Dottorato di Ricerca in Matematica
XII Ciclo

Università degli Studi di Milano

Tesi di Dottorato

Conserved phase-field models with memory in history spaces

Federico Mario Vegni

Relatore: Prof. Maurizio Grasselli
Dipartimento di Matematica "F. Brioschi"
Politecnico di Milano

Correlatore: Prof. Vittorino Pata
Dipartimento di Matematica "F. Brioschi"
Politecnico di Milano

October 25, 2000

Contents

Chapter 1

Introduction

In the framework of phase-field theories (see, e.g., Brokate-Sprekels [11] and Visintin [58]), materials which exhibit phase transitions due to temperature variations are usually described by two state variables: the (absolute) temperature field θ and the so-called order parameter (or phase variable) χ accounting for phase changes.

Here we introduce and analyze a phase transition model for a material with thermal memory. We recall that memory (or relaxation) effects in the heat propagation are related, e.g., to certain liquids of high viscosity (see Jäckle [43] and Joseph-Preziosi [44, 45] and references therein).

Our model basically consists of an integrodifferential evolution equation ruling the temperature coupled with a nonlinear fourth-order evolution equation governing the order parameter.

This system is phenomenologically derived by combining the theory of heat conduction in materials with memory (see, for instance, Joseph-Preziosi [44, 45]) with the Ginzburg-Landau theory for phase transitions (cf. Brokate-Sprekels [11] and Novick–Cohen [50]).

We then associate with the evolution system a set of initial and boundary conditions.

The well-posedness of the corresponding initial and boundary value problem as well as the long time behavior analysis will be the core of this work.

Let $\Omega \subset \mathbb{R}^3$ be a fixed, bounded domain occupied by an isotropic, rigid and homogeneous heat conductor. We consider only small variations of the absolute temperature and its gradient, and we suppose that at each point $x \in \Omega$ and at each time $t \in (\tau, +\infty)$ the state of the material is described by the triplet $(\vartheta, \chi, \vartheta^t)$. Here, ϑ is the *temperature variation field*,

$$\vartheta = \frac{\theta - \theta_c}{\theta_c}$$

where θ_c is the reference temperature at which transition occurs. Moreover, we recall that the *phase variable* χ describes the macroscopic solid-liquid transition, and

$$\vartheta^t(s) = \vartheta(t - s)$$

is the *past history* of ϑ up to time t.

The evolution of the temperature dependent phenomenon is governed by the *energy* *balance* equation

$$\partial_t e + \nabla \cdot \mathbf{q} = f \tag{1.1}$$

where e is the *internal energy*, $\mathbf{q}$ is the *heat flux vector*, and f is the external heat supply.

Taking into consideration a linearized version of the Coleman-Gurtin theory, we assume that e and $\mathbf{q}$ are described by the following constitutive equations (see Coleman-Gurtin [17] and Giorgi-Grasselli-Pata [32])

$$e(x,t) = e_c + c_v\theta_c\vartheta(x,t) + \int_0^\infty a(\sigma)\vartheta^t(x,\sigma)d\sigma + \theta_c\lambda(\chi(x,t)) \tag{1.2}$$

$$\mathbf{q}(x,t) = -k\nabla\vartheta(x,t) - \int_0^\infty b(\sigma)\nabla\vartheta^t(x,\sigma)d\sigma. \tag{1.3}$$

for $(x,t) \in \Omega\times\mathbb{R}$. Here a, b are smooth positive functions such that a', a'', b, b' are summable and $a(0) > 0$. The positive constants e_c, c_v, and k are the internal energy at equilibrium, the specific heat and the *instantaneous conductivity*, respectively. As a consequence, Fourier heat conduction law is recovered when memory term in (1.3) is neglected.

