Exercise 1: Lagrange interpolation in MATLAB

Write `p = lagrange_def(xn,fn,x)` that evaluates $\Pi_n g(x) = \sum_{i=0}^{n} g(x_i) L_i(x)$ directly from $L_i(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}$ (the node x_i is `xn(i+1)`). Check that $\sum_i L_i = 1$ with `fn = ones`.

Interpolate $g(x) = \sin(2\pi x)$ on $[0, 1]$ with two sets of $n + 1$ nodes:

$$x_i = \frac{i}{n} \quad \text{(endpoints included)}, \qquad x_i = \frac{i + 1/2}{n + 1} \quad \text{(shifted, endpoints excluded)}.$$

- For $n = 1, 2, \ldots, 15$ compute $\max_{[0,1]} |g - \Pi_n g|$ on a fine grid and plot it with `semilogy`, one curve per node set.
- Look at $n = 1$ and $n = 2$ with the first set: the error is exactly 1. Plot $\Pi_n g$ and explain what happened (what are the data $g(x_i)$?). Does the second set fail for the same degrees?
- Why does the error decrease so fast for larger n? Use the error formula with $g^{(n+1)}$.

Chebyshev nodes

Error: $g(x) - \Pi_n g(x) = \dfrac{g^{(n+1)}(\xi)}{(n+1)!}\,\omega(x)$, $\omega(x) = \prod_{i=0}^{n}(x - x_i)$. We cannot control $g^{(n+1)}$, but we can choose the nodes to make $\max|\omega|$ small.

- Uniform nodes: ω is huge near the endpoints, where the error oscillates.
- **Chebyshev nodes** are the zeros of $T_{n+1}(t) = \cos\big((n+1)\arccos t\big)$ and minimise $\max_{[-1,1]}|\omega| = 2^{-n}$.
- Picture: project equally spaced points of a semicircle with diameter $[a, b]$ onto the axis; the nodes cluster near the endpoints.

Chebyshev nodes on $[a, b]$

Input: a, b, n.
for i $= 0, 1, \ldots, n$
$\quad t_i = \cos\big((2i+1)\pi/(2n+2)\big)$ (in $[-1, 1]$)
$\quad x_i = (a+b)/2 + (b-a)\,t_i/2$ (in $[a, b]$)
end
Output: $x_0, \ldots, x_n$ (sorted increasingly).

Exercise 2: uniform versus Chebyshev nodes

Use the two node sets on $[-1,1]$: $x_i = -1 + \frac{2i}{n}$ (uniform) and the Chebyshev nodes above, for

$$g_1(x) = \frac{1}{1+25x^2} \quad \text{(Runge)}, \qquad g_2(x) = \tanh(10x).$$

- For $n = 2, 4, \ldots, 40$ compute $\max |g - \Pi_n g|$ on a fine grid (2001 points) and plot it with `semilogy`, for each function and each node set.
- Plot g and $\Pi_n g$ for $n = 16$. Where is the uniform error largest?
- g_2 is smooth (it has no singularity on the real axis), yet it behaves like Runge's function. Why? Both have singularities in the complex plane close to $[-1,1]$: g_1 at $\pm i/5$, g_2 at $\pm i\pi/20$.
- The Chebyshev error decays geometrically. Estimate the rate from the plot and compare it with $\rho = \frac{1+\sqrt{26}}{5} \approx 1.22$ for g_1.