Examples of exponential instability for elliptic inverse problems

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Abstract

Following a recent paper by N. Mandache (Inverse Problems **17** (2001), pp. 1435–1444), we establish a general procedure for determining the instability character of inverse problems. We apply this procedure to many elliptic inverse problems concerning the determination of defects of various types by different kinds of boundary measurements and we show that these problems are exponentially ill-posed.

1 Introduction

Many inverse problems associated to partial differential equations concern the problem of determining a parameter of the equation, for example either a coefficient of the equation (*coefficient identification*) or the geometry (that is the boundary) of the region where the phenomenon modelled by the equation occurs (*boundary identification*). In order to determine this parameter one needs additional information on the solutions to the partial differential equation, usually constituted of measurements of the solutions on an accessible (and therefore known) part of the boundary of the region in which the phenomenon takes place.

As an example of a coefficient identification problem, we may think of the *inverse conductivity problem*, whose formulation is due to A. P. Calderón, see [8]. In this problem, electrostatic measurements of voltage and current are collected on the boundary of a conducting body and by these data one tries to obtain information about the conductivity inside the body.

For what concerns boundary identification problems, we consider the following examples. First, determination of a defect inside a conducting body by electrostatic measurements on the boundary. The defect can be of many different types: it can be an *inclusion*, that is a region where the conductivity is different from the background conductivity, see for instance [6] and [13]; it can be a *crack*, that is a fracture, as it has been introduced in [12], see also [4] and [19]; it can be a *cavity* or a *boundary material loss*, due to corrosion for example, see, for instance, [3] and [19]. Then, also the determination of an *obstacle* by acoustic measurements in the far-field can be considered as a problem of this kind, see [10].

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It has been noted several times that these kinds of inverse problems are ill-posed; in fact, even if the amount of data collected is sufficient to guarantee uniqueness, the coefficient or the defect, respectively, usually does not depend continuously, that is in a stable way, from the measured data.

For the numerical treatment of inverse problems, the ill-posedness constitutes a severe difficulty. The second main difficulty is usually due to the fact that inverse problems are tipically non-linear, even if the direct problem, in the examples above a boundary value problem for an elliptic partial differential equation, is a linear one. An accurate knowledge of the character of the ill-posedness is an advantage for devising efficient numerical methods. Since the problem is ill-posed, in order to recover some kind of stability, we need to apply a regularization procedure, that is to restrict the space of admissible unknowns (either the coefficients or the defects) by assuming that they satisfy a priori conditions involving usually some kind of smoothness assumptions. With this a priori information, it is possible to prove that the unknowns depend in a continuous way from the measured data. However, an explicit knowledge of such a continuous dependence is crucial for several reasons. First, it provides us with a quantitative information on how much ill-posed the inverse problem is, thus how much difficult it is to solve it numerically; second, a precise knowledge of the modulus of continuity of the dependence of the unknowns from the measured data indicates the optimal rate of convergence for regularization schemes and can be useful also for tuning the regularization parameter, see for instance [11].

The determination of the modulus of continuity has to be done in two steps. First, we have to establish *stability estimates* conditioned to some a priori assumptions on the unknowns; second, we have to show that these stability estimates are *optimal* or at least essentially optimal. In order to fulfil the second part of this program, we need to construct examples which show that the inverse problem has an instability character of the same order, or at least of the same kind, that is of logarithmic or Hölder type, for instance, of the stability estimates already established. We say that our inverse problem is *exponentially ill-posed*, or *severely ill-posed*, if such a modulus of continuity is of logarithmic type. In other words, exponential instability corresponds to the fact that optimal stability estimates are at most of logarithmic type.

The first of these examples has been constructed in [2] and deals with the problem of the determination of a boundary material loss in a planar conductor. This example shows that the stability estimates developed in [19] are essentially optimal; since these estimates are of logarithmic type, this kind of problem is therefore exponentially ill-posed. An example similar to the one in [2] has been constructed for the problem of cavities, still in two dimensions, in [7]. These two examples are explicit in the sense that a family of solution showing the instability character of the problem is given by explicit formulas, choosing defects whose boundaries are highly oscillating. The construction of a family satisfying the instability property looked for is not an easy task for other inverse problems.

Recently, however, N. Mandache has proved in [17] that the inverse conductivity problem is also exponentially unstable, showing at the same time that the estimates given in [1] are optimal. The procedure used in [17] does not depend on an explicit construction, it is instead constituted by a purely topological argument, which follows from the work of A. N. Kolmogorov and V. M. Tihomirov, [16]. We wish to illustrate the argument as follows. Let $F : X \mapsto Y$ be a function, X and Y being metric spaces. As a model of an inverse problem,

X represents the space of unknowns, Y the space of the measured data and F is the forward map representing the direct problem. Let us assume that there exists $x_0 \in X$ so that for every $\varepsilon > 0$ the ball $B(x_0, \varepsilon)$ contains $f(\varepsilon)$ disjoint balls of radius $\varepsilon/2$, $f(\varepsilon)$ being an integer depending on ε. Furthermore, we assume that for every $\delta > 0$ there exists an integer $g(\delta)$ so that $F(X)$ can be covered by $g(\delta)$ balls of radius δ. If we can find $\varepsilon_1 > 0$ and $\delta(\varepsilon)$ so that for every ε, $0 < \varepsilon < \varepsilon_1$, $f(\varepsilon) > g(\delta(\varepsilon))$, then we can find x_1 and x_2 in $B(x_0, \varepsilon)$ so that $d_X(x_1, x_2) \geq \varepsilon$ and $d_Y(F(x_1), F(x_2)) \leq 2\delta(\varepsilon)$. Thus, $\delta(\varepsilon)$ provides an indication of the instability of the inverse to the map F. Hence, it appears clear that establishing this instability character depends on an accurate counting either of the maximal amount of disjoint balls with fixed radius that can be found in a given ball of the space X or of the minimal amount of balls with fixed radius required to cover the image through F of X.

This procedure immediately appears to be very general and very well suited to be applied in the context of inverse problems. In fact, the space of unknowns has, in general, a richer structure with respect to that of the data, since usually in inverse problems the forward map F is compact. A first application of this procedure to ill-posed problems is developed in [21].

Following the topological arguments of [16] and the procedure described in [17], we have extracted a general method for determining instability, to be applicable to many different inverse problems. In Theorem 3.1 below, we have stated in a rather abstract framework the outline of the method, in one of its possible formulations (for slightly different but analogous formulations we refer, for instance, to the discussion of the inverse scattering case, see Subsection 5.4).

Then we have applied our abstract result to many inverse boundary value problems of elliptic type. We have shown that all the kinds of boundary identification problems briefly described above are exponentially ill-posed. We also wish to remark that, as in the explicit examples of [2] and [7], the ill-posedness is of exponential type no matter which and how many measurements we take. Our examples in fact deal with the ideal case of performing all possible measurements. This fact is somewhat surprising since in these boundary identification problems a much lesser amount of data is required to have unique identification of a defect and also to have stability estimates; usually a finite number of measurements is enough. This shows the difficulty of the problem and that performing more measurements or different ones does not solve the problem of ill-posedness.

The plan of the paper is as follows. In order to point out to the reader all the inverse problems to which we have successfully applied the method, we first describe, in Section 2, all the instability results that are contained in the paper. Then we proceed with the proofs of these results. The proofs are divided into three sections. In Section 3 we state and prove an abstract result, Theorem 3.1, which provides the general procedure for obtaining the instability examples and therefore constitutes the key ingredient and crucial part of the proofs of all the results described in Section 2. In fact, the proofs are in general obtained as straightforward applications of this abstract theorem. In order to apply the abstract theorem, what is essentially needed is to choose a suitable orthonormal basis and to check that all the hypotheses of the abstract theorem are satisfied. Concerning orthonormal basis, we shall employ eigenfunctions corresponding to eigenvalue problems of Stekloff type. We have collected all the information we shall need about these orthonormal basis in Section 4. Then, in Section 5, the proofs of the instability results are concluded. Using the orthonormal basis introduced in Section 4, we verify that the abstract result applies to the problems we consider and we prove their exponential instability.

In details, in Section 2, first we need to introduce some notations which will be used repeatedly in the paper. In particular we define, and investigate the structure of, the metric spaces of the unknowns. Then, we list the problems for which we have obtained the instability examples, together with the precise formulation of the instability results. We observe that, for the sake of brevity, we usually refer to the bibliography for a more detailed description of the problems considered. We begin with the problem of determination of defects of different types by electrostatic boundary measurements. In Subsection 2.1 we treat the problem of determination of an inclusion, in Subsection 2.2 the determination of cracks is considered, in Subsection 2.3 we deal with the inverse problem of cavities, in Subsection 2.4 we treat the case of cracks reaching the boundary of the domain, that is surface cracks, and in Subsection 2.5 we study the problem of a boundary material loss. Finally, in Subsection 2.6, we deal with inverse scattering problems, in particular with the determination of obstacles (either of sound-soft or of sound-hard type) by far-field acoustic measurements. In Section 3 the abstract result is stated and proved. In Section 4 we study two different eigenvalue problems of Stekloff type and we investigate the asymptotic properties of either their eigenvalues or eigenfunctions, in particular this is done for three different domains of our interest where the solutions can be computed almost explicitly. In Section 5, the conclusions of proofs of all the instability results are developed.

2 Statement of the instability results

Before stating the main results, we need to introduce some notations about the Sobolev spaces we shall use and to describe the spaces of the unknowns.

For any $N \geq 2$, any $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$ and any $r > 0$, we denote $B_N(x,r) = \{y \in \mathbb{R}^N : \ \|y - x\| < r\}$. We set $S^{N-1}(x,r) = \partial B_N(x,r)$. Furthermore, we set $S^{N-1} = \partial B_N(0,1)$, and $S_+^{N-1} = \{y \in S^{N-1} : \ y_N \geq 0\}$, and, analogously, $S_-^{N-1} = \{y \in S^{N-1} : \ y_N \leq 0\}$. Finally, we denote $B'_{N-1}(x,r) = \{y \in B_N(x,r) : \ y_N = x_N\}$.

We need, furthermore, to introduce the following definition.

Definition 2.1 Let (Y, d_Y) be a metric space. For a given positive δ, Y_1, a subset of Y, is said to be a δ-*net* for Y if for every $y \in Y$ there exists $y_1 \in Y_1$ so that $d_Y(y, y_1) \leq \delta$.

Given ε positive, $Y_2 \subset Y$ is ε-*discrete* if for any two distinct points y_2, y'_2 in Y_2 we have $d_Y(y_2, y'_2) \geq \varepsilon$.

Notations on Sobolev spaces

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain and let $\partial\Omega$ be its boundary. About regularity, we assume that there exists a homeomorphism $\chi : B_N(0,1) \mapsto \Omega$ such that, for a positive constant C, we have

$$
\begin{aligned}
&\|\chi(\tilde{x}) - \chi(\tilde{y})\| \leq C\|\tilde{x} - \tilde{y}\| && \text{for any } \tilde{x}, \tilde{y} \in B_N(0,1), \\
&\|\chi^{-1}(x) - \chi^{-1}(y)\| \leq C\|x - y\| && \text{for any } x, y \in \Omega.
\end{aligned}
\tag{2.1}
$$

Furthermore, we shall consider two internally disjoint subsets of $\partial\Omega$, Γ_A and Γ_I, so that $\Gamma_A \cup \Gamma_I = \partial\Omega$. We assume either that $\Gamma_A = \partial\Omega$ and $\Gamma_I = \emptyset$, or that Γ_A and Γ_I are not empty and are assumed to be regular enough, namely there exists a homeomorphism $\chi : B_N(0,1) \mapsto \Omega$ satisfying (2.1), so that, if we still denote with χ its extension by continuity to $\overline{B_N(0,1)}$, then $\Gamma_A = \chi(S_+^{N-1})$ and $\Gamma_I = \chi(S_-^{N-1})$.

We introduce the following Sobolev spaces. Let $H^1(\Omega) = \{u \in L^2(\Omega) : \nabla u \in L^2(\Omega)\}$, where ∇u denotes the gradient of u in the sense of distributions. We recall that $H^1(\Omega)$ is a Hilbert space with scalar product $(u,v)_{H^1(\Omega)} = \int_\Omega \nabla u \cdot \nabla v + uv$. With $H^{1/2}(\Gamma_A)$ we denote the space of traces of $H^1(\Omega)$ functions on Γ_A, which can be endowed in a canonical way with a scalar product induced by the one of $H^1(\Omega)$ so that $H^{1/2}(\Gamma_A)$ is a Hilbert space. By $H^{-1/2}(\Gamma_A)$ we shall denote the dual space to $H^{1/2}(\Gamma_A)$. We recall that $H^{1/2}(\Gamma_A) \subset L^2(\Gamma_A) \subset H^{-1/2}(\Gamma_A)$. We shall also make use of the following spaces. Let ${}_0H^{1/2}(\Gamma_A) = \{\psi \in H^{1/2}(\Gamma_A) : \int_{\Gamma_A} \psi = 0\}$. Its dual is given by the space ${}_0H^{-1/2}(\Gamma_A) = \{\eta \in H^{-1/2}(\Gamma_A) : \langle \eta, 1\rangle = 0\}$, where $\langle\cdot,\cdot\rangle$ denotes the duality pairing.

If Γ_I is not empty, we set $H_0^1(\Omega,\Gamma_I)$ and $H^1_{const}(\Omega,\Gamma_I)$ as the closed subspaces of $H^1(\Omega)$ constituted by the functions $u \in H^1(\Omega)$ so that $u = 0$ in a weak sense on Γ_I and $u = constant$ in a weak sense on Γ_I, respectively. With $H_0^{1/2}(\Gamma_A,\Omega)$ and $H^{1/2}_{const}(\Gamma_A,\Omega)$ we denote the closed subspaces of $H^{1/2}(\Gamma_A)$ constituted by the traces of $H_0^1(\Omega,\Gamma_I)$ and $H^1_{const}(\Omega,\Gamma_I)$ functions on Γ_A, respectively.

For our purposes, we need to introduce on the Sobolev spaces defined above suitable scalar products, which are different but topologically equivalent to the canonical ones. We wish to remark that the definitions of these scalar products do not take into account the fact that the spaces $H^{-1/2}$ and $H^{1/2}$ are dual one to each other.

For any ψ, $\varphi \in H^{1/2}(\Gamma_A)$, we set $\tilde\psi \in H^1(\Omega)$ as the solution to

$$\begin{cases} \Delta\tilde\psi = 0 & \text{in } \Omega, \\ \tilde\psi = \psi & \text{on } \Gamma_A, \\ \frac{\partial\tilde\psi}{\partial\nu} = 0 & \text{on } \Gamma_I, \end{cases} \tag{2.2}$$

and $\tilde\varphi$ as the solution to the same boundary value problem with ψ replaced by φ, and the scalar product we use on $H^{1/2}(\Gamma_A)$ is given by

$$(\psi,\varphi)_{H^{1/2}(\Gamma_A)} = \int_\Omega \nabla\tilde\psi \cdot \nabla\tilde\varphi + \int_{\Gamma_A} \psi\varphi. \tag{2.3}$$

We observe that ${}_0H^{1/2}(\Gamma_A)$ coincides with the subspace which is orthogonal, with respect to this scalar product, to the constant function 1.

Any $\eta \in H^{-1/2}(\Gamma_A)$ can be decomposed, in a unique way, into the sum of $\hat\eta$, an element of ${}_0H^{-1/2}(\Gamma_A)$, and a constant function $c(\eta)$. Furthermore, to $\hat\eta$ we can associate $\tilde\eta \in H^1(\Omega)$ that solves

$$\begin{cases} \Delta\tilde\eta = 0 & \text{in } \Omega, \\ \frac{\partial\tilde\eta}{\partial\nu} = \hat\eta & \text{on } \Gamma_A, \\ \frac{\partial\tilde\eta}{\partial\nu} = 0 & \text{on } \Gamma_I. \end{cases} \tag{2.4}$$

If, in the same way, we associate to $\phi \in H^{-1/2}(\Gamma_A)$ the functions $\hat\phi$, $c(\phi)$ and

$\tilde{\phi}$, then the scalar product on $H^{-1/2}(\Gamma_A)$ may be defined as

$$(2.5) \qquad (\eta, \phi)_{H^{-1/2}(\Gamma_A)} = \int_\Omega \nabla\tilde{\eta}\cdot\nabla\tilde{\phi} + c(\eta)c(\phi).$$

We remark that, with respect to this scalar product, ${}_0H^{-1/2}(\Gamma_A)$ is the orthogonal subspace to the constant function 1.

We take Γ_I not empty. If ψ belongs to $H^{1/2}_{const}(\Gamma_A,\Omega)$, then there exist (and are unique) $\hat{\psi}\in H^{1/2}_0(\Gamma_A,\Omega)$ and a constant function $c(\psi)$ so that $\psi = \hat{\psi}+c(\psi)$. Let $\tilde{\psi}\in H^1(\Omega)$ solve

$$(2.6) \qquad \begin{cases} \Delta\tilde{\psi} = 0 & \text{in } \Omega,\\ \tilde{\psi} = \hat{\psi} & \text{on } \Gamma_A,\\ \tilde{\psi} = 0 & \text{on } \Gamma_I.\end{cases}$$

Then, if we associate to $\varphi \in H^{1/2}_{const}(\Gamma_A,\Omega)$ its corresponding decomposition given by $\hat{\varphi}$ and $c(\varphi)$, and its corresponding function $\tilde{\varphi}$, on $H^{1/2}_{const}(\Gamma_A,\Omega)$ we introduce the scalar product

$$(2.7) \qquad (\psi,\varphi)_{H^{1/2}_{const}(\Gamma_A,\Omega)} = \int_\Omega \nabla\tilde{\psi}\cdot\nabla\tilde{\varphi} + c(\psi)c(\varphi).$$

Such a scalar product obviously induces a scalar product on $H^{1/2}_0(\Gamma_A,\Omega)$, which is the closed subspace of $H^{1/2}_{const}(\Gamma_A,\Omega)$ orthogonal to the constant function 1.

