Checking stability for Implicit Euler

Applying **Implicit Euler** method to the model problem gives
($y_{n+1} = y_n + \Delta t f(t_{n+1}, y_{n+1})$ with $f(t_{n+1}, y_{n+1}) = a y_{n+1}$)

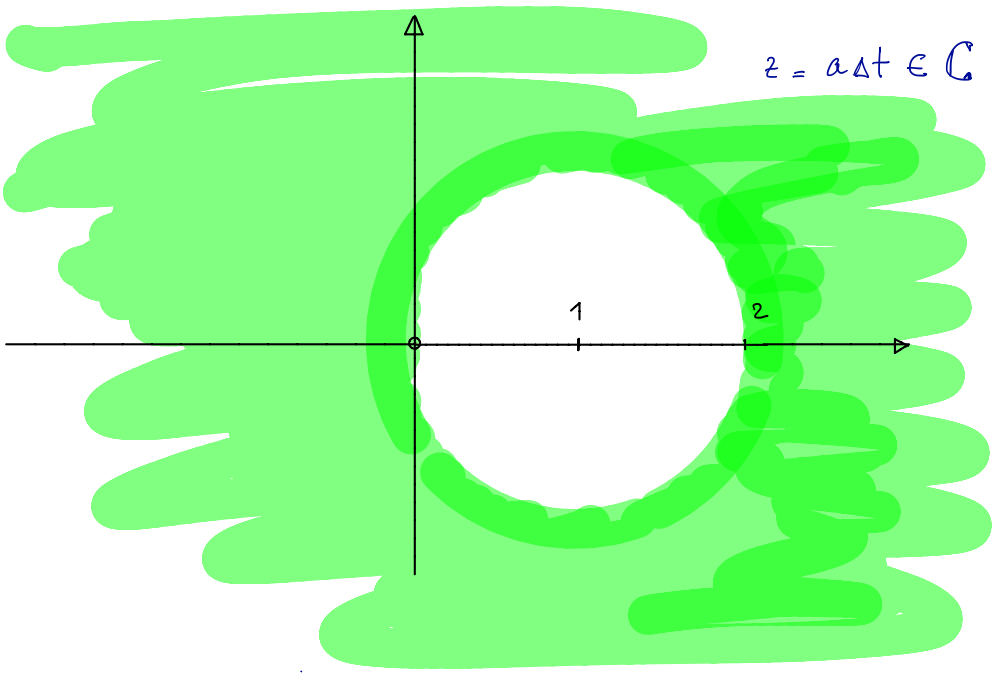
$$y_{n+1} = y_n + a\Delta t y_{n+1} \quad n = 0, 1, ... \implies y_{n+1} = \frac{1}{1 - a\Delta t} y_n = G\, y_n.$$

We have $|G| < 1 \iff |1 - a\Delta t| > 1$ and this happens as soon as $a\Delta t$ is outside the unit circle centered in 1.
If $\mathrm{Re}(a) < 0$, then $|G| < 1$ and we say that Implicit Euler is A-stable: the A-stability condition can be written as

$$\boxed{\text{If Re } a < 0 \text{ then } |G| < 1 \quad =: \textit{A-stability}}$$

However, for Implicit Euler method, the A-stability region is too large: if $a\Delta t$ is outside the unit circle centered in 1, y_n decays even if the exact solution do not.

Stability for Implicit Euler: the complex setting



The scheme is A-stable:

$|G| < 1 \Leftrightarrow |1 - a\Delta t| > 1 \Leftrightarrow a\Delta t$ is outside the circle above $\Leftarrow \mathrm{Re}(a) < 0$