When the influence of the instantaneous heat flow is negligible, we obtain a pure memory heat flux law; namely, the so-called Gurtin-Pipkin law (see Gurtin-Pipkin [39])

$$\mathbf{q}(x,t) = -\int_0^\infty b(\sigma)\nabla\vartheta^t(x,\sigma)d\sigma.$$

The energy balance (1.2) now reads

$$\begin{aligned} &c_v\theta_c\partial_t\vartheta + a(0)\vartheta + \int_0^\infty a'(\sigma)\vartheta(x,t-\sigma)d\sigma + \theta_c\lambda'(\chi)\partial_t\chi \\ &-k\Delta\vartheta - \int_0^\infty b(\sigma)\Delta\vartheta(x,t-\sigma)d\sigma = f \end{aligned} \tag{1.4}$$

where $k \geq 0$.

We couple this equation with a Cahn-Hilliard type equation which rules the phase evolution (see, e.g., Brokate-Sprekels [11], Caginalp [12] and Novick–Cohen [50] for its justification)

$$\partial_t\chi - \Delta w = 0 \tag{1.5}$$

where

$$w = -\Delta\chi + \chi^3 + \gamma'(\chi) - \lambda(\chi)\vartheta \tag{1.6}$$

represents the so-called chemical potential and γ is a smooth function.

Concerning initial condition, given at time τ, besides the values of ϑ and χ, it will be necessary to know the past history of ϑ as well

$$\begin{aligned} \vartheta(\tau) &= \vartheta_0 && \text{in } \Omega \\ \chi(\tau) &= \chi_0 && \text{in } \Omega \\ \vartheta(\tau-s) &= \vartheta_0(s) && \text{in } \Omega\times(0,+\infty). \end{aligned}$$

Provided that χ may represent the density of some substance (e.g., a component in an alloy) then the total amount of the substance does not change in time, provided that the mass flux through the boundary is null. Then, the most natural boundary condition associated with (1.5) is the homogeneous Neumann condition for both χ and the chemical potential w

$$\begin{aligned} \partial_{\mathbf{n}}\chi &= 0 \qquad \text{on } \partial\Omega \times (\tau, +\infty) \\ \partial_{\mathbf{n}} w &= 0 \qquad \text{on } \partial\Omega \times (\tau, +\infty). \end{aligned}$$

Therefore, a formal spatial integration of (1.5) yields the conservation of $\displaystyle\int_\Omega \chi d\Omega$. Phase-field models with this feature are commonly named *conserved* models.

As far as ϑ is concerned, we suppose that the Dirichlet homogeneous boundary condition is satisfied on $\Gamma_0 \subseteq \partial\Omega$, while the adiabatic boundary condition holds on $\Gamma_1 = \partial\Omega \setminus \Gamma_0$

$$\begin{aligned} \vartheta &= 0 \qquad \text{on } \Gamma_0 \times (\tau, +\infty) \\ \mathbf{q}\cdot\mathbf{n} &= 0 \qquad \text{on } \Gamma_1 \times (\tau, +\infty). \end{aligned}$$

Notice that the latter condition depends on the explicit expression of the heat flux vector, i.e., on the heat conduction law under consideration.

It is worth mentioning that a detailed analysis of the standard conserved phase-field model without memory effects has been done in Brochet-Hilhorst-Novick–Cohen [10]. In the hyperbolic case with $a \equiv 0$, well-posedness results have also been obtained by Novick–Cohen [51], Colli-Gilardi-Laurençot-Novick–Cohen [26] and, for more general nonlinearities, by Colli-Gilardi-Grasselli-Schimperna [24, 25]. Moreover, in Colli-Gilardi-Laurençot-Novick–Cohen [26], the asymptotic behavior of a single solution is also investigated.

Nevertheless, in all this previous papers, the problem is formulated by incorporating the given past history of the temperature in the heat source term. As pointed out in, e.g., Giorgi-Grasselli-Pata [33] and Giorgi-Marzocchi-Pata [34], this approach does not allow to interpret the problem as a dynamical system.