Spaces of smooth perturbations of a given set

We shall consider the following examples. Let us fix integers $N\geq 2$ and $m\geq 1$ and positive constants ε and β. Let us also fix $x\in\mathbb{R}^N$ and $r>0$.

To any real function f defined on $\overline{B'_{N-1}(x,r)}$, where $B'_{N-1}(x,r) = \{y\in B_N(x,r) : \ y_N = x_N\}$, we associate its *graph*, that is $\mathrm{graph}(f) = \{y\in\mathbb{R}^N : \ y_N = f(y_1,\ldots,y_{N-1},x_N),\ (y_1,\ldots,y_{N-1},x_N)\in\overline{B'_{N-1}(x,r)}\}$, and, assuming $f\geq x_N$, its *subgraph*, that is $\mathrm{subgraph}(f) = \{y\in\mathbb{R}^N : \ x_N\leq y_N\leq f(y_1,\ldots,y_{N-1},x_N),\ (y_1,\ldots,y_{N-1},x_N)\in\overline{B'_{N-1}(x,r)}\}$.

With the notation $X_{m\beta\varepsilon}(B'_{N-1}(x,r))$ we indicate the space $\{\mathrm{graph}(f) : \ f\in C^m_0(B'_{N-1}(x,r)),\ \|f\|_{C^m(B'_{N-1}(x,r))}\leq\beta$ and $x_N\leq f\leq x_N+\varepsilon\}$ and with $Y_{m\beta\varepsilon}(B'_{N-1}(x,r))$ we indicate the space obtained by taking the subgraphs of all the functions belonging to the same class as before. We consider the spaces $X_{m\beta\varepsilon}(B'_{N-1}(x,r))$ and $Y_{m\beta\varepsilon}(B'_{N-1}(x,r))$ as metric spaces with the Hausdorff distance.

To any strictly positive function g defined on $S^{N-1}(x,r) = \partial B_N(x,r)$, we denote its *radial graph* as $\mathrm{graph}_{rad}(g) = \{y\in\mathbb{R}^N : \ y = x+g(\omega)\cdot\left(\frac{\omega-x}{r}\right),\ \omega\in S^{N-1}(x,r)\}$ and its *radial subgraph* as $\mathrm{subgraph}_{rad}(g) = \{y\in\mathbb{R}^N : \ y = x+\rho\cdot\left(\frac{\omega-x}{r}\right),\ 0\leq\rho\leq g(\omega),\ \omega\in S^{N-1}(x,r)\}$.

Then, with the notation $X_{m\beta\varepsilon}(S^{N-1}(x,r))$ we denote the space given by $\{\mathrm{graph}_{rad}(g) : \ g\in C^m(S^{N-1}(x,r)),\ \|g\|_{C^m(S^{N-1}(x,r))}\leq\beta$ and $r\leq g\leq r+\varepsilon\}$ and with $Y_{m\beta\varepsilon}(S^{N-1}(x,r))$ we denote the space of radial subgraphs of all the functions belonging to the same class used before. Also the spaces $X_{m\beta\varepsilon}(S^{N-1}(x,r))$ and $Y_{m\beta\varepsilon}(S^{N-1}(x,r))$ are metric spaces endowed with the Hausdorff distance.

It is an easy remark the fact that $X_{m\beta\varepsilon}(B'_{N-1}(x,r))$ and $X_{m\beta\varepsilon}(S^{N-1}(x,r))$ are contained in the closed ball, with respect to the Hausdorff distance between closed sets, of radius ε centred at $\overline{B'_{N-1}(x,r)}$ and $S^{N-1}(x,r)$, respectively. Analogously, $Y_{m\beta\varepsilon}(B'_{N-1}(x,r))$ and $Y_{m\beta\varepsilon}(S^{N-1}(x,r))$ are contained in the closed ball, again with respect to the Hausdorff distance, of radius ε and centre $\overline{B'_{N-1}(x,r)}$ and $\overline{B_N(x,r)}$, respectively. Maybe more interesting and significant is the fact that the elements of $Y_{m\beta\varepsilon}(S^{N-1}(x,r))$ are all compact subsets which are star-shaped with respect to a common point $x \in \mathbb{R}^N$. The determination of star-shaped sets is usually considered to be more stable than the determination of other kinds of sets. Nevertheless many of our examples show that even with a star-shapedness assumption the instability is still of exponential type.

We would like to study properties of ε-discrete sets of $X_{m\beta\varepsilon}(B'_{N-1}(x,r))$, $Y_{m\beta\varepsilon}(B'_{N-1}(x,r))$ and $X_{m\beta\varepsilon}(S^{N-1}(x,r))$, $Y_{m\beta\varepsilon}(S^{N-1}(x,r))$. We have the following proposition.

Proposition 2.2 *Let us fix integers $N \geq 2$ and $m \geq 1$ and positive constants β and r. We also fix $x \in \mathbb{R}^N$. Fixed $\varepsilon > 0$, let X_ε be equal to one of the following four metric spaces: $X_{m\beta\varepsilon}(B'_{N-1}(x,r))$, $Y_{m\beta\varepsilon}(B'_{N-1}(x,r))$, $X_{m\beta\varepsilon}(S^{N-1}(x,r))$ or $Y_{m\beta\varepsilon}(S^{N-1}(x,r))$.*

Then, there exists a positive constant ε_0, depending on N, m, β and r only, so that for any ε, $0 < \varepsilon < \varepsilon_0$, we can find Z_ε satisfying the following properties. We have that the set Z_ε is contained in X_ε; Z_ε is ε-discrete, with respect to the Hausdorff distance; and, finally, Z_ε has at least $\exp(2^{-N}\varepsilon_0^{(N-1)/m}\varepsilon^{-(N-1)/m})$ elements.

Proof. The proof can be obtained, with slight modifications, along the lines of the proof of Lemma 2 in [17]. □

2.1 Inverse inclusion problem

Let us assume that the domain $\Omega = B_N(0,1)$, $N \geq 2$, is occupied by a conducting body. Let us assume that an *inclusion* D is present inside the otherwise homogeneous conductor; that is, there exist two different positive constants a and b and a set D which is compactly contained in Ω (that is $\overline{D}$ is a compact subset of Ω) so that the conductivity inside D is constantly equal to a and the conductivity outside D, that is in $\Omega\backslash D$, is constantly equal to b. For the sake of simplicity, we normalize b so that $b = 1$ and we take a to be positive and different from 1.

The electrostatic potential u inside Ω is a solution to the following partial differential equation

$$\operatorname{div}((1 + (a-1)\chi_D)\nabla u) = 0 \quad \text{in } \Omega, \tag{2.8}$$

where χ_D denotes the characteristic function of the domain D.

Furthermore, u satisfies a boundary condition on $\partial\Omega = S^{N-1}$ which depends on whether we prescribe the voltage $\psi \in H^{1/2}(\partial\Omega)$ on the boundary or we assign the current density $\eta \in {}_0H^{-1/2}(\partial\Omega)$ on the boundary. Namely, in the first case the boundary condition is given by

$$u = \psi \text{ on } \partial\Omega; \tag{2.9}$$

in the second case by

$$\frac{\partial u}{\partial \nu} = \eta \text{ on } \partial\Omega; \qquad \int_{\partial\Omega} u = 0; \tag{2.10}$$

where we have added a normalization condition.

We have existence and uniqueness of a (weak) solution for both the boundary value problems (2.8)-(2.9) and (2.8)-(2.10).

The inverse problem we consider is the one of recovering the shape and the location of an unknown inclusion D, by performing current and voltage measurements at the boundary, that is either by prescribing voltages and measuring the corresponding current densities or viceversa.

In the literature, a lot of attention has been devoted to the determination of D by a single measurement; in this case the problem has been often referred to as the inverse conductivity problem with one measurement. A global uniqueness result is still missing, see [6] and its references for a more detailed discussion on this topic. We remark that if all possible measurements are performed, then the inclusion can be uniquely determined, see [13]. However, up to our knowledge, even if all measurements are considered, no explicit stability estimate for this problem has been established.

We produce an example showing that, even if we make many measurements, actually all possible measurements, the optimal stability for this inverse problem is at the best of logarithmic type. We treat the case when voltages are prescribed and currents are measured and the case in which current densities are assigned and voltages are measured, as well.

We fix two positive integers, m and N, $N \geq 2$, and two positive constants, β and a, $a \neq 1$. We consider the metric space (X, d) where $X = Y_{m\beta(1/4)}(S^{N-1}(0, 1/2))$ and d is the Hausdorff distance. If D belongs to X, then we can define the following two operators.

The operator $\Lambda(D) : H^{1/2}(\partial\Omega) \mapsto H^{-1/2}(\partial\Omega)$ is defined as

$$\langle \Lambda(D)\psi, \varphi \rangle = \langle \frac{\partial u}{\partial \nu}|_{\partial\Omega}, \varphi \rangle = \int_{\Omega} (1 + (a-1)\chi_D)\nabla u \cdot \nabla \tilde{\varphi}$$

where ψ, $\varphi \in H^{1/2}(\partial\Omega)$, u is the solution to (2.8)-(2.9) and $\tilde{\varphi}$ is any $H^1(\Omega)$ function whose trace on $\partial\Omega$ is equal to φ. Since the operator $\Lambda(D)$ associates the Dirichlet datum to the corresponding Neumann datum, it is usually called the *Dirichlet-to-Neumann* map.

Viceversa, the operator $\mathcal{N}(D) : {}_0H^{-1/2}(\partial\Omega) \mapsto {}_0H^{1/2}(\partial\Omega)$ is given by

$$\mathcal{N}(D)\eta = u|_{\partial\Omega}$$

where $\eta \in {}_0H^{-1/2}(\partial\Omega)$ and u is the solution to (2.8)-(2.10). For the same reasons, the map $\mathcal{N}(D)$ is called the *Neumann-to-Dirichlet* map.

It is easy to show that for any $D \in X$, the maps $\Lambda(D)$ and $\mathcal{N}(D)$ are linear and bounded operators between a Hilbert space and its dual. In the sequel, their norms will be always assumed to be the canonical ones as bounded operators between Hilbert spaces. We state the instability result.

Proposition 2.3 *We fix integers $N \geq 2$ and $m \geq 1$ and a positive constant β. We also fix $0 < a \neq 1$. Let (X, d) be a metric space, with $X =$*

$Y_{m\beta(1/4)}(S^{N-1}(0,1/2))$ and d being the Hausdorff distance. Then we can find positive constants ε_1 and C, which depend on N, m, β and a only, so that for any ε, $0<\varepsilon<\varepsilon_1$, there exist D_1 and D_2 belonging to X such that

$$d(D_j,\overline{B_N(0,1/2)})\leq\varepsilon,\ \text{for any } j=1,2;\quad d(D_1,D_2)\geq\varepsilon; \tag{2.11}$$

and

$$\|\Lambda(D_1)-\Lambda(D_2)\|\leq C\exp(-\varepsilon^{-(N-1)/(2mN)}); \tag{2.12}$$

$$\|\mathcal{N}(D_1)-\mathcal{N}(D_2)\|\leq C\exp(-\varepsilon^{-(N-1)/(2mN)}). \tag{2.13}$$

The proof of this proposition is postponed to Subsection 5.1.

Experimental measurements

In Proposition 2.3, the inverse inclusion problem is stated to be exponentially ill-posed even if we perform all possible measurements of current and voltage type at the boundary. It is not surprising, therefore, that the inverse inclusion problem is exponentially ill-posed also with respect to measurements which can be actually obtained from the experiments. We shall refer to this kind of measurements as the *experimental measurements*. The model which we shall follow is the one developed in [22], which we briefly describe, referring to the original paper for more details.

The model is the following. On the boundary of the conductor Ω, we attach L *electrodes*. The *contact regions* between the electrodes and the conductor are subsets of $\partial\Omega$ and will be denoted by e_l, $l=1,\ldots,L$. We assume that the subsets e_l, $l=1,\ldots,L$, are open, connected, with a smooth boundary and so that their closures are pairwise disjoint. We remark that we identify any electrode with its contact region. A current is sent to the body through the electrodes and the corresponding voltages are measured on the same electrodes. For each l, $l=1,\ldots,L$, the current applied to the electrode e_l will be denoted by I_l and the voltage measured on the electrode will be denoted by V_l. The column vector I whose components are I_l, $l=1,\ldots,L$, is a *current pattern* if the condition $\sum_{l=1}^{L}I_l=0$ is satisfied. The corresponding *voltage pattern*, that is the column vector V whose components are V_l, $l=1,\ldots,L$, is determined up to an additive constant and we always choose to normalize it in such a way that $\sum_{l=1}^{L}V_l=0$. The voltage pattern depends on the current pattern in a linear way, through an $L\times L$ symmetric matrix R which is called the *resistance matrix*, that is $V=RI$.

The following model can be used to determine the resistance matrix R. We assume that at each electrode e_l, $l=1,\ldots,L$, a *surface impedance* is present and we denote it with z_l. Let us assume that there exists $Z>0$ so that for each l, $l=1,\ldots,L$, $z_l\geq Z$. Let D be as before an inclusion in Ω. The conductivity in D is a, where a is a positive constant different from 1, and the conductivity outside D is equal to 1. If we apply the current pattern I on the electrodes, then the voltage u inside the body satisfies the following boundary value problem

$$\begin{cases}\mathrm{div}((1+(a-1)\chi_D)\nabla u)=0 & \text{in } \Omega,\\ u+z_l\frac{\partial u}{\partial\nu}=U_l & \text{on } e_l,\ l=1,\ldots,L,\\ \frac{\partial u}{\partial\nu}=0 & \text{on } \partial\Omega\backslash\bigcup_{l=1}^{L}e_l,\\ \int_{e_l}\frac{\partial u}{\partial\nu}=I_l & \text{for any } l=1,\ldots,L,\end{cases} \tag{2.14}$$

where U_l, $l = 1, \ldots, L$, are constants to be determined. We call U the column vector whose components are given by U_l, $l = 1, \ldots, L$.

For any l, $l = 1, \ldots, L$, V_l, a component of the voltage pattern V, is given by $V_l = \int_{e_l} u$, thus, by (2.14),

$$V_l = |e_l| U_l - z_l I_l,$$

where $|e_l|$ denotes the surface measure of e_l.

By [22, Theorem 3.3], we infer that there exists a unique couple (u, U), u being in $H^1(\Omega)$ and U being a column vector with L components so that $\sum_{l=1}^{L} |e_l| U_l - z_l I_l = 0$, such that (2.14) is satisfied. Thus the current pattern I uniquely determines the voltage pattern V, if this is normalized in such a way that $\sum_{l=1}^{L} V_l = 0$. Furthermore, it has been proved in [22] that the relation between I and V is linear, thus the resistance matrix $R(D)$ is well defined. Finally, it has been shown that $R(D)$ is actually symmetric. We remark that we shall assume, without loss of generality, that $R(D)[1] = 0$, where $[1]$ denotes the column vector whose components are all equal to 1. Also, we recall that the norm of $R(D)$ will always be the norm of linear operators from $\mathbb{R}^L$ into itself.

The following instability result will be proved in Subsection 5.1.

Proposition 2.4 *We fix integers $N \geq 2$ and $m \geq 1$ and a positive constant β. We also fix $0 < a \neq 1$ and $Z > 0$. Let (X, d) be a metric space, with $X = Y_{m\beta(1/4)}(S^{N-1}(0, 1/2))$ and d being the Hausdorff distance. Let us assume that $L \geq 2$ electrodes e_l, $l = 1, \ldots, L$, and their surface impedances z_l, $l = 1, \ldots, L$, are fixed and satisfy the previously stated assumptions. Then we can find positive constants ε_1 and $\tilde{C}$, which depend on N, m, β, a, Z and the electrodes only, so that for any ε, $0 < \varepsilon < \varepsilon_1$, there exist D_1 and D_2 belonging to X such that*

$$\begin{aligned} &d(D_j, \overline{B_N(0, 1/2)}) \leq \varepsilon, \textit{ for any } j = 1, 2; \quad d(D_1, D_2) \geq \varepsilon; \\ &\|R(D_1) - R(D_2)\| \leq \tilde{C} \exp(-\varepsilon^{-(N-1)/(2mN)}). \end{aligned} \tag{2.15}$$

2.2 Inverse crack problem

Let $\Omega = B_N(0, 1)$, $N \geq 2$, be the region occupied by a homogeneous conducting body. Let us assume that inside the conductor there is a *crack* σ, that is a closed set inside Ω so that $\Omega \backslash \sigma$ is connected and, locally, σ can be represented by the graph of a smooth function. We can consider two different types of cracks, *perfectly insulating* and *perfectly conducting*, and we can prescribe on the (exterior) boundary of Ω either the voltage or the current density. Thus, the electrostatic potential u in Ω satisfies either

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \backslash \sigma, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \sigma; \end{cases} \tag{2.16}$$

if σ is perfectly insulating, or, when σ is assumed to be perfectly conducting,

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \backslash \sigma, \\ u = \textit{constant} & \text{on } \sigma. \end{cases} \tag{2.17}$$

We remark that, in (2.16), on $\partial\sigma$ means on either sides of σ. On the boundary the potential satisfies either

$$u = \psi \text{ on } \partial\Omega; \quad \langle \frac{\partial u}{\partial \nu}|_{\partial\Omega}, 1 \rangle = 0; \tag{2.18}$$

where $\psi \in H^{1/2}(\partial\Omega)$ is the prescribed voltage at the boundary, or, if we prescribe the current density on the boundary to be $\eta \in {}_0H^{-1/2}(\partial\Omega)$,

$$\frac{\partial u}{\partial \nu} = \eta \text{ on } \partial\Omega; \quad \int_{\partial\Omega} u = 0; \tag{2.19}$$

we wish to remark that normalization conditions have been added to the boundary conditions.