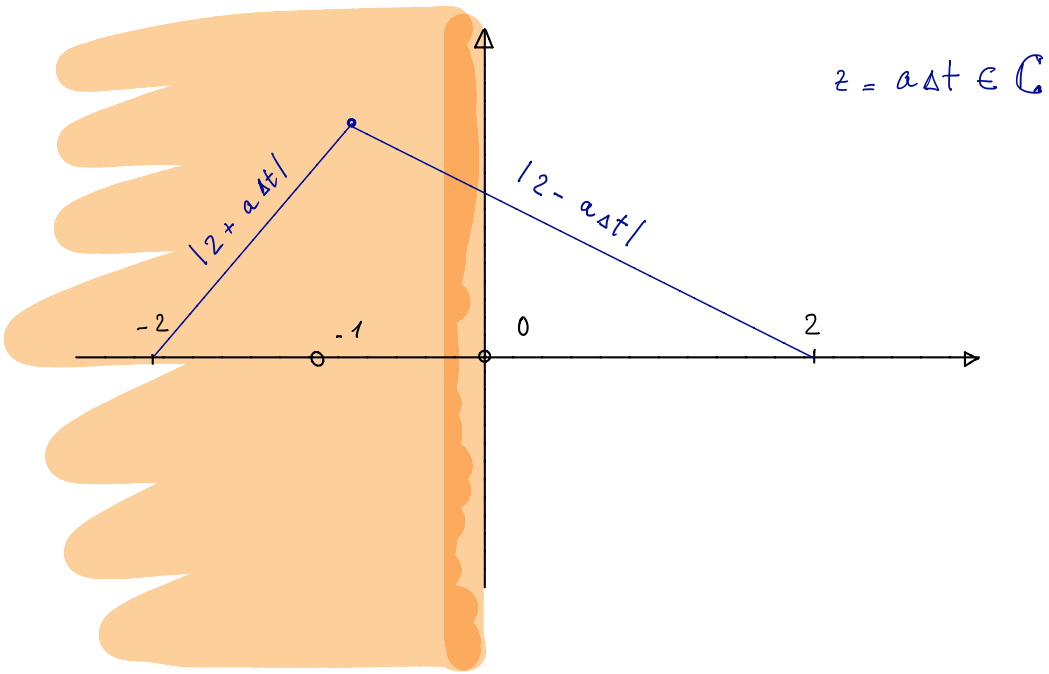
Checking stability for Crank-Nicolson

Crank-Nicolson gives

$$y_{n+1} = y_n + \frac{a\Delta t}{2}(y_n + y_{n+1}) \quad n = 0, 1, ... \implies y_{n+1} = \frac{1 + \frac{a\Delta t}{2}}{1 - \frac{a\Delta t}{2}} y_n = G\, y_n.$$

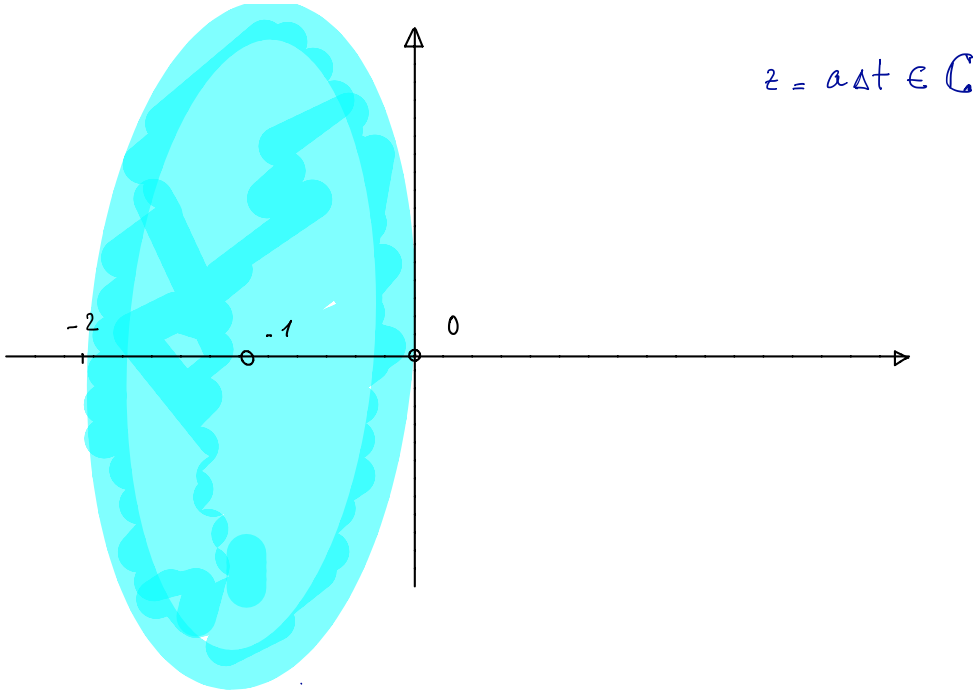
The scheme is A-stable:

$$|G| < 1 \Leftrightarrow |2 + a\Delta t| < |2 - a\Delta t| \Leftrightarrow \mathrm{Re}(a)\Delta t < 0 \Leftrightarrow \mathrm{Re}(a) < 0$$



Checking stability for Heun

Heun method is not A-stable, only conditionally A-stable like Explicit Euler. In fact, $G = 1 + a\Delta t + \frac{(a\Delta t)^2}{2}$, and the condition $|G| < 1$ is satisfied if $\Delta t < \Big(2/|a|\Big)$.



Stability regions: comparison

For each of these methods we have defined the stability region in the complex plane as

$$A := \{a\Delta t \in \mathbb{C} : \lim_{n\to\infty} |y_n| = 0\} \equiv \{a\Delta t \in \mathbb{C} : |G| < 1\}$$