In order to construct a dynamical system associated with our initial and boundary value problem, we need to formulate the problem in a history space setting. Following Dafermos [30] (cf. also Giorgi-Grasselli-Pata [33], Giorgi-Marzocchi-Pata [34]), we introduce an additional variable, the *summed past history* of ϑ

$$\eta^t(s) = \int_0^s \vartheta^t(y)dy = \int_{t-s}^t \vartheta(y)dy \tag{1.7}$$

and we can check that η (formally) satisfies the first order linear evolution equation

$$\partial_t \eta^t(s) + \partial_s \eta^t(s) = \vartheta(t) \qquad \text{in } \Omega \times (\tau, +\infty) \times (0, +\infty)$$

along with the initial and boundary conditions

$$\begin{aligned} \eta^\tau &= \eta_0 \qquad \text{in } \Omega \times (0, +\infty) \\ \eta^t(0) &= 0 \qquad \text{in } \Omega \times (\tau, +\infty) \end{aligned}$$

where

$$\eta_0(s) = \int_0^s \vartheta_0(y)dy \qquad \text{in } \Omega \times (0, +\infty)$$

is the *initial summed past history* of ϑ and the homogeneous boundary condition is a direct consequence of definition (1.7).

On account of (1.4), we observe that a formal integration by parts yields

$$\begin{aligned} \int_0^\infty b(\sigma)\nabla\vartheta^t(\sigma)d\sigma &= -\int_0^\infty b'(\sigma)\nabla\eta^t(\sigma)d\sigma && \text{in } \Omega \times (\tau, +\infty) \\ \int_0^\infty a'(\sigma)\vartheta^t(\sigma)d\sigma &= -\int_0^\infty a''(\sigma)\eta^t(\sigma)d\sigma && \text{in } \Omega \times (\tau, +\infty). \end{aligned}$$

Then we set

$$\begin{aligned} \mu(s) &= -b'(s) \\ \nu_0\nu(s) &= -a''(s) \end{aligned}$$

for every $s > 0$, where $\nu_0 = 1$ or $\nu_0 = -1$ according to the following choices of a

$$a \text{ is bounded, non-decreasing, and concave} \tag{1.8}$$

or

$$a \text{ is summable, non-increasing, and convex.} \tag{1.9}$$

Both the cases are thermodynamically consistent (see Giorgi-Grasselli-Pata [33] and references therein).

For the sake of simplicity, we set all the constants appearing in equation (1.4) but k equal to 1, in particular $a(0) = 1$.

We observe that the introduction of the summed past history as a third state variable and the related formulation in a history space setting seem unavoidable if one is interested in a structural stability analysis with respect to the initial data and the heat supply.

Then we can reformulate the original initial and boundary value problem as follows.

Problem $\mathbf{P}_k$. Find (ϑ, χ, η) solution to the system

$$\partial_t\vartheta + \partial_t\lambda(\chi) + \vartheta - k\Delta\vartheta + \nu_0 \int_0^\infty \nu(\sigma)\eta(\sigma)d\sigma - \int_0^\infty \mu(\sigma)\Delta\eta(\sigma)d\sigma = f$$

$$\partial_t\chi - \Delta w = 0$$

$$w = -\Delta\chi + \chi^3 + \gamma'(\chi) - \lambda(\chi)\vartheta$$

$$\partial_t\eta^t(s) + \partial_s\eta^t(s) = \vartheta(t)$$

in Ω, for any $t > \tau$, and any $s > 0$, which satisfies the initial and boundary conditions

$$\begin{aligned} \vartheta(\tau) &= \vartheta_0 && \text{in } \Omega \\ \chi(\tau) &= \chi_0 && \text{in } \Omega \\ \eta^\tau &= \eta_0 && \text{in } \Omega \times (0, +\infty) \end{aligned}$$

$$\begin{aligned}
\eta^t(0) &= 0 && \text{in } \Omega \times (\tau, +\infty) \\
\vartheta &= 0 && \text{on } \Gamma_0 \times (\tau, +\infty) \\
\partial_{\mathbf{n}}\chi &= 0 && \text{on } \partial\Omega \times (\tau, +\infty) \\
\partial_{\mathbf{n}}w &= 0 && \text{on } \partial\Omega \times (\tau, +\infty) \\
\eta &= 0 && \text{on } \Gamma_0 \times (\tau, +\infty) \times (0, +\infty) \\
k\partial_{\mathbf{n}}\vartheta(x,t) + \int_0^\infty \mu(\sigma)\partial_{\mathbf{n}}\eta(\sigma)d\sigma &= 0 && \text{on } \Gamma_1 \times (\tau, +\infty) \times (0, +\infty).
\end{aligned}$$