We have that all the direct problems (2.16)-(2.18), (2.16)-(2.19), (2.17)-(2.18) and (2.17)-(2.19) admit a unique (weak) solution.

The inverse crack problem consists of recovering the shape and location of an unknown crack σ by performing electrostatic measurements at the boundary.

In this subsection we shall state the instability character of such an inverse problem, in all the possible cases, that is when we consider either insulating or conducting cracks, and when either we prescribe voltages and measure corresponding currents or we prescribe currents and measure corresponding voltages.

For a detailed analysis of uniqueness and stability of this problem we refer to [19], for the two-dimensional case, and to [4], for the three-dimensional case, and to their bibliographies. We wish to remark that, for what concerns uniqueness and stability results, these have been obtained with a finite number of boundary measurements, usually with two suitably chosen measurements. Our instability example shows the optimality of the stability estimates previously obtained and that the stability can not be improved by taking different or more measurements.

The framework of our example is as follows. Let $N \geq 2$ and m, positive integers, and β, a positive constant, be fixed. Let $X = X_{m\beta(1/4)}(B'_{N-1}(0,1/2))$ with the Hausdorff distance. To any $\sigma \in X$, we can associate the following four operators.

Let $\Lambda_1(\sigma) : H^{1/2}(\partial\Omega) \mapsto H^{-1/2}(\partial\Omega)$ be given by

$$\langle \Lambda_1(\sigma)\psi, \varphi\rangle = \langle \frac{\partial u}{\partial \nu}|_{\partial\Omega}, \varphi\rangle = \int_\Omega \nabla u \cdot \nabla\tilde{\varphi},$$

where ψ, $\varphi \in H^{1/2}(\partial\Omega)$, u solves (2.16)-(2.18) and $\tilde{\varphi}$ is any $H^1(\Omega\backslash\sigma)$ function whose trace on $\partial\Omega$ coincides with φ.

Let $\mathcal{N}_1(\sigma) : {}_0H^{-1/2}(\partial\Omega) \mapsto {}_0H^{1/2}(\partial\Omega)$ be given by

$$\mathcal{N}_1(\sigma)\eta = u|_{\partial\Omega},$$

where $\eta \in {}_0H^{-1/2}(\partial\Omega)$ and u solves (2.16)-(2.19).

Let $\Lambda_2(\sigma) : H^{1/2}(\partial\Omega) \mapsto H^{-1/2}(\partial\Omega)$ be given by

$$\langle \Lambda_2(\sigma)\psi, \varphi\rangle = \langle \frac{\partial u}{\partial \nu}|_{\partial\Omega}, \varphi\rangle = \int_\Omega \nabla u \cdot \nabla\tilde{\varphi},$$

where $\psi, \varphi \in H^{1/2}(\partial\Omega)$, u solves (2.17)-(2.18) and $\tilde{\varphi}$ is any $H^1_{const}(\Omega,\sigma)$ function whose trace on $\partial\Omega$ coincides with φ.

Let $\mathcal{N}_2(\sigma) : {}_0H^{-1/2}(\partial\Omega) \mapsto {}_0H^{1/2}(\partial\Omega)$ be given by

$$\mathcal{N}_2(\sigma)\eta = u|_{\partial\Omega},$$

where $\eta \in {}_0H^{-1/2}(\partial\Omega)$ and u solves (2.17)-(2.19).

Let us remark that for any $\sigma \in X$, each $\Lambda_i(\sigma)$ and $\mathcal{N}_i(\sigma)$, $i = 1, 2$, is a bounded linear operator between a Hilbert space and its dual, it is self-adjoint and its norm is always assumed to be the canonical one of bounded operators between these two Hilbert spaces. Keeping in mind these notations and this remark, we are able to state our instability result.

Proposition 2.5 *Let us fix integers $N \geq 2$ and $m \geq 1$ and a positive constant β. Let (X, d) be the metric space where $X = X_{m\beta(1/4)}(B'_{N-1}(0, 1/2))$ and d is the Hausdorff distance. Let us fix $T \in \{\Lambda_1, \mathcal{N}_1, \Lambda_2, \mathcal{N}_2\}$. Then there exists a positive ε_1, depending on N, m and β only, so that for any ε, $0 < \varepsilon < \varepsilon_1$, there exist two cracks σ_1, σ_2 belonging to X satisfying*

$$(2.20) \qquad \begin{aligned} & d(\sigma_j, \overline{B'_{N-1}(0, 1/2)}) \leq \varepsilon, \text{ for any } j = 1, 2; \quad d(\sigma_1, \sigma_2) \geq \varepsilon; \\ & \|T(\sigma_1) - T(\sigma_2)\| \leq 2\exp(-\varepsilon^{-(N-1)/(2mN)}). \end{aligned}$$

For the proof of this proposition we refer to Subsection 5.2.

2.3 Inverse cavity problem

The inverse cavity problem can be treated if we substitute, in the previous subsection, the set of cracks inside Ω, $X = X_{m\beta(1/4)}(B'_{N-1}(0, 1/2))$, with the set of *cavities* inside Ω given by $X = Y_{m\beta(1/4)}(S^{N-1}(0, 1/2))$. With almost no modification in the proof, a result completely analogous to the one described in Proposition 2.5 can be obtained. So also the inverse cavity problem shows an exponential instability character.

We recall that, concerning the inverse cavity problem, stability estimates of logarithmic type have been obtained for the two dimensional case in [7] and for the higher dimensional case in [3]. For the planar case, an explicit example developed in [7] shows the exponential instability character of the inverse cavity problem and, consequently, that the stability estimates therein contained are essentially optimal. Our results here confirm this fact and extend it to the higher dimensional case, thus providing the essential optimality of the estimates proved in [3].

2.4 Inverse surface crack problem

Let $\Omega = B_N(0, 1) \backslash \{x \in B_N(0, 1) : \ x_{N-1} \geq 0 \text{ and } x_N = 0\}$, $N \geq 2$, and let $\Gamma_I = \{x \in \overline{B_N(0, 1)} : \ x_{N-1} \geq 0 \text{ and } x_N = 0\}$. Inside Ω, we consider the geodesic distance between two points as the infimum of the lengths of smooth paths contained in Ω connecting the two points. If we consider the boundary of Ω with respect to this distance, we notice that this boundary contains two overlapping copies of Γ_I, one obtained by approaching Γ_I with points x in Ω such that $x_N > 0$ and the other obtained by approaching it with points $x \in \Omega$ so that $x_N < 0$. The set Γ_A is obtained from Γ_I by taking the closure, in the topology of $\partial\Omega$ induced by the geodesic distance defined above, of $\partial\Omega \backslash \Gamma_I$. We remark that Γ_A coincides, from a set point of view, with S^{N-1}, but each point belonging to the intersection of Γ_I and S^{N-1} should be counted with multiplicity two, as for points of Γ_I.

With $H^{1/2}(\Gamma_A)$ we denote the space of traces of $H^1(\Omega)$ functions on Γ_A and with $H^{-1/2}(\Gamma_A)$ we shall denote its dual. On these two spaces, we consider scalar products which are defined exactly as we have done before for regular domains, in (2.3) and (2.5), respectively. We notice that $H^{1/2}(\Gamma_A) \subset L^2(S^{N-1}) \subset$

$H^{-1/2}(\Gamma_A)$. Finally, we notice that the spaces ${}_0H^{1/2}(\Gamma_A)$ and ${}_0H^{-1/2}(\Gamma_A)$ are the orthogonal subspaces, respectively in $H^{1/2}(\Gamma_A)$ and $H^{-1/2}(\Gamma_A)$, to the constant function 1 and are dual one to each other.

We observe that the spaces $H^1_0(\Omega,\Gamma_I)$ and $H^1_{const}(\Omega,\Gamma_I)$ are given by the spaces of $H^1(B_N(0,1))$ functions which are, respectively, identically zero or constant in a weak sense on Γ_I. The spaces of traces on Γ_A of functions belonging to $H^1_0(\Omega,\Gamma_I)$ and $H^1_{const}(\Omega,\Gamma_I)$, respectively, are again denoted by $H^{1/2}_0(\Gamma_A,\Omega)$ and $H^{1/2}_{const}(\Gamma_A,\Omega)$. On these two last spaces, a scalar product is defined in the same fashion as we have done for regular domains in (2.7).

If $N=2$, let $\sigma_0=\{x\in\overline{B_2(0,1)}:\ x_1\geq -1/2 \text{ and } x_2=0\}$. If $N\geq 3$, let $f\in C^\infty_0(B'_{N-2}(0,1/4))$ so that $-1/4\leq f\leq 0$. Let $\sigma_0=\Gamma_I\cup\{y\in B'_{N-1}(0,1):\ f((y_1,\ldots,y_{N-2},0))\leq y_{N-1}\leq 0,\ (y_1,\ldots,y_{N-2},0)\in B'_{N-2}(0,1/4)\}$. By definition if $N=2$, and by a suitable choice of f if $N\geq 3$, we can always assume that $B'_{N-1}(\tilde{x}_0,1/16)$ is contained in σ_0, where $\tilde{x}_0=(0,\ldots,0,-1/8,0)$.

Then we fix a positive integer m and a positive constant β and we define X as the set

$$X=\{\sigma=(\sigma_0\backslash B'_{N-1}(\tilde{x}_0,1/16))\cup\sigma':\ \sigma'\in X_{m\beta(1/4)}(B'_{N-1}(\tilde{x}_0,1/16))\}. \tag{2.21}$$

We remark that each $\sigma\in X$ is a connected closed set inside $\overline{B_N(0,1)}$ so that $\Gamma_I\subset\sigma$ and $\sigma\backslash\Gamma_I\subset B_N(0,4/5)$.

If we assume that $B_N(0,1)$ is occupied by a homogeneous conductor, we can think any $\sigma\in X$ as a *surface crack* inside $B_N(0,1)$. We can distinguish between two different kinds of surface cracks, namely *insulating* and *conducting*.

Let us assume that $\sigma\in X$ is an insulating surface crack and that we prescribe on Γ_A the current density to be equal to $\eta\in{}_0H^{-1/2}(\Gamma_A)$. Then the electrostatic potential u in $B_N(0,1)$ satisfies

$$\begin{cases}\Delta u=0 & \text{in } B_N(0,1)\backslash\sigma,\\ \frac{\partial u}{\partial\nu}=0 & \text{on either sides of } \sigma,\\ \frac{\partial u}{\partial\nu}=\eta & \text{on } \Gamma_A,\\ \int_{\Gamma_A}u=0, & \end{cases} \tag{2.22}$$

where we have also added a normalization condition. We have that u is a weak solution to (2.22) if and only if $u\in H^1(B_N(0,1)\backslash\sigma)$, $\int_{\Gamma_A}u=0$, and

$$\int_{B_N(0,1)\backslash\sigma}\nabla u\cdot\nabla w=\eta(w|_{\Gamma_A}),\quad \text{for any } w\in H^1(B_N(0,1)\backslash\sigma).$$

Clearly such a function u exists and is unique. We have that $u|_{\Gamma_A}$ belongs to ${}_0H^{1/2}(\Gamma_A)$ and that the operator $\mathcal{N}_3(\sigma):{}_0H^{-1/2}(\Gamma_A)\mapsto{}_0H^{1/2}(\Gamma_A)$ so that, for any $\eta\in{}_0H^{-1/2}(\Gamma_A)$, $\mathcal{N}_3(\sigma)\eta=u|_{\Gamma_A}$, u solution to (2.22), is linear, bounded and self-adjoint.

When, otherwise, $\sigma\in X$ is a conducting surface crack in $B_N(0,1)$ and we prescribe the voltage on Γ_A to be $\psi\in H^{1/2}_{const}(\Gamma_A,\Omega)$, then the potential u in $B_N(0,1)$ solves

$$\begin{cases}\Delta u=0 & \text{in } B_N(0,1)\backslash\sigma,\\ u=c(\psi) & \text{on } \sigma,\\ u=\psi & \text{on } \Gamma_A,\end{cases} \tag{2.23}$$

where $c(\psi)$ is a constant so that $\hat{\psi} = \psi - c(\psi) \in H_0^{1/2}(\Gamma_A, \Omega)$. Let $\tilde{\psi}$ be any $H_0^1(B_N(0,1)\backslash\sigma,\sigma)$ function so that $\tilde{\psi}|_{\Gamma_A} = \hat{\psi}$. Then u solves in a weak sense (2.23) if and only if $u - c(\psi) - \tilde{\psi} \in H_0^1(B_N(0,1)\backslash\sigma)$ and

$$\int_{B_N(0,1)\backslash\sigma} \nabla u \cdot \nabla w = 0, \quad \text{for any } w \in H_0^1(B_N(0,1)\backslash\sigma).$$

By standard elliptic equations methods we infer that u, solution to (2.23), exists and it is unique. To such a solution we can associate $\frac{\partial u}{\partial \nu}|_{\Gamma_A} \in (H^{1/2}_{const}(\Gamma_A,\Omega))'$ as follows. For any $\varphi \in H^{1/2}_{const}(\Gamma_A,\Omega)$, let the constant $c(\varphi)$ be so that $\hat{\varphi} = \varphi - c(\varphi) \in H_0^{1/2}(\Gamma_A,\Omega)$, and let $\tilde{\varphi}$ be any $H_0^1(B_N(0,1)\backslash\sigma,\sigma)$ function so that $\tilde{\varphi}|_{\Gamma_A} = \hat{\varphi}$. Then,

$$\langle \frac{\partial u}{\partial \nu}|_{\Gamma_A}, \varphi\rangle = \int_{B_N(0,1)\backslash\sigma} \nabla u \cdot \nabla \tilde{\varphi}.$$

The operator $\Lambda_3(\sigma) : H^{1/2}_{const}(\Gamma_A,\Omega) \mapsto (H^{1/2}_{const}(\Gamma_A,\Omega))'$ so that, for any $\psi \in H^{1/2}_{const}(\Gamma_A,\Omega)$, $\Lambda_3(\sigma)\psi = \frac{\partial u}{\partial \nu}|_{\Gamma_A}$, u solution to (2.23), is linear, bounded and self-adjoint.

The inverse surface crack problem consists of the determination of an unknown surface crack from suitable information on the operator $\mathcal{N}_3$ or Λ_3, respectively. The operators $\mathcal{N}_3$ and Λ_3 correspond to electrostatic boundary measurements. Many papers have treated this problem when a finite number of measurements is performed, that is when either $\mathcal{N}_3(\eta)$ is measured for a finite number of different η or $\Lambda_3(\psi)$ is measured for a finite number of different ψ. We refer to [19] and its bibliography for a detailed description of the problem in the planar case. In [19] uniqueness and stability estimates of logarithmic type are established. In [5] the uniqueness issue for the three-dimensional case is treated. In the next proposition we show that also this inverse problem is exponentially unstable, thus proving the essential optimality of the stability estimates of [19].

Proposition 2.6 *We fix integers $N \geq 2$ and $m \geq 1$ and a positive constant β. Let X be the set of closed sets described in* (2.21) *and let (X,d) be a metric space with the Hausdorff distance. Let σ_0 be defined as before. Let us fix $T \in \{\mathcal{N}_3, \Lambda_3\}$. Then we can find $\varepsilon_1 > 0$, that depends on N, m and β only, so that for any ε, $0 < \varepsilon < \varepsilon_1$, there exist two surface cracks σ_1, σ_2 belonging to X so that*

$$\begin{array}{l} d(\sigma_j,\sigma_0) \leq \varepsilon, \text{ for any } j = 1,2; \quad d(\sigma_1,\sigma_2) \geq \varepsilon; \\ \|T(\sigma_1) - T(\sigma_2)\| \leq 2\exp(-\varepsilon^{-(N-1)/(2mN)}). \end{array} \tag{2.24}$$

We prove this result in Subsection 5.3.

2.5 Inverse boundary material loss problem

Let $\Omega = \{x \in B_N(0,1) : x_N > 0\}$, and let $\Gamma_A = \{x \in \partial B_N(0,1) : x_N \geq 0\} = S_+^{N-1}$ and $\Gamma_I = \{x \in B_N(0,1) : x_N = 0\} = B'_{N-1}(0,1)$. Fixed a positive integer m and a positive constant β, let

$$X = \{\sigma = \Gamma_I \cup \sigma' : \sigma' \in Y_{m\beta(1/4)}(B'_{N-1}(0,1/2))\}. \tag{2.25}$$

Then every $\sigma \in X$ is a closed subset contained in $\overline{\Omega}$ so that $(\sigma\backslash\Gamma_I) \subset B_N(0,4/5)$.