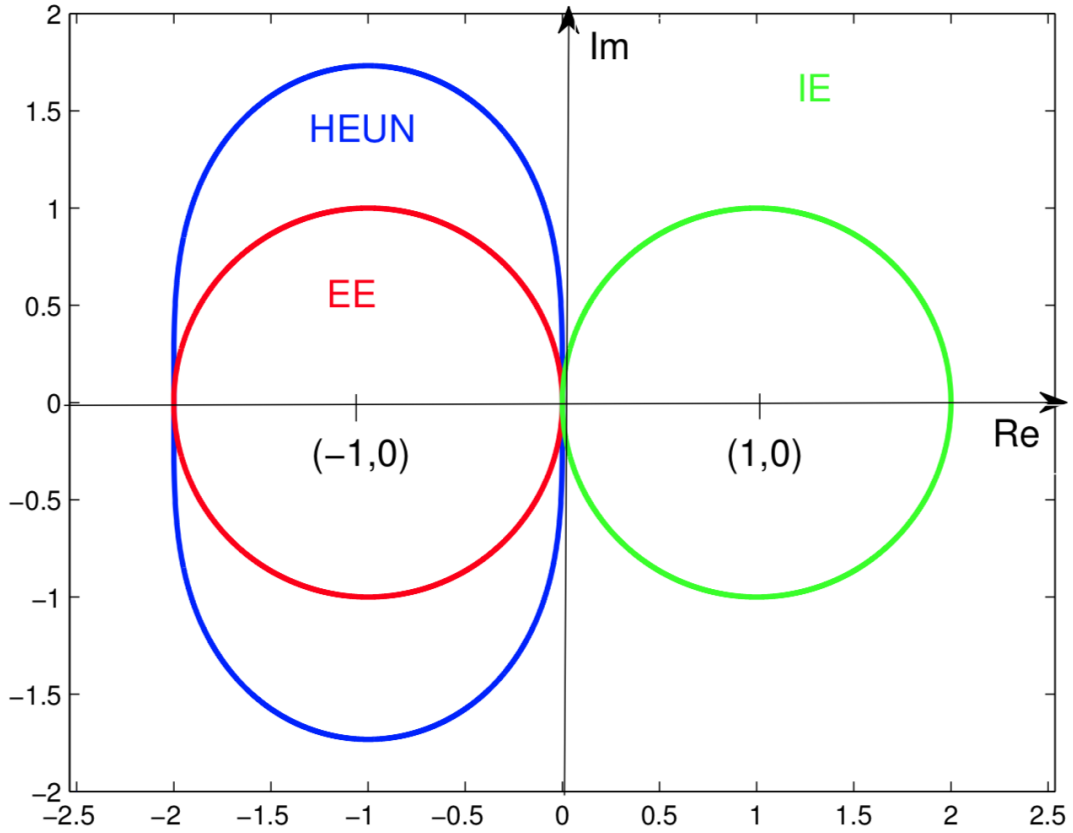
and compared it with the stability region of the continuous problem: the half plane Re $a < 0$.

For Explicit Euler $A = \{a\Delta t \in \mathbb{C} : |1 + a\Delta t| < 1\}$ is a circle with center $(-1, 0)$ and radius 1 (too small!).

For Implicit Euler $A = \{a\Delta t \in \mathbb{C} : |1 - a\Delta t| > 1\}$ is the whole plane minus a circle with center $(1, 0)$ and radius 1 (too big!)

For Crank-Nicolson the region is the left-half plane, exactly as for the true solution (the best you can have).

Stability regions



A-stability regions for EE (the region inside the red circle), Heun (the region inside the blue ellipse), IE (green, the region outside the circle with center $(1, 0)$ and radius 1

Conclusions (for the four basic methods)

In general, explicit schemes are never A-stable, only conditionally A-stable, meaning that to satisfy the A-stability property they need to proceed by small time steps. Some implicit schemes are A-stable.

For the method we have considered:

Method	Consistency	Stability
Explicit Euler	yes, order 1	conditionally A-stable
Implicit Euler	yes, order 1	A-stable
Crank-Nicolson	yes, order 2	A-stable
Heun	yes, order 2	conditionally A-stable

Generalizing Heun idea: Runge-Kutta methods

The celebrated Runge-Kutta methods are compound 1-step methods.

The basic idea is very simple: choose a high precision quadrature formula for $\int f$ on each interval $[t_n, t_{n+1}]$. Then, since the values at the quadrature nodes are not known, we predict them someway (and this is where the detailed description could become quite complicated).

The simplest explicit RK is Heun: the starting point is the trapezoidal rule for $\int_{t_n}^{t_{n+1}} f$, and since we want to go explicit, instead of the value y_{n+1} (that would be needed in the trapezoidal rule) we use $y_n + \Delta t\, f(t_n, y_n)$, that is, the value predicted by Explicit Euler.

Runge-Kutta methods

Heun scheme:

$$\begin{cases} y_0 \text{ given} \\ y_{n+1}^* = y_n + \Delta t\, f(t_n, y_n) \\ y_{n+1} = y_n + \dfrac{\Delta t}{2}\Big(f(t_n, y_n) + f(t_{n+1}, y_{n+1}^*)\Big) \quad n = 0, 1, \cdots, N-1 \end{cases}$$

Denoting by K_1 and K_2 the values of f at the two nodes t_n and $t_{n+1} = t_n + \Delta t$ we can rewrite Heun method as:

$$K_1 = f(t_n, y_n), \quad K_2 = f(t_n + \Delta t, y_n + \Delta t\, K_1)$$
$$y_{n+1} = y_n + \frac{\Delta t}{2}\Big(K_1 + K_2\Big) \quad n = 0, 1, \cdots$$

Runge-Kutta methods

The more famous version of Runge-Kutta, **RK4**, is compounded *four times*, it is based on *Simpson rule*, and achieves order $p = 4$. Let us see how it is obtained.

