$$F(x) := (2x - 1)\left(3x^2 - 2x + 1\right)$$

Recall that the generic Newton iteration is given as follows

$$x^{(k+1)} = x^{(k)} - \frac{F(x^{(k)})}{F'(x^{(k)})},$$

so firstly compute F'

$$F'(x) = 2(3x^2 - 2x + 1) + (2x - 1)(6x - 2).$$

So, compute the first iteration of the Newton method

$$x^{(1)} = x^{(0)} - \frac{F(x^{(0)})}{F'(x^{(0)})} = 1 - \frac{2}{8} = 1 - \frac{1}{4} = \frac{3}{4}.$$

Exercise 6

Given the Cauchy problem:

$$\begin{cases} y'(t) = -2ty(t) + 2t^3 \text{ for } t > 0 \\ y(0) = -1; \end{cases}$$

compute two steps by the implicit Euler method, with $\Delta t = 1$, in order to approximate $y(2)$. Report the intermediate computations.

Solution: We denote by $y_n := y(t_n)$, where $t_n = t_0 + n\Delta t$, and here we have $t_0 = 0,\ t_1 = 1$ and $t_2 = 2$. The Implicit Euler (IE) method is given by

$$\frac{y_{n+1} - y_n}{\Delta t} = f(t_{n+1}, y_{n+1}),$$

and for this exercise, we have $f(t, y(t)) = -2ty(t)+2t^3$. Thus, the IE method applied to our Cauchy problem leads to solving, for each step, the following equation:

$$\frac{y_{n+1} - y_n}{\Delta t} = -2t_{n+1}y_{n+1} + 2t_{n+1}^3, \quad \text{(Implicit in the unknown } y_{n+1}\text{)}.$$

Therefore we manipulate it as

$$\begin{aligned} y_{n+1} + 2\Delta t \cdot t_{n+1} \cdot y_{n+1} &= y_n + 2\Delta t \cdot t_{n+1}^3; \\ y_{n+1} &= \frac{y_n + 2\Delta t \cdot t_{n+1}^3}{1 + 2\Delta t \cdot t_{n+1}}, \quad \text{(Explicitated in } y_{n+1}\text{)}. \end{aligned}$$

Finally, the computation of the first step is

$$y_1 = \frac{y_0 + 2\Delta t \cdot t_1^3}{1 + 2\Delta t \cdot t_1} = \frac{-1+2}{1+2} = \frac{1}{3},$$

and the computation of the second step is

$$y_2 = \frac{y_1 + 2\Delta t \cdot t_2^3}{1 + 2\Delta t \cdot t_2} = \frac{1/3 + 16}{1+4} = \frac{1+48}{15} = \frac{49}{15},$$

Exercise 7

Write the pseudo-code of the composite trapezoidal quadrature rule, then use the composite trapezoidal quadrature rule to compute an approximation of

$$\int_0^{2\pi} \sin^2(t)\, dt$$

by splitting the integration interval $[0, 2\pi]$ into four uniform subintervals. Report the intermediate computations.

Solution: The composite trapezoidal quadrature rule amounts to approximating the function to be integrated with the Lagrange piecewise-linear approximation and integrating it. The extema of the four subintervals are $t_1 = 0$, $t_2 = \frac{\pi}{2}$, $t_3 = \pi$, $t_4 = \frac{3\pi}{2}$, $t_5 = 2\pi$. The width of such subintervals is $\frac{\pi}{2}$. So

$$\int_0^{2\pi} \sin^2(t)dt \simeq \frac{\pi}{4} \sum_{i=1}^{4} \int_{t_i}^{t_{i+1}} \Pi_1(\sin^2)(t)dt = \frac{\pi}{4} \sum_{i=1}^{4} [\sin^2(t_i) + \sin^2(t_{i+1})].$$

Compute each term of the sum individually:

- $\sin^2(0) + \sin^2(\frac{\pi}{2}) = 1$,
- $\sin^2(\frac{\pi}{2}) + \sin^2(\pi) = 1$,
- $\sin^2(\pi) + \sin^2(\frac{3\pi}{2}) = 1$,
- $\sin^2(\frac{3\pi}{2}) + \sin^2(2\pi) = 1$,

combining all the terms yields

$$\int_0^{2\pi} \sin^2(t)dt \simeq \frac{\pi}{4}(1 + 1 + 1 + 1) = \pi.$$