We assume that Ω is the region occupied by a homogeneous conductor and $\sigma \in X$ is a *boundary material loss*, which might be due to a corrosion phenomenon, for instance. We assume that Γ_A is an accessible part of the boundary of our conductor, whereas $\Gamma_\sigma = \partial(\Omega\backslash\sigma)\backslash\Gamma_A$, that is the other part of the boundary where the material loss occurs, is not. Also in this case we distinguish two kinds of boundary material losses, insulating and conducting. In the first case, no current passes through Γ_σ, the part of boundary of $\Omega\backslash\sigma$ which is contained in σ. In the second case, the voltage is constant on σ. More precisely, we have that if σ is an insulating boundary material loss and if we prescribe the current density on Γ_A to be equal to $\eta \in {}_0H^{-1/2}(\Gamma_A)$, then the electrostatic potential u inside $\Omega\backslash\sigma$ is the unique solution to

$$\begin{cases} \Delta u = 0 & \text{in } \Omega\backslash\sigma, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_\sigma, \\ \frac{\partial u}{\partial \nu} = \eta & \text{on } \Gamma_A, \\ \int_{\Gamma_A} u = 0, & \end{cases} \tag{2.26}$$

where the last line is a normalization condition. Otherwise, if σ is conducting, then the electrostatic potential u in Ω is given by

$$\begin{cases} \Delta u = 0 & \text{in } \Omega\backslash\sigma, \\ u = c(\psi) & \text{on } \sigma, \\ u = \psi & \text{on } \Gamma_A, \end{cases} \tag{2.27}$$

where $\psi \in H^{1/2}_{const}(\Gamma_A, \Omega)$ is the prescribed voltage on Γ_A and $c(\psi)$ is a constant so that $\psi - c(\psi)$ belongs to $H^{1/2}_0(\Gamma_A, \Omega)$.

In the insulating case, for every $\sigma \in X$, we define $\mathcal{N}_4(\sigma) : {}_0H^{-1/2}(\Gamma_A) \mapsto {}_0H^{1/2}(\Gamma_A)$ so that for any $\eta \in {}_0H^{-1/2}(\Gamma_A)$, then $\mathcal{N}_4(\sigma)\eta = u|_{\Gamma_A}$, u being the unique solution to (2.26). We have that $\mathcal{N}_4(\sigma)$ is a linear, bounded and self-adjoint operator.

In the conducting case, if $\sigma \in X$, let us define $\Lambda_4(\sigma) : H^{1/2}_{const}(\Gamma_A, \Omega) \mapsto (H^{1/2}_{const}(\Gamma_A, \Omega))'$ as follows. For any ψ, $\varphi \in H^{1/2}_{const}(\Gamma_A, \Omega)$,

$$\langle \Lambda_4(\sigma)\psi, \varphi \rangle = \langle \frac{\partial u}{\partial \nu}|_{\Gamma_A}, \varphi \rangle = \int_{\Omega\backslash\sigma} \nabla u \cdot \nabla \tilde{\varphi},$$

where u solves (2.27) and $\tilde{\varphi}$ is any $H^1_{const}(\Omega, \sigma)$ so that $\tilde{\varphi}|_{\Gamma_A} = \varphi$. Also $\Lambda_4(\sigma)$ is a linear, bounded and self-adjoint operator, for any $\sigma \in X$.

The inverse problem consists of the determination of the shape and the location of an unknown boundary material loss σ from electrostatic measurements performed on the accessible part of the boundary, that is Γ_A. The case of a single electrostatic measurement is particularly interesting and uniqueness and stability estimates have been obtained for this kind of problem, see [19], and its references, for the two-dimensional case and [3] and also [9] for the higher dimensional case. The stability estimates obtained in [19] and [3] are of logarithmic type and they are essentially optimal. In two dimensions, this has been shown through an explicit example provided in [2]. In the next proposition, proven in Subsection 5.3, we confirm that logarithmic stability is essentially optimal in any dimension, no matter how many and which measurements we perform.

Proposition 2.7 *Let $N \geq 2$ and $m \geq 1$ be integers and β be a positive constant. Let X be defined as in (2.25), endowed with the Hausdorff distance d. Let us fix $T \in \{\mathcal{N}_4, \Lambda_4\}$. Then there exists a constant $\varepsilon_1 > 0$, that depends on N, m and β only, so that for any ε, $0 < \varepsilon < \varepsilon_1$, there exist two boundary material losses σ_1, σ_2 belonging to X so that*

$$\begin{aligned} &d(\sigma_j, \overline{B'_{N-1}(0,1)}) \leq \varepsilon, \text{ for any } j = 1,2; \quad d(\sigma_1, \sigma_2) \geq \varepsilon; \\ &\|T(\sigma_1) - T(\sigma_2)\| \leq 2\exp(-\varepsilon^{-(N-1)/(2mN)}). \end{aligned} \tag{2.28}$$

2.6 Inverse scattering problem

We turn our attention to inverse scattering problems, in particular to the determination of *obstacles* inside a medium by acoustic far-field measurements. For a detailed description of this kind of inverse problems we refer to [10].

Let us fix integers $N \in \{2,3\}$ and $m \geq 1$ and two positive constants β and a.

Let $X = Y_{m\beta(1/2)}(S^{N-1}(0,1))$. We assume that $D \in X$ is an obstacle in an otherwise homogeneous medium.

The incident field is determined by a time-harmonic acoustic plane wave and is given by $u^i(x;\omega;a) = \mathrm{e}^{\mathrm{i}\sqrt{a}x\cdot\omega}$, where $x \in \mathbb{R}^N$, $\sqrt{a}$ is the wave number, and $\omega \in S^{N-1}$ is the direction of propagation. The direct scattering problem consists of finding the total field $u(x;\omega;a)$, $x \in \mathbb{R}^N \backslash D$, which is the sum of the incident field $u^i(x;\omega;a)$ and of the scattered field $u^s(x;\omega;a)$, which is due to the presence of the obstacle D. The total field u satisfies

$$\begin{cases} \Delta u + au = 0 & \text{in } \mathbb{R}^N \backslash D, \\ u(x;\omega;a) = \mathrm{e}^{\mathrm{i}\sqrt{a}x\cdot\omega} + u^s(x;\omega;a) & \text{for any } x \in \mathbb{R}^N \backslash D, \end{cases} \tag{2.29}$$

with a boundary condition on ∂D which depends on the nature of the obstacle. Namely, if the obstacle is *sound-soft*, then

$$u = 0 \quad \text{on } \partial D, \tag{2.30}$$

if the obstacle is *sound-hard*, then

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial D. \tag{2.31}$$

Furthermore, the scattered field satisfies the so-called Sommerfeld radiation condition

$$\lim_{r \to \infty} r^{(N-1)/2} \left(\frac{\partial u^s}{\partial r} - \mathrm{i}\sqrt{a} u^s \right) = 0, \tag{2.32}$$

where $r = \|x\|$ and the limit holds uniformly for all directions $\hat{x} = x/\|x\| \in S^{N-1}$. The Sommerfeld radiation condition characterizes outgoing waves and implies that the asymptotic behaviour of the scattered field is given by

$$u^s(x;\omega;a) = \frac{\mathrm{e}^{\mathrm{i}\sqrt{a}\|x\|}}{\|x\|^{(N-1)/2}} \left\{ u^s_\infty(\hat{x};\omega;a) + O\left(\frac{1}{\|x\|}\right) \right\}, \tag{2.33}$$

as $\|x\|$ goes to ∞, uniformly in all directions $\hat{x} = x/\|x\| \in S^{N-1}$. The function u^s_∞ is called the *far-field pattern* related to the solution to (2.29)-(2.30)-(2.32), or to (2.29)-(2.31)-(2.32) respectively, for the direction of propagation ω and the wave number $\sqrt{a}$.

Therefore, if $D \in X$ is sound-soft, then we denote with $\mathcal{A}_s(D) : S^{N-1} \times S^{N-1} \times (0,\infty) \mapsto \mathbb{C}$ its *far-field map*, that is, for any $\hat{x},\ \omega \in S^{N-1}$ and any $a > 0$,

$$\mathcal{A}_s(D)(\hat{x},\omega,a) = u^s_\infty(\hat{x};\omega;a), \tag{2.34}$$

where u^s_∞ is the far-field pattern related to the solution to (2.29)-(2.30)-(2.32).

In an analogous way, assuming $D \in X$ to be sound-hard, we can associate to D its far-field map, given by $\mathcal{A}_h(D) : S^{N-1} \times S^{N-1} \times (0,\infty) \mapsto \mathbb{C}$, so that, for any $(\hat{x},\omega,a) \in S^{N-1} \times S^{N-1} \times (0,\infty)$,

$$\mathcal{A}_h(D)(\hat{x},\omega,a) = u^s_\infty(\hat{x};\omega;a), \tag{2.35}$$

where u^s_∞ is now the far-field pattern related to the solution to (2.29)-(2.31)-(2.32).

The inverse scattering problem consists of the determination of an unknown obstacle D from some information about its far-field map. More precisely, we assume that we a priori know whether the unknown obstacle is sound-soft or sound-hard. Then, from a, usually partial, knowledge of $\mathcal{A}_s(D)$ or $\mathcal{A}_h(D)$, respectively, we try to determine the obstacle D. We remark that information about the far-field map can be collected by performing suitable far-field acoustic measurements.

In the next proposition we show that also this inverse problem is exponentially unstable. We recall that stability estimates for the determination of a sound-soft obstacle have been obtained by V. Isakov, see [14, 15].

Proposition 2.8 *Let $N \in \{2,3\}$. Let m and j be positive integers and β be a positive constant. Let $\underline{a}$ and $\overline{a}$ be two constants so that $0 < \underline{a} \leq \overline{a}$. We denote*

$$I_N = \begin{cases} [\underline{a},\overline{a}] & \text{if } N = 2; \\ (0,\overline{a}] & \text{if } N = 3; \end{cases}$$

and we fix any j real numbers $a_1,\ldots,a_j$ so that $a_i \in I_N$ for any i, $i = 1,\ldots,j$. Let (X,d) be the metric space with $X = Y_{m\beta(1/2)}(S^{N-1}(0,1))$ and d the Hausdorff distance. Then we can find a constant $\varepsilon_1 > 0$, that depends on N, m, j, β and I_N only, so that for any ε, $0 < \varepsilon < \varepsilon_1$, there exist obstacles D_1, D_2, D_3 and D_4 all belonging to X such that

$$\begin{aligned} &d(D_j,\overline{B_N(0,1)}) \leq \varepsilon, \quad \text{for any } j = 1,\ldots,4; \\ &d(D_1,D_2) \geq \varepsilon; \quad d(D_3,D_4) \geq \varepsilon; \end{aligned} \tag{2.36}$$

and, for the sound-soft case,

$$\sup_{a \in \{a_1,\ldots,a_j\}} \|(\mathcal{A}_s(D_1) - \mathcal{A}_s(D_2))(\cdot,\cdot,a)\|_{L^2(S^{N-1}\times S^{N-1})} \leq 2\exp(-\varepsilon^{-\frac{N-1}{2mN}}); \tag{2.37}$$

and, for the sound-hard case,

$$\sup_{a \in \{a_1,\ldots,a_j\}} \|(\mathcal{A}_h(D_3) - \mathcal{A}_h(D_4))(\cdot,\cdot,a)\|_{L^2(S^{N-1}\times S^{N-1})} \leq 2\exp(-\varepsilon^{-\frac{N-1}{2mN}}). \tag{2.38}$$

We observe that the result is slightly different depending whether $N = 2$ or $N = 3$. If $N = 2$, the real numbers a_i, $i = 1, \dots, j$, satisfy $a_i \geq \underline{a}$ and the constant ε_1 depends on $\underline{a}$ as well. If $N = 3$, we do not need a lower bound for the numbers a_i, apart from the fact that they are positive, and the constant ε_1 does not depend on $\underline{a}$. We shall point out during the proof, which will be developed in Subsection 5.4, where the hypothesis $a_i \geq \underline{a}$ becomes necessary.

3 The abstract theorem

Let (X, d) be a metric space and let H be a separable Hilbert space with scalar product $(\cdot,\cdot)$. As usual we denote with H' its dual and for any $v' \in H'$ and any $v \in H$ we denote by $\langle v', v\rangle$ the duality pairing between v' and v. With $\mathcal{L}(H, H')$ we denote the space of bounded linear operators between H and H' with the usual operators norm. We shall also fix $\gamma : H\backslash\{0\} \mapsto [0, +\infty]$ such that

$$\gamma(\lambda v) = \gamma(v) \quad \text{for any } v \in H\backslash\{0\} \text{ and any } \lambda \in \mathbb{R}\backslash\{0\}. \tag{3.1}$$

Let us remark that the function γ may attain both the values 0 and $+\infty$ and can be thought of as a suitable kind of Rayleigh quotient.

Let F be a function from X to $\mathcal{L}(H, H')$, that is, for any $x \in X$, $F(x)$ will denote a linear and bounded operator between H and H'. We also fix a reference operator $F_0 \in \mathcal{L}(H, H')$ and a reference point x_0 in X. We wish to point out that no connection is required between x_0 and F_0, in particular F_0 might be different from $F(x_0)$. For any $\varepsilon > 0$, let $X_\varepsilon = \{x \in X : \ d(x, x_0) \leq \varepsilon\}$.

Recalling the notations introduced in Definition 2.1, we can formulate the following exponential instability result related to the map F.

Theorem 3.1 *Let us assume that the following conditions are satisfied.*

i) There exist positive constants ε_0, C_1 and α_1 such that for any ε, $0 < \varepsilon < \varepsilon_0$, we can find an ε-discrete set Z_ε contained in X_ε with at least $\exp(C_1\varepsilon^{-\alpha_1})$ elements.

ii) There exist three positive constants p, C_2 and α_2 and an orthonormal basis in H, $\{v_k\}_{k=1}^{+\infty}$, such that the following conditions hold.

For any $k \in \mathbb{N}$, we have that $\gamma(v_k) < \infty$, and for any $n \in \mathbb{N}$,

$$\#\{k \in \mathbb{N} : \ \gamma(v_k) \leq n\} \leq C_2(1 + n)^p \tag{3.2}$$

where # denotes the number of elements.

For any $x \in X$ and any $(k, l) \in \mathbb{N} \times \mathbb{N}$ we have

$$|\langle (F(x) - F_0)v_k, v_l\rangle| \leq C_2 \exp(-\alpha_2 \max\{\gamma(v_k), \gamma(v_l)\}). \tag{3.3}$$

Then there exists a positive constant ε_1, depending on ε_0, C_1, C_2, α_1, α_2 and p only, so that for every ε, $0 < \varepsilon < \varepsilon_1$, we can find x_1 and x_2 satisfying

$$\begin{aligned} &x_1, x_2 \in X_\varepsilon; \quad d(x_1, x_2) \geq \varepsilon; \\ &\|F(x_1) - F(x_2)\|_{\mathcal{L}(H,H')} \leq 2\exp(-\varepsilon^{-\alpha_1/2(p+1)}). \end{aligned} \tag{3.4}$$

A crucial point in the proof of Theorem 3.1 is constituted by the following lemma in which we construct δ-nets in the image through F of X with a control in terms of δ of the number of their elements.

Lemma 3.2 *Under assumption ii) of Theorem* 3.1*, there exists a positive constant* C_3*, depending on* p*,* C_2 *and* α_2 *only, such that for every* δ*,* $0 < \delta < 1/\mathrm{e}$*, we can find a* δ*-net* Y_δ *for* $F(X)$ *with at most* $\exp(C_3(-\log\delta)^{2p+1})$ *elements.*

Proof. Let G be an element of $\mathcal{L}(H, H')$ and let $\{v_k\}_{k=1}^{+\infty}$ be the orthonormal basis in H defined in assumption *ii)* of Theorem 3.1. For any pair $(k, l) \in \mathbb{N}\times\mathbb{N}$ we associate to G the real number $a_{k,l} = \langle Gv_k, v_l\rangle$. Let $\|G\|_Y = \sup_{k,l} |a_{k,l}|(2 + \max\{\gamma(v_k), \gamma(v_l)\})^{p+1}$ and let Y be the normed space

$$Y = \{G \in \mathcal{L}(H, H') : \ \|G\|_Y < \infty\}$$

with norm $\|\cdot\|_Y$. First, we notice that, for any $x \in X$, $F(x) - F_0$ is contained in Y. This is an immediate consequence of (3.3); in fact $|\langle(F(x) - F_0)v_k, v_l\rangle| \leq C_2 \exp(-\alpha_2 \max\{\gamma(v_k), \gamma(v_l)\})$ and hence

$$\|F(x) - F_0\|_Y \leq \sup_n C_2 \exp(-\alpha_2(n - 1))(2 + n)^{p+1} < \infty.$$

Second, if we set $C_4 = C_2 \left(\sum_n (1 + n)^{-2}\right)^{1/2}$, for any $G \in Y$ we have

$$\|G\|_{\mathcal{L}(H,H')} \leq C_4\|G\|_Y. \tag{3.5}$$

This follows from the following computation

$$\begin{aligned}\|G\|_{\mathcal{L}(H,H')} &\leq \left(\sum_{k,l} |a_{k,l}|^2\right)^{1/2} \leq \\ &\leq \left(\sum_{k,l} |a_{k,l}|^2 \tfrac{(2+\max\{\gamma(v_k),\gamma(v_l)\})^{2p+2}}{(2+\max\{\gamma(v_k),\gamma(v_l)\})^{2p+2}}\right)^{1/2} \leq \\ &\leq \left(\sum_{k,l} \tfrac{1}{(2+\max\{\gamma(v_k),\gamma(v_l)\})^{2p+2}}\right)^{1/2} \|G\|_Y.\end{aligned}$$

Let us show that C_4 is a bound for $\left(\sum_{k,l} \frac{1}{(2+\max\{\gamma(v_k),\gamma(v_l)\})^{2p+2}}\right)^{1/2}$. For any positive integer n, the number of pairs (k, l) so that $n-1 \leq \max\{\gamma(v_k), \gamma(v_l)\} < n$ is bounded by $C_2^2(1 + n)^{2p}$, therefore

$$\sum_{k,l} \frac{1}{(2 + \max\{\gamma(v_k), \gamma(v_l)\})^{2p+2}} \leq \sum_n C_2^2(1 + n)^{2p} \frac{1}{(2 + (n - 1))^{2p+2}} = C_4^2.$$

Let us now fix δ, $0 < \delta < 1/\mathrm{e}$. Let $\tilde{n}$ be the smallest positive integer so that for any real number $t \geq \tilde{n}$ it holds $C_2 \exp(-\alpha_2(t - 1))(2 + t)^{p+1} \leq \delta/(2C_4)$. There exists a positive constant C_5, depending on p, C_2 and α_2 only, such that $\tilde{n} \leq C_5 \log \delta^{-1}$.