Simpson rule uses on the interval $[t_n, t_{n+1}]$ three nodes t_n, $t_{n+1/2} = t_n + \Delta t/2$, and $t_{n+1} = t_n + \Delta t$, and would give

$$\int_{t_n}^{t_{n+1}} f(t, y(t))dt \simeq \frac{\Delta t}{6}\Big(f_n + 4f_{n+1/2} + f_{n+1}\Big).$$

$(f_n = f(t_n, y_n), f_{n+1/2} = f(t_{n+1/2}, y_{n+1/2}), f_{n+1} = f(t_{n+1}, y_{n+1}))$

The value f_n is known, so we set $K_1 = f(t_n, y_n)$; then write $4f_{n+1/2}$ as $2f^{(1)}_{n+1/2} + 2f^{(2)}_{n+1/2}$, and we choose two different "predictions" for $y_{n+1/2}$:

$$f^{(1)}_{n+1/2} = f(t_{n+1/2}, y_n + \frac{\Delta t}{2}K_1) =: K_2$$

$$f^{(2)}_{n+1/2} = f(t_{n+1/2}, y_n + \frac{\Delta t}{2}K_2) =: K_3$$

RK4

We had

$K_1 = f(t_n, y_n),\ K_2 = f^{(1)}_{n+1/2} = f(t_{n+1/2}, y_n + \frac{\Delta t}{2} K_1),$

$K_3 = f^{(2)}_{n+1/2} = f(t_{n+1/2}, y_n + \frac{\Delta t}{2} K_2)$

For f_{n+1} we take here $f_{n+1} = f(t_{n+1}, y_n + \Delta t\, K_3)$ (although other choices could have been possible), so that the scheme is:

$$K_1 = f(t_n, y_n), \quad K_2 = f(t_{n+1/2}, y_n + \frac{\Delta t}{2} K_1)$$

$$K_3 = f(t_{n+1/2}, y_n + \frac{\Delta t}{2} K_2), \quad K_4 = f(t_{n+1}, y_n + \Delta t\, K_3)$$

$$y_{n+1} = y_n + \frac{\Delta t}{6}\Big(K_1 + 2K_2 + 2K_3 + K_4\Big) \quad n = 0, 1, \cdots$$

Explicit Runge-Kutta

The family of explicit Runge-Kutta methods is a generalisation of the RK4 scheme above:

$$y_0 \text{ given} \qquad y_{n+1} = y_n + \Delta t \sum_{i=1}^{s} b_i K_i, \qquad n = 0, 1, \cdots \tag{1}$$

where

$$K_i = f(t_n + c_i \Delta t, y_n + \Delta t \sum_{j=1}^{i-1} a_{ij} K_j)$$

To specify a particular method one needs to provide the integer s (the number of stages), and the coefficients a_{ij}, b_i, c_i. The matrix $[a_{ij}]$ is called *Runge-Kutta matrix*, while the b_i and c_i are called *weights* and *nodes*, respectively. These coefficients are usually arranged in the *Butcher tableau*

Butcher tableau

$$
\begin{array}{c|ccccc}
0 & & & & & \\
c_2 & a_{2,1} & & & & \\
c_3 & a_{3,1} & a_{3,2} & & & \\
\vdots & & \vdots & \ddots & & \\
c_s & a_{s,1} & a_{s,2} & \cdots & a_{s,s-1} & \\
\hline
 & b_1 & b_2 & \cdots & b_{s-1} & b_s
\end{array}
$$

Consistency conditions

For consistency the coefficients must verify algebraic conditions;

$$\boxed{\text{a RK scheme is consistent } \textbf{iff } \sum_{i=1}^{s} b_i = 1.}$$

(This condition is always verified since the b_i are the weights of the quadrature formula used, which has to be exact at least on constants).

For higher order of consistency, other relations must be satisfied. For instance, for a 2 stage explicit RK to have order 2 we need, together with $b_1 + b_2 = 1$, also $b_2 c_2 = 1/2$ (check!)

Accuracy and stages of RK ** NOT FOR THE EXAM **

Theorem 1

An explicit s-stages Runge-Kutta method cannot have order of accuracy p greater than s. Moreover, there are no known explicit s-stages RK methods with order $p = s$ for $s \geq 5$.

order p	1	2	3	4	5	6	7	8
s_{min}	1	2	3	4	**6**	7	9	11

RK methods are very successful and widely used in the codes for their ductility: the time step can easily be modified from one interval to another if needed, the initial value y_0 is all what is needed to start the method, and they have high accuracy.

A first multistep method

Let us start with an example. Let $t_0, t_1, \cdots, t_N = T$ be a set of *equally spaced* points in $[t_0, T]$, and let $\Delta t = \dfrac{T - t_0}{N}$ be the time step (this time the points **must** be equally spaced).

We want to construct an explicit scheme of order 2 going back two steps:

$$y_{n+1} = y_n + \Delta t\Big(\alpha f(t_n, y_n) + \beta f(t_{n-1}, y_{n-1})\Big), n = 1, 2, \cdots$$

This requires values at time $t_{n-1} = t_0 + (n-1)\Delta t$ as well as at time t_n. Therefore the initial value y_0 is not enough to start the procedure and we need to compute y_1 with a 1-step method. Then we have to find α and β such that the scheme has order 2.

