We also have a convergence results for the approximation of the eigenvalue λ_1.

Corollary

Let $A \in \mathbb{C}^{n\times n}$ be a diagonalizable matrix. Assume $|\lambda_1| > |\lambda_2|$ and $v_0 = \sum_{i=1}^{n} \alpha_i u_i$, with $\alpha_1 \neq 0$. Then it holds

$$|\mu_k - \lambda_1| = O\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right), \qquad \text{for } k \to +\infty.$$

For symmetric real matrices, we have a better convergence results:

Corollary

Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Assume $|\lambda_1| > |\lambda_2|$ and $v_0 = \sum_{i=1}^{n} \alpha_i u_i$, with $\alpha_1 \neq 0$. Then it holds

$$|\mu_k - \lambda_1| = O\left(\left|\frac{\lambda_2}{\lambda_1}\right|^{2k}\right), \qquad \text{for } k \to +\infty.$$

Some observations

One of the hypothesis of the previous results is $\alpha_1 \neq 0$, where α_i are defined such that $v_0 = \sum_{i=1}^{n} \alpha_i u_i$. Clearly, $u_1, \ldots, u_n$ are unknown and we cannot check if v_0 satisfies this hypothesis.
Practically this is not a real obstacle. If unfortunately we choose v_0 s.t $\alpha_1 = 0$, there are two possible cases:

- in *exact arithmetic*, we get $\lim_{k \to +\infty} \widetilde{v}_k = u_2$ and $\lim_{k \to +\infty} \mu_k = \lambda_2$, as long as $|\lambda_2| > |\lambda_3|$ and $\alpha_2 \neq 0$.
- in *finite arithmetic*, during the iterations of the Power Method, round-off errors can cause the appearance of a non-zero component in the direction of u_1 in a certain v_k. If this happens, the method immediately starts to converge towards the dominant eigenvalue λ_1 and its corresponding eigenvector u_1.

Stopping criterion

A simple stopping criterion for the power method is based on the residual:

$$\text{Stop when} \qquad \|Av_k - \mu_k v_k\| \leq \texttt{tol}$$

How can we compute other eigenvalues and eigenvectors?

Let $\mu \in \mathbb{C}$ a user-specified parameter that is not an eigenvalue of A, we want to approximate the closest eigenvalue of A to μ, i.e.

$$\lambda_J = \underset{i}{\operatorname{argmin}} |\mu - \lambda_i|$$

Inverse Power method

Input: $A \in \mathbb{C}^{n \times n}$, $v_0 \in \mathbb{C}^n$ with $\|v_0\| = 1$, MAXITER $\in \mathbb{N}$, tol $\in \mathbb{R}^+$.

for k $= 1, 2, \ldots,$ MAXITER

$w = (A - \mu I)^{-1} v_{k-1}$ (equivalently, solve $(A - \mu I) w = v_{k-1}$)

$v_k = w / \|w\|$

$\mu_k = (v_k)^H A v_k$ (Rayleigh quotient with A)

Check the Stopping criterion

end

Output: μ_k and v_k.

Beware

Since μ is not an eigenvalue of A, the matrix $A - \mu I$ is non singular.

Since $Au_i = \lambda_i u_i$, then $(A - \mu I)u_i = (\lambda_i - \mu)u_i$, and then $\frac{1}{\lambda_i - \mu} u_i = (A - \mu I)^{-1} u_i$. Let λ_J be the eigenvalue of A closest to μ, the largest (in module) eigenvalue of $(A - \mu I)^{-1}$ is then $\frac{1}{\lambda_J - \mu}$, and the relative eigenvector is u_J. The inverse power method is just a power method applied to $(A - \mu I)^{-1}$, and the previous results apply: $\widetilde{v}_k$ converges to u_J. Since the Rayleigh quotient μ_k is computed with A instead of $(A - \mu I)^{-1}$, it converges to λ_J.

Theorem

Assume $|\mu - \lambda_J| < |\mu - \lambda_i| \ \forall\ i = 1, \ldots, n,\ i \neq J$ and $v_0 = \sum_{i=1}^{n} \alpha_i u_i$, with $\alpha_J \neq 0$. Then

$$\lim_{k \to +\infty} \mu_k = \lambda_J$$

and

$$\lim_{k \to +\infty} \|\widetilde{v}_k - u_J\|_2 = 0, \qquad \text{where } \widetilde{v}_k = \frac{\|A^k v_0\|}{\alpha_1 \lambda_1^k} v_k.$$

Note that if $\mu = 0$, the method approximates the eigenvalue of A that is smallest in module.