Let $\delta' = \frac{(2+\tilde{n})^{-(p+1)}}{2C_4}\delta$ and let $\Psi_\delta = \delta'\mathbb{Z} \cap [-C_2, C_2]$. We remark that Ψ_δ is a finite subset of $\mathbb{R}$ and we have that $\#\Psi_\delta \leq C_6/\delta'$, where C_6 can be chosen as $2C_2 + 1$.

Let us define the following subset of $\mathcal{L}(H, H')$. Let $\tilde{Y}_\delta = \{G \in \mathcal{L}(H, H') : a_{k,l} \in \Psi_\delta$ if $\max\{\gamma(v_k), \gamma(v_l)\} \leq \tilde{n}$ and $a_{k,l} = 0$ otherwise$\}$.

Let us count the number of elements of $\tilde{Y}_\delta$. If we set

$$s = \#\{(k,l) : \max\{\gamma(v_k), \gamma(v_l)\} \leq \tilde{n}\}$$

then we have that $\#\tilde{Y}_\delta = (\#\Psi_\delta)^s$ and hence

$$\begin{aligned} \#\tilde{Y}_\delta &\leq (2C_6C_4(2+\tilde{n})^{p+1}\delta^{-1})^{C_2^2(1+\tilde{n})^{2p}} \leq \\ &\leq (2C_6C_4(2+C_5\log\delta^{-1})^{p+1}\delta^{-1})^{C_2^2(1+C_5\log\delta^{-1})^{2p}} \end{aligned}$$

and then by straightforward computations we can find a positive constant C_3, depending on p, C_2 and α_2 only, so that $\#\tilde{Y}_\delta \leq \exp(C_3(-\log\delta)^{2p+1})$.

Then we need to show that for every $x \in X$ there exists $G \in \tilde{Y}_\delta$ so that $\|F(x) - F_0 - G\|_{\mathcal{L}(H,H')} \leq \delta/2$.

We fix $x \in X$ and we set $b_{k,l} = \langle (F(x) - F_0)v_k, v_l\rangle$; we recall that $F(X) - F_0 \subset Y$ and, by (3.5), it is enough to determine $G \in \tilde{Y}_\delta$ so that $\|F(x) - F_0 - G\|_Y \leq \delta/(2C_4)$. We observe that, by (3.3), $b_{k,l} \in [-C_2, C_2]$ for every (k,l). We construct such a G as follows. We set $a_{k,l} = \langle Gv_k, v_l\rangle$. For any (k,l) so that $\max\{\gamma(v_k), \gamma(v_l)\} \leq \tilde{n}$, we prescribe $a_{k,l}$ to be the element of Ψ_δ that is closest to $b_{k,l}$. If, otherwise, (k,l) is so that $\max\{\gamma(v_k), \gamma(v_l)\} > \tilde{n}$, then we set $a_{k,l} = 0$. We have that G belongs to $\tilde{Y}_\delta$ by construction and we can evaluate $\|F(x) - F_0 - G\|_Y$ as follows. For any (k,l) so that $\max\{\gamma(v_k), \gamma(v_l)\} \leq \tilde{n}$, $|a_{k,l} - b_{k,l}| \leq \delta'$, that is $|a_{k,l} - b_{k,l}|(2 + \max\{\gamma(v_k), \gamma(v_l)\})^{p+1} \leq (2+\tilde{n})^{p+1}\frac{(2+\tilde{n})^{-(p+1)}}{2C_4}\delta \leq \delta/(2C_4)$. If (k,l) is such that $t = \max\{\gamma(v_k), \gamma(v_l)\} > \tilde{n}$, then $a_{k,l} = 0$ and $|b_{k,l}|(2+t)^{p+1} \leq C_2\exp(-\alpha_2 t)(2+t)^{p+1} \leq \delta/(2C_4)$ by the definition of $\tilde{n}$.

Having established this property, it is easy to find a subset Y_δ of $F(X)$ with the same number of elements as $\tilde{Y}_\delta$ which is a δ-net for $F(X)$ and hence the proof is concluded. □

Proof of Theorem 3.1. The proof is obtained by combining assumption *i)* of Theorem 3.1 with Lemma 3.2 as follows. Let ε satisfy $0 < \varepsilon < \varepsilon_0$ and δ satisfy $0 < \delta < 1/\mathrm{e}$. Let $Z_\varepsilon \subset X_\varepsilon$ be, as in assumption *i)* of Theorem 3.1, an ε-discrete set with at least $\exp(C_1\varepsilon^{-\alpha_1})$ elements. The same procedure employed in Lemma 3.2 allows us to find $Y_\delta \subset F(X_\varepsilon)$ which is a δ-net for $F(X_\varepsilon)$ with at most $\exp(C_3(-\log\delta)^{2p+1})$ elements. If $\#Z_\varepsilon > \#Y_\delta$, then there exist x_1 and $x_2 \in X_\varepsilon$ so that $d(x_1,x_2) \geq \varepsilon$ and $\|F(x_1) - F(x_2)\|_{\mathcal{L}(H,H')} \leq 2\delta$.

In order to have that $\#Z_\varepsilon > \#Y_\delta$, it suffices to have that $\exp(C_1\varepsilon^{-\alpha_1}) > \exp(C_3(-\log\delta)^{2p+1})$. Let us define $\delta(\varepsilon) = \exp(-\varepsilon^{-\alpha_1/2(p+1)})$. Then there exists a constant ε_1, $0 < \varepsilon_1 \leq \varepsilon_0$, depending on ε_0, C_1, C_2, α_1, α_2 and p only, such that for every ε, $0 < \varepsilon < \varepsilon_1$, we have $\delta(\varepsilon) < 1/\mathrm{e}$ and $\exp(C_1\varepsilon^{-\alpha_1}) > \exp(C_3(-\log\delta(\varepsilon))^{2p+1})$ and so the result follows. □

Remark 3.3 We wish to remark that it is easy to see that the order of instability can be improved up to $\exp(-\varepsilon^{-\alpha_1/(2p+1+\alpha_3)})$, for any $\alpha_3 > 0$. However, in this case, the constant ε_1 depends on α_3, too. For the sake of simplicity, we have stated the theorem when α_3 is chosen to be equal to 1.

4 Stekloff eigenvalue problems

In this section we collect some results which will be repeatedly used later, when we shall apply the abstract theorem to find instability examples for inverse problems. Most of the results described in this section are obtained by standard methods, thus, for the sake of brevity, we do not enter into any detail and we limit ourselves to fix the notation and to state the results which will be needed later, referring to the literature when necessary.

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain and let Γ_A and Γ_I be two internally disjoint subsets of $\partial\Omega$, so that $\Gamma_A \cup \Gamma_I = \partial\Omega$. About the regularity and the properties of Ω, Γ_A and Γ_I, we shall consider the same assumptions used at the beginning of Section 2, at page 4.

The following eigenvalue problems of Stekloff type will be discussed; first

$$\text{(EP1)} \qquad \begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda u & \text{on } \Gamma_A, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_I, \end{cases}$$

and then, assuming Γ_I not empty,

$$\text{(EP2)} \qquad \begin{cases} \Delta v = 0 & \text{in } \Omega, \\ \frac{\partial v}{\partial \nu} = \mu v & \text{on } \Gamma_A, \\ v = 0 & \text{on } \Gamma_I. \end{cases}$$

We state the following propositions concerning the eigenvalues and eigenfunctions of (EP1) and (EP2) respectively.

Proposition 4.1 *Under the assumptions on Ω, Γ_A and Γ_I previously made, we have that the eigenvalues of* (EP1)*, counted with their multiplicity, are given by an increasing sequence*

$$0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_k \leq \ldots$$

so that $\lim_{k\to\infty} \lambda_k = \infty$. For any $n \in \mathbb{N}$, we set $N_1(n) = \#\{k \in \mathbb{N} : \ \lambda_k \leq n\}$. Then the asymptotic behaviour of the eigenvalues is as follows. There exists a constant C_1 depending on Ω, Γ_A and Γ_I only so that

$$N_1(n) \leq C_1 n^{N-1}, \quad \text{for any } n \in \mathbb{N}. \tag{4.1}$$

Moreover, there exists a corresponding sequence of eigenfunctions, $\{u_k\}_{k\in\mathbb{N}}$, that is $u_k \in H^1(\Omega)\backslash\{0\}$ and the couple (λ_k, u_k) solves (EP1) *for any $k \in \mathbb{N}$, so that the following three conditions holds*

$$\{u_k|_{\Gamma_A}\}_{k\in\mathbb{N}} \text{ is an orthonormal basis of } L^2(\Gamma_A);$$

$$\left\{\frac{u_k}{\sqrt{1+\lambda_k}}|_{\Gamma_A}\right\}_{k\in\mathbb{N}} \text{ is an orthonormal basis of } H^{1/2}(\Gamma_A);$$

$$\{1|_{\Gamma_A}\} \cup \{\sqrt{\lambda_k} u_k|_{\Gamma_A}\}_{k\geq 2} \text{ is an orthonormal basis of } H^{-1/2}(\Gamma_A);$$

where we have considered the spaces $H^{1/2}(\Gamma_A)$ and $H^{-1/2}(\Gamma_A)$ with the scalar products defined in (2.3) *and* (2.5) *respectively. We remark that u_1 is a constant function not identically equal to zero.*

Proposition 4.2 *Under the assumptions on Ω, Γ_A and Γ_I previously made, and assuming that Γ_I is not empty, then the eigenvalues of* (EP2)*, counted with their multiplicity, constitute an increasing sequence*

$$0 < \mu_1 \leq \mu_2 \leq \ldots \leq \mu_k \leq \ldots$$

so that $\lim_{k\to\infty}\mu_k=\infty$. *For any* $n\in\mathbb{N}$, *we set as before* $N_2(n)=\#\{k\in\mathbb{N}:\mu_k\leq n\}$. *Then the eigenvalues satisfy the following asymptotic behaviour. There exists a constant* C_2 *depending on* Ω, Γ_A *and* Γ_I *only so that*

$$N_2(n)\leq C_2 n^{N-1},\quad \text{for any } n\in\mathbb{N}. \tag{4.2}$$

Furthermore, we can find a sequence $\{v_k\}_{k\in\mathbb{N}}$ *of corresponding eigenfunctions, that is* $v_k\in H^1(\Omega)\backslash\{0\}$ *and the couple* (μ_k,v_k) *is a solution to* (EP2) *for any* $k\in\mathbb{N}$, *so that*

$$\{v_k|_{\Gamma_A}\}_{k\in\mathbb{N}} \text{ is an orthonormal system of } L^2(\Gamma_A);$$

$$\{1|_{\Gamma_A}\}\cup\left\{\frac{v_k}{\sqrt{\mu_k}}|_{\Gamma_A}\right\}_{k\in\mathbb{N}} \text{ is an orthonormal basis of } H^{1/2}_{const}(\Gamma_A,\Omega);$$

where we have considered the space $H^{1/2}_{const}(\Gamma_A,\Omega)$ *with the scalar product defined in* (2.7).

Beyond the asymptotic behaviour of the eigenvalues, we are interested in the asymptotic behaviour of the eigenfunctions, in particular in a kind of exponential decay, in terms of the eigenvalues, of the eigenfunctions away from Γ_A. In the next examples, we present some particular cases in which such kind of decay holds.

Example 4.3 Let $\Omega=B_N(0,1)$ and $\partial\Omega=S^{N-1}$, and let $\Gamma_A=\partial\Omega$ and $\Gamma_I=\emptyset$. In this case the problem (EP1) is a classical Stekloff eigenvalue problem and it is well-known that the orthonormal basis of $L^2(S^{N-1})$ constituted by the traces of eigenfunctions, as described in Proposition 4.1, coincides with

$$\{f_{jp}:\ j\geq 0 \text{ and } 1\leq p\leq p_j\} \tag{4.3}$$

where each f_{jp} is a *spherical harmonic* of degree j, j being a nonnegative integer. We have that the function

$$u_{jp}(x)=\|x\|^j f_{jp}(x/\|x\|) \tag{4.4}$$

is harmonic in $\mathbb{R}^N$ and solves the eigenvalue problem (EP1) with eigenvalue $\lambda=j$. So, the sequence $\{u_{jp}:\ j\geq 0 \text{ and } 1\leq p\leq p_j\}$ coincides with the sequence of eigenfunctions we have described in Proposition 4.1. The integers p_j are the dimensions of the spaces of spherical harmonics of degree j and we have that, see for instance [18, page 4],

$$p_j=\begin{cases}1 & \text{if } j=0,\\ \frac{(2j+N-2)(j+N-3)!}{j!(N-2)!} & \text{if } j\geq 1,\end{cases}$$

so that

$$p_j\leq 2(j+1)^{N-2},\quad j\geq 0,$$

and

$$N_1(n)\leq\sum_{j=0}^{n}p_j\leq\sum_{j=0}^{n}2(j+1)^{N-2}\leq 2(n+1)^{N-1},\quad \text{for any } n\in\mathbb{N}.$$

Furthermore, for any r_0, $0<r_0<1$, there exist two positive constants, $C(r_0,N)$ and $\alpha(r_0)$, so that for any u_{jp} as in (4.4) it holds

$$\|u_{jp}\|_{H^1(B_N(0,r_0))}\leq C(r_0,N)\exp(-\alpha(r_0)j). \tag{4.5}$$

Example 4.4 Let $\Omega = \{x \in B_N(0,1) : x_N > 0\}$, and let $\Gamma_A = \overline{\{x \in \partial B_N(0,1) : x_N \geq 0\}} = \overline{S^{N-1}_+}$ and $\Gamma_I = \{x \in \overline{B_N(0,1)} : x_N = 0\} = \overline{B'_{N-1}(0,1)}$. First of all, we notice that the hypoteses of Proposition 4.1 and Proposition 4.2 are satisfied, so the conclusions of Proposition 4.1 and of Proposition 4.2 hold for the eigenvalues and eigenfunctions related to problem (EP1) and problem (EP2) with these data, respectively.

The following exponential decay property can be obtained, as well.

We have that if $u \in H^1(\Omega)\backslash\{0\}$ solves (EP1) for a constant λ, then, by a reflection argument, it follows that there exist j, a nonnegative integer, and f, a spherical harmonic function on S^{N-1} of degree j, so that $u(x) = \|x\|^j f(x/\|x\|)$ for any $x \in \Omega$ and $\lambda = j$. Thus, if we assume that $\|f\|_{L^2(\Gamma_A)} = 1$, we can conclude that for any r_0, $0 < r_0 < 1$,

$$\|u\|_{H^1(B_N(0,r_0)\cap\Omega)} \leq C(r_0,N)\exp(-\alpha(r_0)\lambda), \tag{4.6}$$

where the constants $C(r_0,N)$ and $\alpha(r_0)$ coincide with the ones obtained in Example 4.3.

Again by a reflection argument, we have that if $v \in H^1(\Omega)\backslash\{0\}$ and a constant μ solve (EP2) then there exist j, a positive integer, and f, a spherical harmonic function on S^{N-1} of degree j, so that $v(x) = \|x\|^j f(x/\|x\|)$ for any $x \in \Omega$ and $\mu = j$. Thus, if we assume as before that $\|f\|_{L^2(\Gamma_A)} = 1$, we immediately infer that for any r_0, $0 < r_0 < 1$,

$$\|v\|_{H^1(B_N(0,r_0)\cap\Omega)} \leq C(r_0,N)\exp(-\alpha(r_0)\mu), \tag{4.7}$$

with the same constants $C(r_0,N)$ and $\alpha(r_0)$ as before.

Example 4.5 Let $\Omega = B_N(0,1)\backslash\{x \in B_N(0,1) : x_{N-1} \geq 0 \text{ and } x_N = 0\}$, and let $\Gamma_I = \{x \in \overline{B_N(0,1)} : x_{N-1} \geq 0 \text{ and } x_N = 0\}$, as in Subsection 2.4. We set Γ_A as in Subsection 2.4, as well. Also the notations concerning Sobolev spaces on Γ_A are the ones introduced in Subsection 2.4.