Choosing the parameters in a multistep scheme....

The starting point is the same as for 1-step methods:

$$y(t_{n+1}) - y(t_n) = \int_{t_n}^{t_{n+1}} y'(t)dt = \int_{t_n}^{t_{n+1}} f(t, y(t))dt. \qquad (*)$$

The function f is then approximated by its Lagrange interpolant polynomial of degree ≤ 1 with respect to the nodes t_{n-1} and t_n:

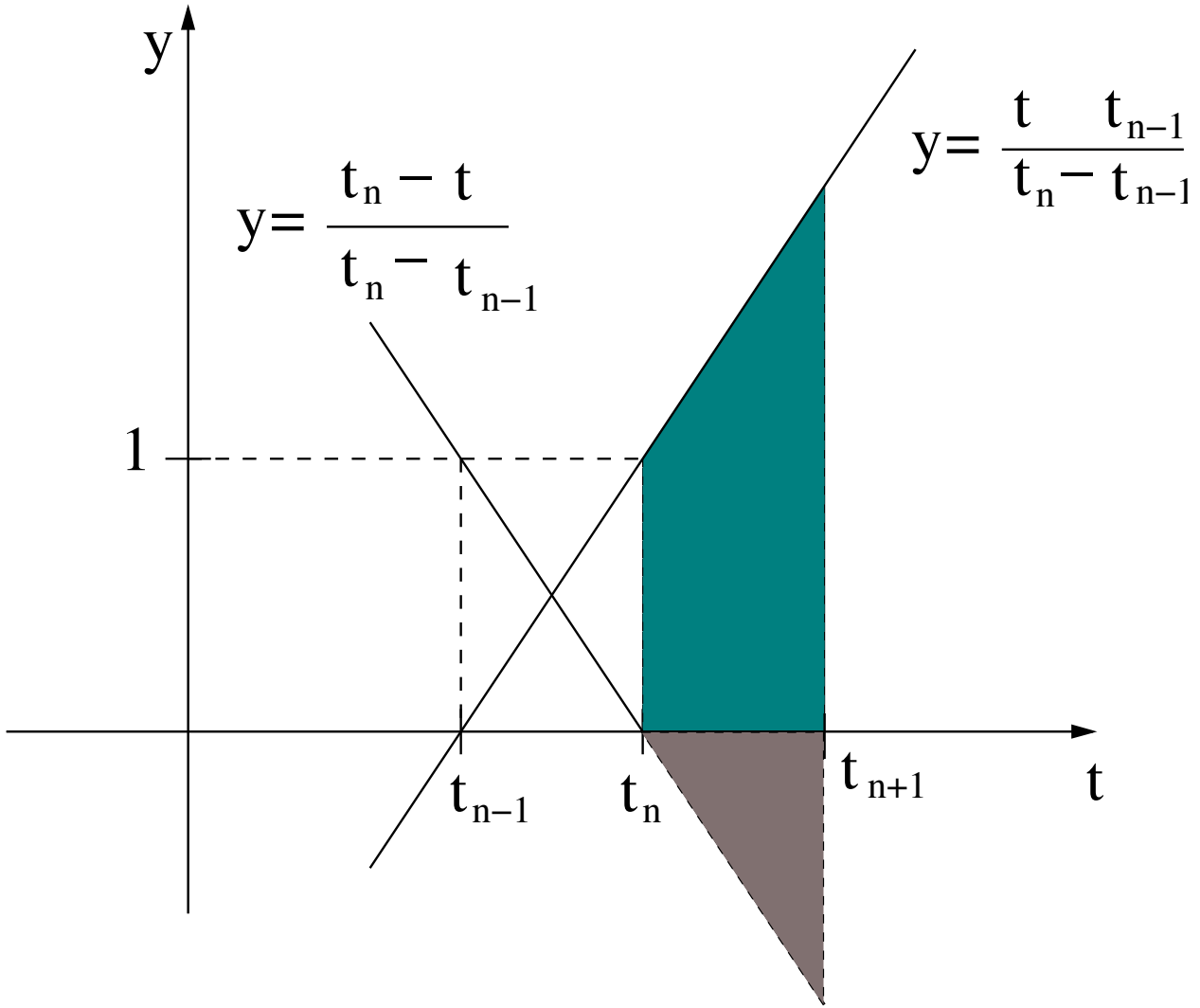
$$f(t, y(t)) \simeq \Pi_1(t) := \frac{t - t_{n-1}}{t_n - t_{n-1}} f(t_n, y_n) + \frac{t_n - t}{t_n - t_{n-1}} f(t_{n-1}, y_{n-1})$$

Consequently,

$$\int_{t_n}^{t_{n+1}} f(t, y(t))dt \sim \int_{t_n}^{t_{n+1}} \Pi_1(t)dt = \frac{3}{2}\Delta t\, f(t_n, y_n) - \frac{1}{2}\Delta t\, f(t_{n-1}, y_{n-1})$$

The order accuracy is 2: if $f \in \mathbb{P}_1$, then $f \equiv \Pi_1$ and $\int f$ is computed exactly. On the other hand, $f \in \mathbb{P}_1$ implies $y \in \mathbb{P}_2$.