Observe that $\mathbf{P}_k$ includes four distinct models according to the possible choices of ν_0 and k. In particular, we shall refer to the *parabolic* case or to the *hyperbolic* case whenever $k > 0$ or $k = 0$, respectively. Indeed, the latter model entails that ϑ propagates at finite speed (see Joseph-Preziosi [44] and [45]).

The goal of this work is to analyze the well-posedness of problem $\mathbf{P}_k$ and the longtime behavior of its solution. More precisely, we prove some continuous dependence and uniqueness result, which allow to interpret $\mathbf{P}_k$ as a dynamical system. Then, we show the dissipativity of the dynamical system by constructing an absorbing set which is uniform with respect to f. In the parabolic case, we also prove the existence of a uniform attractor of finite fractal (and Hausdorff) dimension.

This analysis extends to the so-called conserved case characterized by (1.5) the results obtained in Giorgi-Grasselli-Pata [33], Giorgi-Marzocchi-Pata [34] and Grasselli-Pata [38] in the non-conserved case, characterized by

$$\partial_t\chi + w = 0.$$

It is also worth mentioning that several other papers have been devoted to the analysis of the non-conserved case with memory effects, but they are concerned with well-posedness issues, asymptotic analysis with respect to certain physical properties, and the longtime behavior of a single solution (cf. Aizicovici-Barbu [1], Bonetti [5], Bonfanti-Luterotti [6, 7], Chiusano-Colli [16], Colli-Gilardi-Grasselli [18, 19, 20, 21], Colli-Laurençot [27, 28], Grasselli [36]).

The plan of the thesis goes as follows. In Chapter 2 we fix some notation and we recall useful basic tools for the reader's convenience. Chapter 3 is devoted to the rigorous variational formulation of problem $\mathbf{P}_k$. Existence is treated in Chapter 4, while Chapter 5 is devoted to continuous dependence and uniqueness. The first sections of the last three chapters contain a concise presentation of dissipative dynamical systems, with definitions and results needed and used in the thesis (we referred and address the reader to classic Haraux [42], Temam [56] and Vishik [57]). In Chapter 6 we prove the existence of a uniform absorbing set both for the parabolic and the hyperbolic case. The existence of a uniform attractor for the parabolic case is showed in Chapter 7, while the question is still open in the hyperbolic case. The finite fractal dimension of the attractor is demonstrated in Chapter 8.

The final Appendix is intended to be helpful for a careful reading of the thesis. It is devoted to recall some well-known compactness results and Gronwall type lemmas.

Chapter 2

Notation and basic tools

Given a real Banach space $(X, \|\cdot\|_X)$, we denote by $\langle \cdot, \cdot \rangle_X$ the duality pairing between the dual space X^* and X, and by $(\cdot, \cdot)_X$ the scalar product in X, whenever X is Hilbert, respectively. Let H_1 and H_2 be two Hilbert spaces such that $H_1 \subset H_2$ with dense and continuous injection. Then, we recall that H_2 can be identified with a subspace of H_1^* by setting

$$\langle u, v \rangle_{H_1} = (u, v)_{H_2}, \qquad \forall\, u \in H_2, \quad \forall\, v \in H_1. \tag{2.1}$$

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded, and connected domain, whose boundary $\Gamma = \partial\Omega$ is smooth. We denote $|\Omega|$ and $|\Gamma|$ the Lebesgue measure of Ω and Γ, respectively. Besides, Γ_0 denote a fixed open subset of Γ.