The eigenvalue problem (EP1) with these data can be rewritten as

$$\text{(EP1')} \qquad \begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda u & \text{on } \Gamma_A, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on (either sides of) } \Gamma_I, \end{cases}$$

that is, $u \in H^1(\Omega)$ solves (EP1') if

$$\int_\Omega \nabla u \cdot \nabla w = \int_{\Gamma_A} \lambda u w, \quad \text{for any } w \in H^1(\Omega).$$

Then, all the conclusions of Proposition 4.1 still hold true for the eigenvalue problem (EP1'), also with the possibility to replace the space $L^2(\Gamma_A)$ with the space $L^2(S^{N-1})$.

The exponential decay of the eigenfunctions is still valid. By separation of variables, we have that if $u \in H^1(\Omega)\backslash\{0\}$ solves (EP1') with a constant λ, then there exists a function $g \in L^2(S^{N-1})$ so that $u(x) = \|x\|^\lambda g(x/\|x\|)$ for any $x \in \Omega$. Assuming that $\|g\|_{L^2(S^{N-1})} = 1$, we obtain that for any r_0, $0 < r_0 < 1$,

$$\|u\|_{H^1(B_N(0,r_0)\cap\Omega)} \leq C_1(r_0,N)\exp(-\alpha(r_0)\lambda), \tag{4.8}$$

where $C_1(r_0, N)$ is a positive constant not depending on λ and $\alpha(r_0)$ coincides with the one defined in Example 4.3.

For what concerns the eigenvalue problem (EP2) with these data, that is,

$$\text{(EP2')} \qquad \begin{cases} \Delta v = 0 & \text{in } \Omega, \\ \frac{\partial v}{\partial \nu} = \mu v & \text{on } \Gamma_A, \\ v = 0 & \text{on } \Gamma_I, \end{cases}$$

we have that v solves (EP2'), in a weak sense, for a constant μ, if $v \in H_0^1(\Omega, \Gamma_I)$ and

$$\int_\Omega \nabla v \cdot \nabla w = \int_{\Gamma_A} \mu v w, \quad \text{for any } w \in H_0^1(\Omega, \Gamma_I).$$

Then, all the results of Proposition 4.2 are still valid for the eigenvalue problem (EP2'), and we can again replace the space $L^2(\Gamma_A)$ with the space $L^2(S^{N-1})$.

If $v \in H_0^1(\Omega, \Gamma_I)\backslash\{0\}$ and μ are a solution to (EP2'), then, by separation of variables, we can find a function $g \in L^2(S^{N-1})$ so that $v(x) = \|x\|^\mu g(x/\|x\|)$ for any $x \in \Omega$. If we further suppose $\|g\|_{L^2(S^{N-1})} = 1$, we have that for any r_0, $0 < r_0 < 1$,

$$\text{(4.9)} \qquad \|v\|_{H^1(B_N(0,r_0)\cap\Omega)} \leq C_2(r_0, N) \exp(-\alpha(r_0)\mu),$$

where $C_2(r_0, N)$ is a positive constant not depending on μ and $\alpha(r_0)$ is the same as before.

5 Proofs of the main results

In this section we apply the abstract theorem to the inverse problems described in Section 2 and we conclude the proofs of our instability results.

5.1 Inverse inclusion problem

We treat the problem related to the Dirichlet-to-Neumann map, the one related to the Neumann-to-Dirichlet map and the one related to the experimental measurements separately.

Dirichlet-to-Neumann case

Proof of Proposition 2.3 (Dirichlet-to-Neumann case). We apply Theorem 3.1 to prove Proposition 2.3 for the operator Λ. Let us show how the abstract theorem is used in this situation.

We fix $x_0 \in X$ to be equal to $\overline{B_N(0, 1/2)}$ and we notice that Proposition 2.2 implies that X satisfies assumption *i)* of Theorem 3.1, with constants ε_0 and C_1 which depend on N, m and β only, and with the constant $\alpha_1 = (N-1)/m$. Furthermore, about X, we remark the following property. We have that every $D \in X$ is compactly contained in $B_N(0, 4/5)$.

Concerning assumption *ii)* of Theorem 3.1, we observe that for every $D \in X$, the operator $\Lambda(D)$ is a bounded and linear operator between $H^{1/2}(\partial\Omega)$ and its dual. We fix $H = H^{1/2}(\partial\Omega)$ and $F : X \mapsto \mathcal{L}(H, H')$ as $F(D) = \Lambda(D)$ for any $D \in X$. With F_0 we denote the Dirichlet-to-Neumann map associated to the problem (2.8)-(2.9) when $D = \emptyset$, that is when no inclusion is present, the conductor is therefore homogeneous and its conductivity is identically equal to 1 in Ω. Concerning the function γ our choice is the following. For any $\psi \in H^{1/2}(\partial\Omega)\backslash\{0\}$, let

$$\gamma(\psi) = \frac{\|\psi\|^2_{H^{1/2}(\partial\Omega)}}{\|\psi\|^2_{L^2(\partial\Omega)}}. \tag{5.1}$$

Then it remains to choose an orthonormal basis of H, $\{v_k\}_{k\in\mathbb{N}}$, so that $\gamma(v_k)$ is finite for any k and (3.2) and (3.3) are satisfied. Recalling Example 4.3, in particular (4.3), we consider the set

$$\left\{\frac{f_{jp}}{\sqrt{1+j}} :\ j \geq 0 \text{ and } 1 \leq p \leq p_j\right\} \tag{5.2}$$

with the natural order. This set, by Proposition 4.1, is an orthonormal basis of H and it is the one we choose. We also recall that f_{jp} is a spherical harmonic of degree j so that $\|f_{jp}\|_{L^2(\partial\Omega)} = 1$, hence $\gamma(f_{jp}/\sqrt{1+j}) = 1+j$, for any j and p. Fixed $n \in \mathbb{N}$, $\#\{k \in \mathbb{N} :\ \gamma(v_k) \leq n\}$ is clearly bounded from above by $2(1+n)^{N-1}$, see Example 4.3.

We now verify that (3.3) is also satisfied. First of all, we remark that for any $D \in X$ we have that the operator $F(D) - F_0$ is self-adjoint, in the following sense:

$$\langle (F(D) - F_0)\psi, \varphi\rangle = \langle (F(D) - F_0)\varphi, \psi\rangle,$$

for any ψ, $\varphi \in H^{1/2}(\partial\Omega)$, where $\langle\cdot,\cdot\rangle$ is as usual the duality pairing between H' and H. In fact, let u be the solution to (2.8)-(2.9) and u_0 be the solution to the same boundary value problem with D replaced by the empty set, and let v and v_0 be the solutions to the same boundary value problems with ψ replaced by φ. Then

$$\langle (F(D) - F_0)\psi, \varphi\rangle = \int_\Omega (1 + (a-1)\chi_D)\nabla u \cdot \nabla v - \int_\Omega \nabla u_0 \cdot \nabla v_0,$$

which is clearly symmetric, thus the self-adjointness follows.

By self-adjointness, it is enough to show that there exist positive constants C_2 and α_2, depending on N, m, β and a only, so that

$$\left\|(F(D) - F_0)\frac{f_{jp}}{\sqrt{1+j}}\right\|_{H^{-1/2}(\partial\Omega)} \leq C_2 \exp(-\alpha_2(1+j)), \tag{5.3}$$

for any j and p. In fact, by (5.3), we infer that

$$\begin{aligned}\left|\left\langle (F(D) - F_0)\tfrac{f_{jp}}{\sqrt{1+j}}, \tfrac{f_{kq}}{\sqrt{1+k}}\right\rangle\right| &\leq \left\|(F(D) - F_0)\tfrac{f_{jp}}{\sqrt{1+j}}\right\|_{H^{-1/2}(\partial\Omega)} \\ &\leq C_2 \exp(-\alpha_2(1+j)),\end{aligned}$$

where we have used the fact that $\left\|\frac{f_{kq}}{\sqrt{1+k}}\right\|_{H^{1/2}(\partial\Omega)} = 1$. Since $F(D) - F_0$ is self-adjoint, we can reverse the role of j and k and thus obtain

$$\left|\left\langle (F(D) - F_0)\frac{f_{jp}}{\sqrt{1+j}}, \frac{f_{kq}}{\sqrt{1+k}}\right\rangle\right| \leq C_2 \exp(-\alpha_2(1+k))$$

as well, and so (3.3) immediately follows from (5.3).

Let $u_{jp}(D)$ be the solution to (2.8)-(2.9) with the boundary datum ψ replaced by f_{jp}, let u_{jp} be defined as in (4.4) and let $v_{jp} = u_{jp}(D) - u_{jp}$. Since every $D \in X$ is compactly contained in $B_N(0, 4/5)$, we can find a constant C_3, which depends only on N, so that for any $D \in X$

$$\left\|(F(D) - F_0)\frac{f_{jp}}{\sqrt{1+j}}\right\|_{H^{-1/2}(\partial\Omega)} \leq C_3 \left(\int_{\Omega\setminus\overline{B_N(0,4/5)}} \|\nabla v_{jp}\|^2\right)^{1/2}.$$

In fact, there exists a constant C_3, depending on N and $4/5$ only, so that for every $\psi \in H^{1/2}(\partial\Omega)$ there exists $\tilde{\psi} \in H^1(\Omega)$ with the properties that $\tilde{\psi}|_{\partial\Omega} = \psi$, $\tilde{\psi} \equiv 0$ on $B_N(0, 4/5)$ and

$$\|\tilde{\psi}\|_{H^1(\Omega)} = \|\tilde{\psi}\|_{H^1(\Omega\setminus\overline{B_N(0,4/5)})} \leq C_3\|\psi\|_{H^{1/2}(\partial\Omega)}.$$

We have that v_{jp} solves the boundary value problem

$$\text{(5.4)} \qquad \begin{cases} \operatorname{div}((1 + (a-1)\chi_D)\nabla v_{jp}) = -\operatorname{div}((a-1)\chi_D \nabla u_{jp}) & \text{in } \Omega, \\ v_{jp} = 0 & \text{on } \partial\Omega. \end{cases}$$

By the weak formulation of (5.4) and the fact that $\overline{D} \subset B_N(0, 4/5)$, we have that for every $\psi \in H^{1/2}(\partial\Omega)$

$$\langle (F(D) - F_0)f_{jp}, \psi\rangle = \int_\Omega \nabla v_{jp} \cdot \nabla\tilde{\psi},$$

thus

$$\begin{aligned} |\langle (F(D) - F_0)f_{jp}, \psi\rangle| &\leq \left(\textstyle\int_{\Omega\setminus\overline{B_N(0,4/5)}} \|\nabla v_{jp}\|^2\right)^{1/2}\left(\textstyle\int_{\Omega\setminus\overline{B_N(0,4/5)}} \|\nabla\tilde{\psi}\|^2\right)^{1/2} \\ &\leq C_3 \left(\textstyle\int_{\Omega\setminus\overline{B_N(0,4/5)}} \|\nabla v_{jp}\|^2\right)^{1/2} \|\psi\|_{H^{1/2}(\partial\Omega)}. \end{aligned}$$

Again by the weak formulation of (5.4), we have that

$$\begin{aligned} \textstyle\int_\Omega \|\nabla v_{jp}\|^2 &\leq \max\{1, 1/a\} \textstyle\int_\Omega (1 + (a-1)\chi_D)\|\nabla v_{jp}\|^2 \\ &= \max\{1, 1/a\}(-\textstyle\int_D (a-1)\nabla u_{jp} \cdot \nabla v_{jp}) \\ &\leq \max\{1, 1/a\}|a-1| \left(\textstyle\int_\Omega \|\nabla v_{jp}\|^2\right)^{1/2} \left(\textstyle\int_D \|\nabla u_{jp}\|^2\right)^{1/2}. \end{aligned}$$

From here, using again the fact that D is contained in $B_N(0, 4/5)$, it is easy to infer that there exists a constant C_4 depending on a only so that

$$\left(\int_{\Omega\setminus\overline{B_N(0,4/5)}} \|\nabla v_{jp}\|^2\right)^{1/2} \leq C_4 \left(\int_{B_N(0,4/5)} \|\nabla u_{jp}\|^2\right)^{1/2}$$

and so (5.3) is proved by using (4.5). □

Neumann-to-Dirichlet case

PROOF OF PROPOSITION 2.3 (NEUMANN-TO-DIRICHLET CASE). For what concerns the Neumann-to-Dirichlet case, the proposition can be proved as an application of the abstract theorem (see for a similar procedure the proof of the "insulating crack & Neumann-to-Dirichlet case" for the inverse crack problem, page 29). However, it is also possible to argue in the following way. We observe that, for every $D \in X$, $\mathcal{N}(D)$ and $\tilde{\Lambda}(D)$, the restriction of $\Lambda(D)$ to ${}_0H^{1/2}(\partial\Omega)$, are inverse to each other. Since we have already established the Dirichlet-to-Neumann case, there exist $\varepsilon_1 > 0$ and C, depending on N, M, β and a only, so that for any ε, $0 < \varepsilon < \varepsilon_1$, there exists D_1 and D_2 in X so that (2.11) and (2.12) are satisfied. Using the identity

$$\mathcal{N}(D_1) - \mathcal{N}(D_2) = \mathcal{N}(D_2)(\tilde{\Lambda}(D_2) - \tilde{\Lambda}(D_1))\mathcal{N}(D_1), \tag{5.5}$$

we infer that

$$\|\mathcal{N}(D_1) - \mathcal{N}(D_2)\| \leq \|\mathcal{N}(D_2)\| \, \|\tilde{\Lambda}(D_2) - \tilde{\Lambda}(D_1)\| \, \|\mathcal{N}(D_1)\|, \tag{5.6}$$

where the natural norms have been used. We remark that there exists a positive constant C_5, depending on N, M, β and a only, so that for any $D \in X$ we have

$$\|\mathcal{N}(D)\| \leq C_5. \tag{5.7}$$

So (2.13) is proved by (2.12), (5.6) and (5.7). □

Experimental measurements case

Proof of Proposition 2.4. For the basic properties of the problem (2.14), we shall always refer to [22]. Let $H = H^1(\Omega) \times \mathbb{R}^L$. For any $D \in X$, and for any (u, U) and $(w, W) \in H$, let

$$B_D((u,U),(w,W)) = \int_\Omega (1 + (a-1)\chi_D)\nabla u \cdot \nabla w + \sum_{l=1}^{L} \int_{e_l} (u - U_l)(w - W_l).$$

Then, for any $\eta \in {}_0H^{-1/2}(\partial\Omega)$ and any $I \in \mathbb{R}^L$ so that $\sum_{l=1}^{L} I_l = 0$, we have that there exist, and it is unique up to an additive constant, a couple $(u, U) \in H$ so that we have

$$B_D((u,U),(w,W)) = \eta(w|_{\partial\Omega}) + \sum_{l=1}^{L} I_l W_l, \quad \text{for any } (w,W) \in H. \tag{5.8}$$

Furthermore, we have that there exists a constant C_6, depending on N, a, Z and the electrodes only, so that

$$\|\nabla u\|_{L^2(\Omega)} \leq C_6(\|\eta\|_{{}_0H^{-1/2}(\partial\Omega)} + \|I\|). \tag{5.9}$$

We also remark that if (u, U) solves (5.8), then

$$\mathrm{div}((1 + (a-1)\chi_D)\nabla u) = 0 \quad \text{in } \Omega.$$

Therefore, we can rewrite the equation (5.8) on the boundary as follows. We denote $\phi = \frac{\partial u}{\partial \nu}|_{\partial\Omega} \in {}_0H^{-1/2}(\partial\Omega)$ and we recall that $\mathcal{N}(D)$ is the Neumann-to-Dirichlet map associated to the inclusion D. Then, we deduce by straightforward computations that (u, U) satisfies (5.8) if and only if

$$U_l = \frac{z_l}{|e_l|} I_l + \frac{1}{|e_l|} \int_{e_l} u, \quad \text{for any } l = 1, \ldots, L, \tag{5.10}$$

and the following equation holds in ${}_0H^{-1/2}(\partial\Omega)$

$$\phi+\sum_{l=1}^{L}\frac{1}{z_l}\left(\mathcal{N}(D)\phi-\frac{1}{|e_l|}\int_{e_l}\mathcal{N}(D)\phi\right)\chi_{e_l}=\eta+\sum_{l=1}^{L}\left(\frac{I_l}{|e_l|}\chi_{e_l}\right). \tag{5.11}$$

Let $\mathcal{K}(D):{}_0H^{-1/2}(\partial\Omega)\mapsto{}_0H^{-1/2}(\partial\Omega)$ be the operator defined as follows. For any $\phi\in{}_0H^{-1/2}(\partial\Omega)$,

$$\mathcal{K}(D)\phi=\sum_{l=1}^{L}\frac{1}{z_l}\left(\mathcal{N}(D)\phi-\frac{1}{|e_l|}\int_{e_l}\mathcal{N}(D)\phi\right)\chi_{e_l}.$$

We have that $\mathcal{K}(D)$ is a compact linear operator. Since the equation (5.8) admits, up to additive constants, a unique solution, we can infer that (5.11) is uniquely solvable, therefore the operator $Id+\mathcal{K}(D)$ is invertible, where Id denotes the identity operator. Using (5.9), this inverse, which we shall denote with $\tilde{\mathcal{K}}(D)$, satisfies

$$\|\tilde{\mathcal{K}}(D)\|=\|(Id+\mathcal{K}(D))^{-1}\|\leq C_7, \tag{5.12}$$

where the constant C_7 depends on N, a, Z and the electrodes only.