Integral of Π_1



Generalization

With the same approach we can design schemes that use values computed at p previous steps and are p-accurate:

on each interval $[t_n, t_{n+1}]$ the function f is replaced by its Lagrange interpolant polynomial (of degree $\leq p-1$) with respect to the p points $t_n, t_{n-1}, \cdots, t_{n+1-p}$:

$$
\begin{aligned}
& f(t, y(t)) \simeq \Pi_{p-1}(t) \quad \text{with } \Pi_{p-1} \in \mathbb{P}_{p-1} \text{ verifying} \\
& \Pi_{p-1}(t_n) = f(t_n, y_n), \\
& \Pi_{p-1}(t_{n-1}) = f(t_{n-1}, y_{n-1}), \\
& \cdots \\
& \Pi_{p-1}(t_{n+1-p}) = f(t_{n+1-p}, y_{n+1-p}).
\end{aligned}
$$

The $\int_{t_n}^{t_{n+1}} \Pi_{p-1}(t)dt$ is then computed exactly; to complete the p-step scheme we will need to compute $p-1$ "initial values" $y_1, y_2, \cdots, y_{p-1}$ in addition to y_0 (for instance with a 1-step method).

Adams-Bashforth schemes

The resulting scheme will be:

$$\begin{cases} y_0 \text{ given}, \quad y_1, y_2, \cdots, y_{p-1} \text{ to be computed} \\ y_{n+1} = y_n + \Delta t \Big(c_1 f_n + c_2 f_{n-1} + \cdots + c_p f_{n+1-p} \Big), \\ \qquad\qquad\qquad\qquad\qquad\qquad n = p-1, p, p+1, \cdots \end{cases} \tag{2}$$

where $\Delta t\, c_1, \Delta t\, c_2, \cdots$ are the integrals of the characteristic Lagrange polynomials, and $f_n = f(t_n, y_n)$, $f_{n-1} = f(t_{n-1}, y_{n-1})$ and so on.

The multistep methods obtained in this way are "**Adams-Bashforth**" methods: they are **explicit**, p-accurate. In Table 1 below the coefficients of the first four schemes.

Adams-Bashforth schemes ** NOT FOR THE EXAM **

	c_1	c_2	c_3	c_4
$p = 1$	1			
$p = 2$	$3/2$	$-1/2$		
$p = 3$	$23/12$	$-16/12$	$5/12$	
$p = 4$	$55/24$	$-59/24$	$37/24$	$-9/24$

Table: First Adams-Bashforth schemes of order p

Adams-Moulton

Note: A similar construction gives **implicit methods**, called "**Adams-Moulton**. Compared with (2) they have an extra term $c_0 f_{n+1}$ at the new time level. Properly chosen, that adds one extra order of accuracy (as it did for the Crank-Nicolson scheme).

	c_1	c_2	c_3	c_4	A-stability	order
$p=0$	1				yes	Δt
$p=1$	$1/2$	$1/2$			yes	Δt^2
$p=2$	$5/12$	$8/12$	$-1/12$		no	Δt^3
$p=3$	$9/24$	$19/24$	$-5/24$	$1/24$	no	Δt^4

Table: First four Adams-Moulton schemes: order $p+1$ ** NOT FOR THE EXAM **

Consistency of multistep meth.s ** NOT FOR EXAM **

As we have already seen when checking consistency of Heun method, this amounts to impose that the local truncation error is zero when the exact solution of the Cauchy Problem is a polynomial of degree up to 2: hence we must require that $\tau_n \equiv 0$ when the solution of the Cauchy Problem is $1, t, t^2$.

$$\tau_n = \frac{y(t_{n+1}) - y(t_n)}{\Delta t} - \Big(\alpha f(t_n, y(t_n)) + \beta f(t_{n-1}, y(t_{n-1}))\Big) \quad n = 1, 2, \cdots$$

When $y(t) = 1, y' = 0 = f, \Longrightarrow \tau_n = \dfrac{1-1}{\Delta t} = 0;$

When $y(t) = t, y' = 1 = f, \Longrightarrow \tau_n = \dfrac{t_{n+1} - t_n}{\Delta t} - (\alpha + \beta) = 0;$

$$\Longrightarrow \alpha + \beta = 1 \Longrightarrow \beta = 1 - \alpha;$$

When $y(t) = t^2, y' = 2t = f, \Longrightarrow \tau_n = \dfrac{t_{n+1}^2 - t_n^2}{\Delta t} - (2\alpha t_n + 2\beta t_{n-1})$

$$= t_{n+1} + t_n - 2(\alpha t_n + \beta t_{n-1}) = 0$$

$$\Longrightarrow \alpha t_n + \beta t_{n-1} = \frac{t_{n+1} + t_n}{2}.$$

Consistency of multistep meth.s ** NOT FOR EXAM **

$$\boxed{\beta = 1 - \alpha, \quad \alpha t_n + \beta t_{n-1} = \frac{t_{n+1}+t_n}{2}}$$