With standard notation, we introduce the Hilbert spaces $L^p(\Omega)$ and $H^k(\Omega)$. Given a (possibly unbounded) interval $I \subset \mathbb{R}$, we will also make use of the Banach spaces of X-valued functions $C(I; X)$, $L^p(I; X)$, and $H^k(I; X)$ (see, e.g., [31] for details).

For the sake of convenience, we set

$$\begin{aligned}
H &= L^2(\Omega), \\
V &= H^1(\Omega), \\
V_0 &= \Big\{ v \in H^1(\Omega) : v_{|\Gamma} = 0 \quad \text{a.e. on } \Gamma_0 \Big\}, \\
W &= \Big\{ w \in H^2(\Omega) : \partial_{\mathbf{n}} w = 0 \quad \text{a.e. on } \Gamma \Big\}, \\
W_0 &= \Big\{ w \in V_0 \text{ and } \Delta w \in H : \partial_{\mathbf{n}} w = 0 \quad \text{in } (H^{1/2}_{00}(\Gamma_1))^* \Big\},
\end{aligned}$$

where $\Gamma_1 = \Gamma \setminus \Gamma_0$ and (see, e.g., [47])

$$H^{1/2}_{00}(\Gamma_1) = \Big\{ v \in L^2(\Gamma_1) : \exists\, \widetilde{v} \in H^1_{0,\Gamma_0}(\Omega) : \widetilde{v}_{|\Gamma_1} = v \Big\}.$$

Making the standard identification $H \equiv H^*$, we have the following inclusions

$$W \hookrightarrow V \hookrightarrow H \hookrightarrow V^* \hookrightarrow W^* \tag{2.2}$$

$$W_0 \hookrightarrow V_0 \hookrightarrow H \hookrightarrow V_0^* \hookrightarrow W_0^* \tag{2.3}$$

$$V_0 \hookrightarrow V \hookrightarrow H \hookrightarrow V^* \hookrightarrow V_0^*. \tag{2.4}$$

Clearly, when $\Gamma_0 = \emptyset$, $V_0 \equiv V$ and $W_0 \equiv W$.

Given a positive function α defined on $\mathbb{R}^+ = (0, +\infty)$ and a real Hilbert space X, let $L^2_\alpha(\mathbb{R}^+; X)$ be the Hilbert space of X-valued functions on $\mathbb{R}^+$, endowed with the inner product

$$(\varphi, \psi)_{L^2_\alpha(\mathbb{R}^+;X)} = \int_0^\infty \alpha(\sigma)(\varphi(\sigma), \psi(\sigma))_X d\sigma$$

and norm

$$\|\varphi\|_{L^2_\alpha(\mathbb{R}^+;X)} = \left(\int_0^\infty \alpha(\sigma)\|\varphi(\sigma)\|^2_X d\sigma\right)^{1/2}.$$

We will also need to introduce the Hilbert space $H^1_\alpha(\mathbb{R}^+; X)$, defined by

$$H^1_\alpha(\mathbb{R}^+; X) = \Big\{\varphi \in L^2_\alpha(\mathbb{R}^+; X) : \partial_s\varphi \in L^2_\alpha(\mathbb{R}^+; X)\Big\}.$$

We recall that, given a pair of Banach spaces X and Y, the space $X + Y$ turns out to be a Banach space, provided it is endowed with the norm

$$\|u\|_{X+Y} = \inf\big(\|u_1\|_X + \|u_2\|_Y\big)$$

where the infimum is taken among all the possible decompositions $u = u_1 + u_2$, such that $u_1 \in X$ and $u_2 \in Y$.