For any given current pattern I, that is $I\in\mathbb{R}^L$ so that $\sum_{l=1}^{L}I_l=0$, we can define $\tilde{I}=\sum_{l=1}^{L}\left(\frac{I_l}{|e_l|}\chi_{e_l}\right)\in{}_0H^{-1/2}(\partial\Omega)$. Furthermore, there exists a constant C_8, depending on N and the electrodes only, so that, for any $I\in\mathbb{R}^L$ satisfying $\sum_{l=1}^{L}I_l=0$, we have

$$\|\tilde{I}\|_{{}_0H^{-1/2}(\partial\Omega)}\leq C_8\|I\|. \tag{5.13}$$

As it is shown in [22], we have that u solves our direct problem (2.14) for a given current pattern I if and only if (5.8) is satisfied with $\eta=0$. Therefore, if we take $I\in\mathbb{R}^L$ so that $\sum_{l=1}^{L}I_l=0$, we have that $R(D)I=V$ where, for any $l=1,\ldots,L$,

$$V_l=\int_{e_l}\mathcal{N}(D)(\tilde{\mathcal{K}}(D)\tilde{I})+c|e_l|, \tag{5.14}$$

where c is a constant which can be computed by imposing the condition that $\sum_{l=1}^{L}V_l=0$, that is

$$c=-\frac{\sum_{l=1}^{L}\int_{e_l}\mathcal{N}(D)(\tilde{\mathcal{K}}(D)\tilde{I})}{\sum_{l=1}^{L}|e_l|}. \tag{5.15}$$

In order to establish Proposition 2.4, we observe that, by Proposition 2.3, we can find constants $\varepsilon_1>0$ and C, which depend on N, m, β and a only, so that for any ε, $0<\varepsilon<\varepsilon_1$, there exists D_1 and D_2 in X so that (2.11) and (2.13) are satisfied. We show that these inclusions D_1 and D_2 provide us with the instability example also in the experimental measurements case. Let us evaluate the norm of $R(D_1)-R(D_2)$. Therefore, we take $I\in\mathbb{R}^L$ so that $\sum_{l=1}^{L}I_l=0$ and we evaluate $\|(R(D_1)-R(D_2))I\|$. We recall that we have posed $R(D_1)[1]=R(D_2)[1]=0$. By (5.14) and (5.15), we have that

$$\|(R(D_1)-R(D_2))I\|\leq C_9\|\mathcal{N}(D_1)(\tilde{\mathcal{K}}(D_1)\tilde{I})-\mathcal{N}(D_2)(\tilde{\mathcal{K}}(D_2)\tilde{I})\|_{L^2(\partial\Omega)},$$

where C_9 depends on N and the electrodes only. Thus, we can find a constant C_{10}, depending on N and C_9 only, so that

$$\begin{aligned}\|(R(D_1)-R(D_2))I\| &\leq C_{10}\|\mathcal{N}(D_1)(\tilde{\mathcal{K}}(D_1)-\tilde{\mathcal{K}}(D_2))\tilde{I}\|_{H^{1/2}(\partial\Omega)}+\\ &\quad C_{10}\|(\mathcal{N}(D_2)-\mathcal{N}(D_1))(\tilde{\mathcal{K}}(D_2)\tilde{I})\|_{H^{1/2}(\partial\Omega)},\end{aligned}$$

and, by (5.7), (5.12) and (5.13), we also deduce that

$$\begin{aligned}\|(R(D_1)-R(D_2))I\| &\leq C_5C_8C_{10}\|\tilde{\mathcal{K}}(D_1)-\tilde{\mathcal{K}}(D_2)\|\,\|I\|+\\ &\quad C_7C_8C_{10}\|\mathcal{N}(D_2)-\mathcal{N}(D_1)\|\,\|I\|.\end{aligned}\tag{5.16}$$

It remains to evaluate the term $\|\tilde{\mathcal{K}}(D_1)-\tilde{\mathcal{K}}(D_2)\|$. We proceed as follows. Using an identity analogous to (5.5) applied to the operators $Id+\mathcal{K}$ and $\tilde{\mathcal{K}}$, and recalling (5.12), we obtain that

$$\|\tilde{\mathcal{K}}(D_1)-\tilde{\mathcal{K}}(D_2)\| \leq C_7^2\|\mathcal{K}(D_1)-\mathcal{K}(D_2)\| \leq C_{11}\|\mathcal{N}(D_1)-\mathcal{N}(D_2)\|,\tag{5.17}$$

where C_{11} depends on N, a, Z and the electrodes only.

And so the conclusion immediately follows by coupling (5.16) with (5.17) and using (2.13). □

5.2 Inverse crack problem

The proof of Proposition 2.5 follows directly from the abstract theorem stated in Theorem 3.1. We just need to check that all the hypotheses of Theorem 3.1 are satisfied. Therefore, the proof is divided into two steps, each corresponding to one of the hypotheses of Theorem 3.1.

PROOF OF PROPOSITION 2.5 - FIRST STEP. First, let $x_0 \in X$ be $\overline{B'_{N-1}(0,1/2)}$. Then, by Proposition 2.2, X satisfies assumption *i*) of Theorem 3.1, with constants ε_0 and C_1 depending on N, m and β only, and constant $\alpha_1=(N-1)/m$. We recall also that $\sigma \subset B_N(0,4/5)$ for any $\sigma \in X$. □

For what concerns the second step, we turn our attention to assumption *ii*) of Theorem 3.1. Each case, corresponding to operators Λ_i and $\mathcal{N}_i$, $i=1,2$, should be treated separately. We limit ourselves to two cases, namely the cases corresponding to $\mathcal{N}_1$ and Λ_2, in order to show the main points of the proof, and we leave the details concerning the other two cases to the reader.

Insulating crack & Neumann-to-Dirichlet case

PROOF OF PROPOSITION 2.5 - SECOND STEP (INSULATING CRACK & NEUMANN-TO-DIRICHLET CASE). First, we notice that u is a solution to (2.16)-(2.19) if and only if $u \in H^1(\Omega\backslash\sigma)$, $\int_{\partial\Omega} u = 0$, and

$$\int_{\Omega\backslash\sigma} \nabla u \cdot \nabla w = \eta(w|_{\partial\Omega}), \quad \text{for any } w \in H^1(\Omega\backslash\sigma).$$

We observe that for any $\sigma \in X$, $\mathcal{N}_1(\sigma)$ is a bounded and linear operator between ${}_0H^{-1/2}(\partial\Omega)$ and its dual. Hence we take H to be ${}_0H^{-1/2}(\partial\Omega)$ and $F : X \mapsto \mathcal{L}(H,H')$ to be defined as $F(\sigma)=\mathcal{N}_1(\sigma)$ for any $\sigma \in X$. With F_0 we denote in an analogous way the Neumann-to-Dirichlet map related to (2.16)-(2.19) when $\sigma = \emptyset$, that is the Neumann-to-Dirichlet map associated to the body where no crack is present. For any $\eta \in {}_0H^{-1/2}(\partial\Omega)\backslash\{0\}$, we define

$$\gamma(\eta) = \frac{\|\eta\|^2_{L^2(\partial\Omega)}}{\|\eta\|^2_{H^{-1/2}(\partial\Omega)}}. \tag{5.18}$$

Referring to Proposition 4.1, Example 4.3 and (4.3), $\{v_k\}_{k\in\mathbb{N}}$, the orthonormal basis of H we shall employ, is given by

$$\left\{\sqrt{j} f_{jp}:\ j \geq 1 \text{ and } 1 \leq p \leq p_j\right\} \tag{5.19}$$

with the natural order. We have that $\gamma(\sqrt{j} f_{jp}) = j$, for any j and p. Again by our remarks in Example 4.3, we deduce that $\#\{k \in \mathbb{N}:\ \gamma(v_k) \leq n\} \leq 2(1+n)^{N-1}$, for any $n \in \mathbb{N}$,

For what concerns (3.3), we argue in this way. We need a kind of self-adjointness of $F(\sigma) - F_0$ for every $\sigma \in X$. We have that

$$\langle (F(\sigma) - F_0)\eta, \phi\rangle = \langle (F(\sigma) - F_0)\phi, \eta\rangle$$

for any η, $\phi \in {}_0H^{-1/2}(\partial\Omega)$, where $\langle\cdot,\cdot\rangle$ is again the duality pairing between H' and H. In fact, if u solves (2.16)-(2.19), u_0 solves the same boundary value problem with σ replaced by the empty set, v and v_0 solves the same boundary value problems with η replaced by ϕ, then

$$\langle (F(\sigma) - F_0)\eta, \phi\rangle = \int_{\Omega\backslash\sigma} \nabla v \cdot \nabla u - \int_\Omega \nabla v_0 \cdot \nabla u_0.$$

By the self-adjointness of the operator $F(\sigma) - F_0$, for any $\sigma \in X$, in order to prove (3.3) we have to show that there exist positive constants C_2 and α_2, which depend on N, m and β only, so that, for any j and p,

$$\left\|(F(\sigma) - F_0)\sqrt{j} f_{jp}\right\|_{H^{1/2}(\partial\Omega)} \leq C_2 \exp(-\alpha_2 j). \tag{5.20}$$

We can find a constant C_3, depending on N only, so that, for any $\sigma \in X$,

$$\left\|(F(\sigma) - F_0)\sqrt{j} f_{jp}\right\|_{H^{1/2}(\partial\Omega)} \leq C_3 \|v_{jp}\|_{H^1(\Omega\backslash\overline{B_N(0,4/5)})},$$

where v_{jp} satisfies

$$\begin{cases} \Delta v_{jp} = 0 & \text{in } \Omega\backslash\sigma, \\ \frac{\partial v_{jp}}{\partial\nu} = 0 & \text{on } \partial\Omega, \\ \frac{\partial v_{jp}}{\partial\nu} = -j^{-1/2}\frac{\partial u_{jp}}{\partial\nu} & \text{on } \partial\sigma, \\ \int_{\partial\Omega} v_{jp} = 0, \end{cases} \tag{5.21}$$

with u_{jp} given by formula (4.4). Since $\int_{\partial\Omega} v_{jp} = 0$, a Poincaré type inequality implies that there exists a constant C_4, depending on N only, so that, for any $\sigma \in X$, we have

$$\left\|(F(\sigma) - F_0)\sqrt{j} f_{jp}\right\|_{H^{1/2}(\partial\Omega)} \leq C_4 \left(\int_{\Omega\backslash\overline{B_N(0,4/5)}} \|\nabla v_{jp}\|^2\right)^{1/2}.$$

We can estimate the right hand side of the last equation as follows. We fix a cut-off function χ so that $\chi \in C_0^\infty(B_N(0,5/6))$, $0 \leq \chi \leq 1$, $\chi \equiv 1$ on $B_N(0,4/5)$. Without loss of generality, we can assume that for every $x \in \mathbb{R}^N$, $\|\nabla\chi(x)\| \leq C_5$, C_5 being a constant depending on N only. Let us observe that (5.21) means that for every $w \in H^1(\Omega\backslash\sigma)$ we have

$$\int_{\Omega\backslash\sigma} \nabla v_{jp} \cdot \nabla w = -\int_{\Omega\backslash\sigma} j^{-1/2}\nabla u_{jp} \cdot \nabla(\chi w).$$

Then, by taking $w = v_{jp}$, we infer that

$$\int_{\Omega\backslash\sigma} \|\nabla v_{jp}\|^2 = -\int_{\Omega\backslash\sigma} j^{-1/2}\nabla u_{jp} \cdot \nabla(\chi v_{jp}).$$

Straightforward computations allow us to prove that there exists a constant C_6, depending on N only, so that

$$\left(\int_{\Omega\backslash\sigma} \|\nabla v_{jp}\|^2\right)^{1/2} \leq C_6 \left(\int_{B_N(0,5/6)} \|\nabla u_{jp}\|^2\right)^{1/2}.$$

Then we can conclude using (4.5). □

Conducting crack & Dirichlet-to-Neumann case

Proof of Proposition 2.5 - Second step (Conducting crack & Dirichlet-to-Neumann case). We begin with a description of the weak formulation of the boundary value problem (2.17)-(2.18). With $H^1_{const}(\Omega,\sigma)$ we denote the subspace of $H^1(\Omega)$ functions which are constant on σ. For any $c \in \mathbb{R}$, we set $H^1_c(\Omega,\sigma)$ as the subset of $H^1(\Omega)$ functions which are equal to the constant c on σ. For any $c \in \mathbb{R}$, we have that there exists and it is unique a solution to the following boundary value problem

$$\text{(5.22)} \qquad \begin{cases} \Delta u_c = 0 & \text{in } \Omega\backslash\sigma, \\ u_c = c & \text{on } \sigma, \\ u_c = \psi & \text{on } \partial\Omega, \end{cases}$$

that is a function $u_c \in H^1_c(\Omega,\sigma)$ so that $u_c|_{\partial\Omega} = \psi$ and that

$$\int_{\Omega\backslash\sigma} \nabla u_c \cdot \nabla w = 0, \quad \text{for any } w \in H^1_0(\Omega) \cap H^1_0(\Omega,\sigma).$$

Given u_c, solution to (5.22), we can define $\frac{\partial u_c}{\partial\nu}|_{\partial\Omega} \in H^{-1/2}(\partial\Omega)$ as follows

$$\langle \frac{\partial u_c}{\partial \nu}|_{\partial\Omega}, \varphi\rangle = \int_{\Omega\backslash\sigma} \nabla u_c \cdot \nabla\tilde{\varphi},$$

where $\varphi \in H^{1/2}(\partial\Omega)$ and $\tilde{\varphi}$ is any $H^1_0(\Omega,\sigma)$ function so that $\tilde{\varphi}|_{\partial\Omega} = \varphi$.

We claim that there exists a unique $c \in \mathbb{R}$ so that $\langle \frac{\partial u_c}{\partial\nu}|_{\partial\Omega}, 1\rangle = 0$, that is existence and uniqueness of a solution to (2.17)-(2.18).

We have that u solves (2.17)-(2.18) if and only if $u \in H^1_{const}(\Omega,\sigma)$ so that $u|_{\partial\Omega} = \psi$ and that

$$\int_{\Omega\backslash\sigma} \nabla u \cdot \nabla w = 0, \quad \text{for any } w \in H^1_0(\Omega) \cap H^1_{const}(\Omega,\sigma).$$

If we take $\tilde{\psi}$ to be any $H^1_0(\Omega,\sigma)$ function so that $\tilde{\psi}|_{\partial\Omega} = \psi$, we have that u solves (2.17)-(2.18) if and only if $\tilde{u} = u - \tilde{\psi}$ belongs to $H^1_0(\Omega) \cap H^1_{const}(\Omega,\sigma)$ and satisfies

$$\int_{\Omega\backslash\sigma} \nabla\tilde{u}\cdot\nabla w = -\int_{\Omega\backslash\sigma} \nabla\tilde{\psi}\cdot\nabla w, \quad \text{for any } w \in H^1_0(\Omega)\cap H^1_{const}(\Omega,\sigma).$$

Standard elliptic theory provides us with existence and uniqueness of such a solution. By the property $\langle \frac{\partial u}{\partial \nu}|_{\partial\Omega}, 1\rangle = 0$, we can infer that $\frac{\partial u}{\partial \nu}|_{\partial\Omega} \in H^{-1/2}(\partial\Omega)$ can be also defined as

$$\langle \frac{\partial u}{\partial \nu}|_{\partial\Omega}, \varphi\rangle = \int_{\Omega\backslash\sigma} \nabla u\cdot\nabla\tilde{\varphi},$$

where $\varphi \in H^{1/2}(\partial\Omega)$ and $\tilde{\varphi}$ is any $H^1_{const}(\Omega,\sigma)$ function so that $\tilde{\varphi}|_{\partial\Omega} = \varphi$.

Now we can denote with H the space $H^{1/2}(\partial\Omega)$, and we can fix γ as in (5.1) and the orthonormal basis as the one described in (5.2). The map $F : X \mapsto \mathcal{L}(H, H')$ is given by $F(\sigma) = \Lambda_2(\sigma)$, for any $\sigma \in X$, and F_0 denotes the Dirichlet-to-Neumann map corresponding to $\sigma = \emptyset$. We recall that the operator $F(\sigma)$ is self-adjoint for any $\sigma \in X$, as well as F_0 is.

We proceed to verify (3.3) in this case. First, there exists a constant C_7, depending on N only, so that, for any $\sigma \in X$,

$$\left\|(F(\sigma) - F_0)\frac{f_{jp}}{\sqrt{1+j}}\right\|_{H^{-1/2}(\partial\Omega)} \leq C_7\left(\int_{\Omega\backslash\overline{B_N(0,4/5)}} \|\nabla v_{jp}\|^2\right)^{1/2} \tag{5.23}$$

where $v_{jp} = u_{jp}(\sigma) - \frac{u_{jp}}{\sqrt{1+j}}$, $u_{jp}(\sigma)$ being the solution to (2.17)-(2.18) with ψ replaced by $\frac{f_{jp}}{\sqrt{1+j}}$ and u_{jp} being as in (4.4).