Inserting $\beta = 1 - \alpha$ in the second equation we have

$$\begin{aligned}\alpha(t_n - t_{n-1}) &= \frac{t_n + t_{n+1} - 2t_{n-1}}{2} \\ &= \frac{n\Delta t + (n+1)\Delta t - 2(n-1)\Delta t}{2} = \frac{3\Delta t}{2}.\end{aligned}$$

Therefore we obtain $\alpha = \frac{3}{2}$ and $\beta = -\frac{1}{2}$. The 2-step scheme is then

$$\begin{cases} y_0 \text{ given}, \quad y_1 \text{ to be computed} \\ y_{n+1} = y_n + \Delta t\left(\frac{3}{2}f(t_n, y_n) - \frac{1}{2}f(t_{n-1}, y_{n-1})\right), n = 1, 2, \cdots \end{cases}$$

By construction the scheme is consistent of order 2. Being explicit, it will not be A-stable, only conditionally A-stable.

More schemes ** NOT FOR THE EXAM **

Another way of constructing explicit methods with a good accuracy is to choose an implicit scheme, and make it explicit with a very simple and successful trick:

- P: use the explicit formula to *predict* a new y^*_{n+1}
- E: use y^*_{n+1} to *evaluate* $f^*_{n+1} = f(t_{n+1}, y^*_{n+1})$
- C: use f^*_{n+1} in the implicit formula to *correct* the new y_{n+1}

This is the **predictor-corrector** method (see Heun method). The stability is much improved if there is another E step to evaluate f_{n+1} with the corrected y_{n+1}. So PECE is the basic sequence
continue the correction repeating the CE steps until y_{n+1} no longer changes: it has reached its final value for the implicit formula. Often two or three corrections are enough, and this is faster than using Newton's method in solving a single step of the implicit method.

Systems of Ode's

It is much more common to have systems of differential equations than a single equation. The unknown is now a vector $\underline{Y}(t)$, and so is the right-hand side $\underline{F}(t, \underline{Y}(t))$. The problem is: find $\underline{Y}(t)$ solution of

$$\begin{cases} \underline{Y}'(t) = \underline{F}(t, \underline{Y}(t)) \quad t \in [t_0, T] \\ \underline{Y}(t_0) = \underline{Y}^{(0)}. \end{cases} \tag{3}$$

with

$$\underline{Y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_N(t) \end{bmatrix}, \quad \underline{F}(t, \underline{Y}(t)) = \begin{bmatrix} f_1(t, \underline{Y}(t)) \\ f_2(t, \underline{Y}(t)) \\ \vdots \\ f_N(t, \underline{Y}(t)) \end{bmatrix}, \quad \underline{Y}^{(0)} = \begin{bmatrix} y_1^{(0)} \\ y_2^{(0)} \\ \vdots \\ y_N^{(0)} \end{bmatrix}$$

Systems of Ode's

The numerical schemes used for a single equation apply directly to systems of Ode's.
For instance, the two Euler methods become:

$$(EE) \quad \begin{cases} \underline{Y}^{(0)} \text{ given} \\ \underline{Y}^{(n+1)} = \underline{Y}^{(n)} + \Delta t \underline{F}(t_n, \underline{Y}^{(n)}) \quad n = 0, 1, \cdots \end{cases}$$

Ex: $N = 2$ equations, and 2 unknowns y_1, y_2:

$$\begin{cases} y_1^{(0)}, \; y_2^{(0)} \quad \text{given} \\ y_1^{(n+1)} = y_1^{(n)} + \Delta t \, f_1(t_n, y_1^{(n)}, y_2^{(n)}) \quad n = 0, 1, \cdots \\ y_2^{(n+1)} = y_2^{(n)} + \Delta t \, f_2(t_n, y_1^{(n)}, y_2^{(n)}) \quad n = 0, 1, \cdots \end{cases}$$

Systems of Ode's

$$(IE)\ \begin{cases} \underline{Y}^{(0)} \text{ given} \\ \underline{Y}^{(n+1)} = \underline{Y}^{(n)} + \Delta t \underline{F}(t_{n+1}, \underline{Y}^{(n+1)}) \quad n = 0, 1, \cdots \end{cases}$$

Ex: $N = 2$ equations, and 2 unknowns y_1, y_2:

$$\begin{cases} y_1^{(0)},\ y_2^{(0)} \quad \text{given} \\ y_1^{(n+1)} = y_1^{(n)} + \Delta t\, f_1(t_{n+1}, y_1^{(n+1)}, y_2^{(n+1)}) \quad n = 0, 1, \cdots \\ y_2^{(n+1)} = y_2^{(n)} + \Delta t\, f_2(t_{n+1}, y_1^{(n+1)}, y_2^{(n+1)}) \quad n = 0, 1, \cdots \end{cases}$$

much more expensive now: at each step, to go from $\underline{Y}^{(n)}$ to $\underline{Y}^{(n+1)}$ requires the solution of a non-linear *system*!