Moreover, if X and Y are Hilbert, the space $X \cap Y$ is a Hilbert space with the inner product

$$(u, v)_{X\cap Y} = (u, v)_X + (u, v)_Y.$$

In particular, if ν and μ are two positive functions defined on $\mathbb{R}^+$, we introduce as well the Hilbert spaces

$$\begin{aligned}
\mathcal{L} &= L^2_\nu(\mathbb{R}^+; V_0) \cap L^2_\mu(\mathbb{R}^+; W_0)\\
\mathcal{M} &= L^2_\nu(\mathbb{R}^+; H) \cap L^2_\mu(\mathbb{R}^+; V_0)\\
\mathcal{N} &= L^2_\nu(\mathbb{R}^+; V_0^*) \cap L^2_\mu(\mathbb{R}^+; H).
\end{aligned}$$

In order to analyze the longtime behavior of solutions, we also need to consider the Banach space of L^p-translation bounded X-valued functions, namely,

$$\mathcal{T}^p(X) = \left\{f \in L^p_{\mathrm{loc}}(\mathbb{R}; X) : \|f\|_{\mathcal{T}^p(X)} = \sup_{r\in\mathbb{R}}\left(\int_r^{r+1} \|f(\xi)\|^p_X d\xi\right)^{1/p} < +\infty\right\}$$

for every fixed $p \geq 1$. Besides, we recall that a function $g \in L^p_{\mathrm{loc}}(\mathbb{R}; X)$ is said to be *translation compact* in $L^p_{\mathrm{loc}}(\mathbb{R}; X)$ if the set $\{g^r\}_{r\in\mathbb{R}}$ of translates of g, where $g^r(\cdot) = g(\cdot + r)$, is relatively compact in $L^p_{\mathrm{loc}}(\mathbb{R}; X)$. The set

$$\mathrm{H}(g) = \overline{\{g^r\}_{r\in\mathbb{R}}}^{L^p_{\mathrm{loc}}(\mathbb{R};X)}$$

(the closure in $L^p_{\rm loc}(\mathbb{R};X)$) is called the *hull* of g. Hence, a function $f \in \mathrm{H}(g)$ if there exists a sequence $r_n \in \mathbb{R}$ such that

$$g(\cdot + r_n) \longrightarrow f(\cdot) \quad \text{in } L^p_{\rm loc}(\mathbb{R};X).$$

Note that g is translation compact in $L^p_{\rm loc}(\mathbb{R};X)$ if and only if the following two conditions are satisfied (cf. [15] and references therein):

(i) for any $r>0$ the set $\left\{ \int_\xi^{\xi+r} g(y)dy,\ \xi \in \mathbb{R}\right\}$ is relatively compact in X;

(ii) there exists a positive function $\beta(r)$, $r>0$, such that $\beta(r)\to 0$ as $r\to 0^+$, and

$$\int_\xi^{\xi+1} \|g(y)-g(y+r)\|_X^p dy \le \beta(|r|), \qquad \forall\, \xi \in \mathbb{R}.$$

It is apparent that if g is translation compact in $L^p_{\rm loc}(\mathbb{R};X)$, then $g \in \mathcal{T}^p(X)$ and

$$\|f\|_{\mathcal{T}^p(X)} \le \|g\|_{\mathcal{T}^p(X)}, \qquad \forall\, f \in \mathrm{H}(g). \tag{2.5}$$

We point out that the class of translation compact functions in $L^p_{\rm loc}(\mathbb{R};X)$ is quite broad. For example, it contains $L^q(\mathbb{R};X)$ for all $q \ge p$, the constant X-valued functions, and the class of *almost periodic* functions (see [2]).

Since equation (1.5) implies that the medium average of its solution is conserved, it is convenient to define the null-average spaces; that is,

$$V_A = \{v \in V : (v,1)_H = 0\}$$
$$V_A^* = \{v \in V^* : \langle v,1\rangle_V = 0\}.$$

Note first that the inclusions

$$V_A \hookrightarrow V \hookrightarrow V_A^* \hookrightarrow V^*$$

hold. However, V_A^* is not the topological dual of V_A.