Hence, v_{jp} satisfies

$$\begin{cases} \Delta v_{jp} = 0 & \text{in } \Omega\backslash\sigma,\\ v_{jp} = 0 & \text{on } \partial\Omega,\\ v_{jp} = c - \frac{u_{jp}}{\sqrt{1+j}} & \text{on } \partial\sigma,\\ \langle \frac{\partial v_{jp}}{\partial\nu}|_{\partial\Omega}, 1\rangle = 0, & \end{cases} \tag{5.24}$$

where $c = u_{jp}(\sigma)|_\sigma$. We notice that, if χ is the cut-off function previously defined in this subsection, then $w_{jp} = (v_{jp} - c + \chi\frac{u_{jp}}{\sqrt{1+j}}) \in H^1_0(\Omega,\sigma)$ and $v_{jp}|_{\partial\Omega} = -c$. So,

$$\int_{\Omega\backslash\sigma} \nabla v_{jp}\cdot\nabla w_{jp} = \langle \frac{\partial v_{jp}}{\partial\nu}|_{\partial\Omega}, -c\rangle = 0,$$

that is

$$\int_{\Omega\backslash\sigma} \nabla v_{jp}\cdot\nabla v_{jp} = \int_{\Omega\backslash\sigma} \nabla v_{jp}\cdot\nabla(\chi\frac{u_{jp}}{\sqrt{1+j}}),$$

from which we easily deduce that

$$\left(\int_{\Omega\backslash\sigma} \|\nabla v_{jp}\|^2\right)^{1/2} \leq C_8\|u_{jp}\|_{H^1(B_N(0,5/6)}, \tag{5.25}$$

where C_8 depends on N only.

So (3.3) is obtained by combining (5.23), (5.25) and (4.5) and the self-adjointness of the operator $F(\sigma) - F_0$. $\square$

5.3 Inverse cavity problem, inverse surface crack problem and inverse boundary material loss problem

As we have already observed, the inverse problem of cavities can be treated in a way which is completely analogous to the treatment of the inverse crack problem.

PROOF OF PROPOSITION 2.6. It can be obtained along the lines of the proof of Proposition 2.5, with obvious modifications. In particular, the reference point in X is given by σ_0, the orthonormal basis used are those described in Example 4.5, whereas the reference operator is the one related to the domain Ω, Ω as in Example 4.5. □

PROOF OF PROPOSITION 2.7. Also the arguments for the proof of Proposition 2.7 are simple modifications of what we have used to prove Proposition 2.5, clearly making use of the orthonormal basis described in Example 4.4. □

5.4 Inverse scattering problem

The proof of Proposition 2.8 is somehow different from the proofs of the analogous propositions discussed previously. In fact, we can not prove Proposition 2.8 as a straightforward application of Theorem 3.1. Nevertheless, the procedure developed during the proof of Theorem 3.1 can be adjusted in such a way to cover also the inverse scattering case framework. In the sequel, we limit ourselves to the sound-soft case, the sound-hard case can be obtained with minor adjustments. We shall point out the main differences between the sound-soft and the sound-hard case and conclude the proof for the sound-hard case at the end of the subsection. The proof of Proposition 2.8 for the sound-soft case will be divided in two steps.

PROOF OF PROPOSITION 2.8 - FIRST STEP (SOUND-SOFT CASE). First, we fix $x_0 \in X$ to be equal to $\overline{B_N(0,1)}$ and we observe that assumption *i)* of Theorem 3.1 is satisfied, by Proposition 2.2, with constants ε_0 and C_1, depending on N, m and β only, and constant $\alpha_1 = (N-1)/m$. □

The second step deals with the main difference from the previous cases, which is as follows. In the abstract theorem, we have a function F which maps elements of a metric space X into elements of $\mathcal{L}(H, H')$, H being a separable Hilbert space. Now, fixed $a > 0$, we define a map F which associates to each obstacle $D \in X$ a complex-valued function defined on $S^{N-1} \times S^{N-1}$, namely $F(D) = \mathcal{A}_s(D)(\cdot,\cdot,a)$ or, respectively, $F(D) = \mathcal{A}_h(D)(\cdot,\cdot,a)$. In the abstract theorem, fixed a suitable $F_0 \in \mathcal{L}(H, H')$, the operator $F(x)$, $x \in X$, was characterized by the numbers $b_{k,l} = \langle (F(x) - F_0)v_k, v_l\rangle$, where k, $l \in \mathbb{N}$ and $\{v_k\}_{k\in\mathbb{N}}$ is a suitably chosen orthonormal basis of H. The fundamental properties of such a characterization were summarized in assumption *ii)* of Theorem 3.1. In particular, the crucial property was a control on the asymptotic behaviour of the coefficients $b_{k,l}$, which was provided by formulas (3.2) and (3.3). We shall obtain a completely analogous characterization by decomposing the far-field pattern in spherical harmonics.

PROOF OF PROPOSITION 2.8 - SECOND STEP (SOUND-SOFT CASE). We take $\{v_k\}_{k\in\mathbb{N}}$ as the orthonormal basis of $L^2(S^{N-1})$ described in Example 4.3, precisely in (4.3), with the natural order. Therefore, for each $k \in \mathbb{N}$, v_k is a (real-valued) spherical harmonic function on S^{N-1}. We set $\gamma(v_k)$ as the degree of the spherical harmonic function v_k. We have that $\gamma(v_k)$ is an increasing sequence, with respect to k, whose asymptotic behaviour satisfies (3.2) with constants $C_2 = 2$ and $p = N - 1$.

The decomposition of the far-field pattern in spherical harmonics is given by, for any $(\hat{x}, \omega, a) \in S^{N-1} \times S^{N-1} \times (0, \infty)$,

$$\mathcal{A}_s(D)(\hat{x}, \omega, a) = \sum_{k,l} b_{k,l}(a) v_k(\hat{x}) v_l(\omega), \tag{5.26}$$

where the complex-valued coefficients $b_{k,l}(a)$ are given, for any $a \in (0, \infty)$, by

$$b_{k,l}(a) = \iint_{S^{N-1}\times S^{N-1}} \mathcal{A}_s(D)(\hat{x}, \omega, a) v_k(\hat{x}) v_l(\omega) \mathrm{d}\hat{x}\mathrm{d}\omega. \tag{5.27}$$

Furthermore, we use the following characterization

$$b_{k,l}(a) = \int_{S^{N-1}} \tilde{b}_k(\omega, a) v_l(\omega) \mathrm{d}\omega, \tag{5.28}$$

where the complex-valued coefficients $\tilde{b}_k(\omega, a)$ are, for any $\omega \in S^{N-1}$ and any $a \in (0, \infty)$, the Fourier coefficients, with respect to the orthonormal basis $\{v_k\}_{k\in\mathbb{N}}$, of the far-field pattern $u^s_\infty(\cdot; \omega; a)$ corresponding to the scattered field of the solution to (2.29)-(2.30)-(2.32), that is

$$\tilde{b}_k(\omega, a) = \int_{S^{N-1}} \mathcal{A}_s(D)(\hat{x}, \omega, a) v_k(\hat{x}) \mathrm{d}\hat{x}. \tag{5.29}$$

In the next lemma, we establish the asymptotic behaviour of the coefficients $b_{k,l}$, which will play the role of the assumption stated in (3.3).

Lemma 5.1 *Under the previous assumptions and definitions, there exist positive constants C_2 and α_2, depending on N, m, β and I_N only, so that for any $D \in X$, for any $a \in I_N$ and for any $(k, l) \in \mathbb{N} \times \mathbb{N}$, we have*

$$|b_{k,l}(a)| \leq C_2 \exp(-\alpha_2 \max\{\gamma(v_k), \gamma(v_l)\}), \tag{5.30}$$

coefficient $b_{k,l}$ as in (5.27).

Proof. First, we claim that there exist positive constants C_3 and α_3, depending on N, m, β and I_N only, so that for any $D \in X$, for any $a \in I_N$, for any $\omega \in S^{N-1}$ and for any $k \in \mathbb{N}$, we have

$$|\tilde{b}_k(\omega, a)| \leq C_3 \exp(-\alpha_3 \gamma(v_k)), \tag{5.31}$$

$\tilde{b}_k$ defined by (5.29).

By (5.31) and (5.28), we immediately infer that, for any k, $l \in \mathbb{N}$,

$$|b_{k,l}(a)| \leq |S^{N-1}|^{1/2} C_3 \exp(-\alpha_3 \gamma(v_k)), \tag{5.32}$$

$|S^{N-1}|$ being the $(N-1)$-dimensional measure of S^{N-1}.

Then, we make use of the following *reciprocity relation*, see for instance [10, Theorem 3.13]. For any $D \in X$ and any $a \in (0, \infty)$ we have

$$\mathcal{A}_s(D)(\hat{x}, \omega, a) = \mathcal{A}_s(D)(-\omega, -\hat{x}, a), \quad \text{for any } \hat{x}, \omega \in S^{N-1}. \tag{5.33}$$

The reciprocity relation plays the role of self-adjointness for the elliptic operators we have considered before and allows us, using (5.32), to easily conclude the proof of the lemma. Therefore, what remains to be proven is the claim in (5.31).

In order to prove (5.31), we begin with a uniform bound on the scattered field. We notice that, for any $D \in X$, $D \subset B_N(9/5)$. With a procedure which is analogous to the one used first in [14, Lemma 2] and later in [20], and using the fact that the scattered fields are radiating solutions to the Helmholtz equation, we can find a constant C_4, depending on N, m, β and I_N only, so that, for any $D \in X$, any $\omega \in S^{N-1}$ and any $a \in I_N$, we have

$$|u^s(x;\omega;a)| \leq C_4 \|x\|^{-(N-1)/2}, \quad \text{for any } x \in \mathbb{R}^N \backslash B_N(0,2), \tag{5.34}$$

where u^s is the scattered field corresponding to the solution to (2.29)-(2.30)-(2.32). We remark that only in the estimate above the difference between the cases $N = 2$ and $N = 3$ shows up. We refer to [20] for a detailed discussion about uniform estimates of decay at infinity for radiating solutions to the Helmholtz equation.

By Theorem 2.14 in [10], we have that since u^s is a radiating solution to the Helmholtz equation, with coefficient $a > 0$, in $\mathbb{R}^N \backslash \overline{B_N(9/5)}$, then, for any $x \in \mathbb{R}^N \backslash B_N(0,2)$,

$$u^s(x;\omega;a) = \sum_k \hat{b}_k(\omega,a) H^{(1)}_{\gamma(v_k)}(\sqrt{a}\|x\|) v_k(x/\|x\|), \tag{5.35}$$

where $\hat{b}_k$ are complex-valued coefficients given by

$$\hat{b}_k H^{(1)}_{\gamma(v_k)}(\sqrt{a} r) = \int_{S^{N-1}} u^s(r\hat{x};\omega;a) v_k(\hat{x}) \mathrm{d}\hat{x}, \quad \text{for any } r \geq 2, \tag{5.36}$$

where, for any integer $n \geq 0$, $H^{(1)}_n$ denotes the *Hankel function* of first kind and order n.

Theorem 2.15 in [10] provides us with the necessary link between coefficients $\tilde{b}_k$ and $\hat{b}_k$. In fact, it holds that

$$\tilde{b}_k(\omega,a) = (\pi/2)^{(N-3)/2} a^{-(N-1)/4} (-\mathrm{i})^{\gamma_k + (N-1)/2} \hat{b}_k(\omega,a). \tag{5.37}$$

We combine (5.37) with (5.36) and, by using (5.34), we obtain that there exists a constant C_5, depending on C_4 only, so that, for any $k \in \mathbb{N}$, any $\omega \in S^{N-1}$ and any $a \in I_N$,

$$|\tilde{b}_k(\omega,a)| \leq C_5 a^{-(N-1)/4} r^{-(N-1)/2} |H^{(1)}_{\gamma(v_k)}(\sqrt{a} r)|^{-1}, \quad \text{for any } r \geq 2. \tag{5.38}$$

We choose r as follows. For any a, $a \geq 1$, we take $r = 2$, whereas for any a, $0 < a < 1$, we pick $r = 2/\sqrt{a}$. With this choice we infer that for any $a \in I_N$, we have that

$$|\tilde{b}_k(\omega,a)| \leq C_6 |H^{(1)}_{\gamma(v_k)}(\tilde{r})|^{-1} \tag{5.39}$$

where $2 \leq \tilde{r} \leq 2\max\{1,\overline{a}\}$ and the constant C_6 again depends on C_4 only. Then we can establish (5.31) by using the well-known asymptotic behaviour of the Hankel functions. In fact, there exists a constant C_7, depending on N and $\overline{a}$ only, so that for any $\tilde{r}$, $2 \leq \tilde{r} \leq 2\max\{1,\overline{a}\}$,

$$|H_n^{(1)}(\tilde{r})|^{-1} \leq \begin{cases} C_7 & \text{if } n = 0,1, \\ C_7\left(\frac{e\tilde{r}}{2}\right)^n (n-1)^{-(n-1)} & \text{for any } n \geq 2. \end{cases} \tag{5.40}$$

By a straightforward computation, (5.39) together with (5.40) implies the validity of (5.31). □

Now we have what is needed to prove Proposition 2.8 in the sound-soft case. Let us just notice that, for any $D \in X$ and any $a \in (0,\infty)$,

$$\|\mathcal{A}_s(D)(\cdot,\cdot,a)\|_{L^2(S^{N-1}\times S^{N-1})} = \left(\sum_{k,l}|b_{k,l}(a)|^2\right)^{1/2}.$$

Then, with the same procedure used to prove Lemma 3.2, and keeping in mind the fact that we repeat the procedure j times, one for each a_i, $i = 1,\ldots,j$, we can find a constant C_8, depending on N, and the constants C_2 and α_2 of Lemma 5.1 only, so that, for any δ, $0 < \delta < 1/\mathrm{e}$, there exists a subset Y_δ of X with at most $\exp(jC_8(-\log\delta)^{2N-1})$ elements so that for any $D \in X$ there exists $\tilde{D} \in Y_\delta$ satisfying

$$\sup_{a\in\{a_1,\ldots,a_j\}} \|(\mathcal{A}_s(D) - \mathcal{A}_s(\tilde{D}))(\cdot,\cdot,a)\|_{L^2(S^{N-1}\times S^{N-1})} \leq \delta.$$

Then the conclusion of the proof of Proposition 2.8 in the sound-soft case is immediate. □

Proof of Proposition 2.8 (Sound-hard case). We conclude this subsection sketching the proof for the sound-hard case.

First, for any α, $0 < \alpha < 1$, and $\beta > 0$ we define $Y_{(1,\alpha)\beta(1/2)}(S^{N-1}(0,1))$ as we have defined $Y_{m\beta(1/2)}(S^{N-1}(0,1))$, m being an integer, with the only obvious modification of replacing the C^m norm with the $C^{1,\alpha}$ norm. Furthermore, we observe that $Y_{(1,\alpha)\beta(1/2)}(S^{N-1}(0,1))$ satisfies assumption *i)* of Theorem 3.1 with constants ε_0 and C_1, depending on N, α and β only, and constant $\alpha_1 = (N-1)/(1+\alpha)$.

The difference between the sound-soft case and the sound-hard case relies in the estimate contained in (5.34). With arguments which are analogous to the ones used for the sound-soft obstacles, estimate (5.34) can be proved for sound-hard obstacles belonging to $Y_{m\beta(1/2)}(S^{N-1}(0,1))$, for any integer $m \geq 2$, with a constant C_4 depending on N, m, β and I_N only, and for sound-hard obstacles belonging to $Y_{(1,\alpha)\beta(1/2)}(S^{N-1}(0,1))$, with $0 < \alpha < 1$, with a constant C_4 depending on N, α, β and I_N only.

Since the other part of the proof does not depend on the type of boundary conditions used, the result follows, for sound-hard obstacles, for any $m \geq 2$. Some modifications are needed to treat the case when $m = 1$. We have that, for any $0 < \alpha < 1$, $Y_{(1,\alpha)\beta(1/2)}(S^{N-1}(0,1)) \subset Y_{1\beta(1/2)}(S^{N-1}(0,1))$. We apply the procedure described before to the set $Y_{(1,\alpha)\beta(1/2)}(S^{N-1}(0,1))$ and, recalling Remark 3.3, we infer that for any $\gamma > 0$ there exists a constant $\varepsilon_1 > 0$, that depends on N, j, β, I_N, α and γ only, so that for any ε, $0 < \varepsilon < \varepsilon_1$, we can find D_3 and D_4, both belonging to $Y_{(1,\alpha)\beta(1/2)}(S^{N-1}(0,1))$, satisfying $d(D_3,D_4) \geq \varepsilon$ and

$$\sup_{a\in\{a_1,\ldots,a_j\}} \|(\mathcal{A}_h(D_3) - \mathcal{A}_h(D_4))(\cdot,\cdot,a)\|_{L^2(S^{N-1}\times S^{N-1})} \leq 2\exp(-\varepsilon^{-\alpha_1}),$$

where $\alpha_1 = \frac{N-1}{(1+\alpha)(2N-1+\gamma)}$. We can choose, from the very beginning, α and γ in such a way that α and γ depend on N only and $(1+\alpha)(2N-1+\gamma) = 2N$, for instance we can take $\alpha = \frac{1}{4(2N-1)}$ and $\gamma = \frac{3(2N-1)}{4(2N-1)+1}$. Thus the result is established also for the case $m = 1$ and sound-hard obstacles. □

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