** NOT FOR THE EXAM **

$$\boxed{\underline{Y}^{(n+1)} = \underline{Y}^{(n)} + \Delta t \underline{F}(t_{n+1}, \underline{Y}^{(n+1)})}$$

find $\underline{X}$ such that

$$\underline{G}(\underline{X}) := \underline{X} - \Delta t \underline{F}(t_{n+1}, \underline{X}) - \underline{Y}^{(n)} = 0,$$

and set $Y^{(n+1)} := \underline{X}$
Newton would give:

$$\begin{cases} \underline{X}^{(0)} \quad \text{given} \\ JG_{[\underline{X}^{(0)}]} \underline{\delta X} = -\underline{G}(\underline{X}^{(0)}) \\ \underline{X}^{(1)} = \underline{X}^{(0)} + \underline{\delta} X \end{cases}$$

$\underline{X}^{(0)} = ??$ for example: $\underline{X}^{(0)} = \underline{Y}^{(n)} + \Delta t \underline{F}(t_n, \underline{Y}^{(n)})$ (EE)

** NOT FOR THE EXAM **

Or you could use *PECE* with a few cycles of *CE*:

$$P: \quad \underline{X}^{(0)} = \underline{Y}^{(n)} + \Delta t \underline{F}(t_n, \underline{Y}^{(n)}) \quad (EE)$$

$$E: \quad \underline{F}^{(0)}(t_{n+1}, \underline{X}^{(0)})$$

$$C: \quad \underline{X}^{(1)} = \underline{Y}^{(n)} + \Delta t \underline{F}^{(0)}(t_{n+1}, \underline{X}^{(0)})$$

$$E: \quad \underline{F}^{(1)}(t_{n+1}, \underline{X}^{(1)})$$

$$C: \quad \underline{X}^{(2)} = \underline{Y}^{(n)} + \Delta t \underline{F}^{(1)}(t_{n+1}, \underline{X}^{(1)})$$

$$E: \quad \underline{F}^{(2)}(t_{n+1}, \underline{X}^{(2)})$$

Then set:

$$\boxed{\underline{Y}^{(n+1)} = \underline{Y}^{(n)} + \Delta t \underline{F}^{(2)}(t_{n+1}, \underline{X}^{(2)})}$$

Lack of A-stability ** NOT FOR THE EXAM **

For a single equation the lack of A-stability is not a major drawback: to have a good accuracy small Δt have to be used. Instead for systems it could be more severe when the problem has different time scales.

$$\underline{Y}'(t) = A\underline{Y}(t)$$

the eigenvalues λ_j of the square matrix A take the place of the single number a.

Explicit Euler would give

$$\underline{Y}^{(n+1)} = (I + \Delta t\, A)\underline{Y}^{(n)} \quad \forall n \implies \underline{Y}^{(n+1)} = (I + \Delta t\, A)^{n+1}\underline{Y}^{(0)}$$

The growth factor is now a matrix

$$G = (I + \Delta t\, A) \quad \text{with eigenvalues } g_j = 1 + \Delta t\, \lambda_j.$$

Lack of A-stability ** NOT FOR THE EXAM **

Suppose that both A and G are diagonalised. Then,

- each component of the discrete solution grows like g_j^n,
- each component of the continuous solution grows like $e^{\lambda_j t}$.

The continuous solution is stable if all the λ_j are negative (or $\mathrm{Re}\,\lambda_j < 0$): hence $e^{\lambda_j t} \to 0$ for $t \to \infty$.

The discrete solution is stable if all the $|g_j| < 1$, so that $g_j^n \to 0$ for $n \to \infty$.

If the problem has different time scales we are in trouble...

Since Δt is the same for all the components, its size is controlled by the most negative eigenvalue, which corresponds to the fastest decay and dies out first in the true solution.

Stiff systems ** NOT FOR THE EXAM **

When the matrix A has eigenvalues with very different magnitude, the system is called **stiff**.

Let us see a simple example.

$$\underline{Y}'(t) = \begin{bmatrix} -2 & 1 \\ 0 & -100 \end{bmatrix} \underline{Y}(t) \longrightarrow \begin{array}{l} y_1'(t) = -2y_1(t) + y_2(t) \\ y_2'(t) = -100y_2(t) \end{array}$$

solution: $y_1(t) \simeq e^{-2t}, y_2(t) \simeq e^{-100t}$.

If we use Explicit Euler method we need

$$\Delta t < \frac{2}{|\lambda_1|} = 1 \quad \text{and} \quad \Delta t < \frac{2}{|\lambda_2|} = \frac{1}{50}.$$

Stability requires then $\Delta t < \frac{1}{50}$ even though it is e^{-2t} that controls the true solution: in fact, y_2 decays like e^{-100t} and dies out very fast, but its presence forces us to proceed by small time steps even when it has virtually disappeared and we are interested in following the e^{-2t} component.

Matlab functions

Most commonly used Matlab functions:

Non stiff problems:

ode23 (low order RK), ode45 (medium order RK), ode113 (variable order)

Stiff: ode15s (low to medium order), ode23s (low order RK)