We recall that the laplacian $-\Delta$ can be viewed as an operator from V onto V^*. Indeed, for every $v \in V$, we can set

$$\langle -\Delta v, z\rangle_V = \sum_{i=1}^3 \int_\Omega \frac{\partial v}{\partial x_i}\frac{\partial z}{\partial x_i} d\Omega, \qquad \forall\, z \in V. \tag{2.6}$$

Thus $-\Delta$ maps V onto V_A^*. The kernel of $-\Delta$ is the subspace of V of constant functions; it follows that the restriction of $-\Delta$ on V_A maps isomorphically V_A onto V_A^*. Hence we can define the Riesz map $\mathcal{T} : V_A^* \to V_A$ as

$$-\Delta \mathcal{T}\xi = \xi \qquad \forall\, \xi \in V_A^*. \tag{2.7}$$

Thus we can endow V_A^* with the norm

$$\|\xi\|_{V_A^*} = \left(\int_\Omega |\nabla \mathcal{T}\xi|^2 d\Omega\right)^{1/2} = \langle \xi, \mathcal{T}\xi\rangle_V^{1/2}. \tag{2.8}$$

Notice that $\mathcal{T}\xi$ is the (unique) solution with zero mean value to the homogeneous Neumann problem associated with $-\Delta$. Owing to (2.6) and (2.7), the following useful relations hold

$$\langle -\Delta v, \mathcal{T}\xi\rangle_V = \langle \xi, v\rangle_V, \qquad \forall\, v \in V, \quad \forall\, \xi \in V_A^* \tag{2.9}$$

$$\langle \zeta, \mathcal{T}\xi\rangle_V = \langle \xi, \mathcal{T}\zeta\rangle_V = (\mathcal{T}\zeta, \mathcal{T}\xi)_V = (\zeta, \xi)_{V^*}, \qquad \forall\, \zeta, \xi \in V_A^*. \tag{2.10}$$

Analogously, $-\Delta + \mathbb{I}$ is the linear operator from V onto V^* defined by

$$\langle (-\Delta + \mathbb{I})v, z\rangle_V = \sum_{i=1}^{3} \int_\Omega \left(\frac{\partial v}{\partial x_i}\frac{\partial z}{\partial x_i} + v \right) d\Omega, \qquad \forall\, z \in V.$$

Of course, $-\Delta + \mathbb{I}$ maps V onto V^* isomorphically. Hence we can define the Riesz operator $\mathcal{R} : V^* \longrightarrow V$ as

$$(-\Delta + \mathbb{I})\mathcal{R}\xi = \xi, \qquad \forall\, \xi \in V^*.$$

Observe that $\mathcal{R}\xi$ is the (unique) solution to the Neumann problem associated with $-\Delta + \mathbb{I}$, and it satisfies the following relations

$$\langle (-\Delta + \mathbb{I})v, \mathcal{R}\xi\rangle_V = \langle \xi, v\rangle_V, \qquad \forall\, v \in V, \qquad \forall\, \xi \in V^* \tag{2.11}$$

$$\langle \zeta, \mathcal{R}\xi\rangle_V = \langle \xi, \mathcal{R}\zeta\rangle_V = (\mathcal{R}\zeta, \mathcal{R}\xi)_V = (\zeta, \xi)_{V^*}, \qquad \forall\, \xi, \zeta \in V^*. \tag{2.12}$$

We conclude this chapter recalling three well known inequalities, which will be used in the course of the investigation. For the proofs, the reader is referred, for instance, to [9, 56] and [54], respectively (see also LemmaA.2 in Appendix).

Young inequality. For any $a, b \geq 0$, $\varepsilon > 0$ and $p, q > 1$ such that $1/p + 1/q = 1$, there holds

$$ab \leq \frac{\varepsilon}{p} a^p + \frac{1}{q\varepsilon^{q/p}} b^q.$$

Poincaré inequality. There exists a positive constant $c_P = c_P(\Omega)$ (called Poincaré constant) such that

$$\|v\|_H \leq c_P \left\{ \|\nabla v\|_H + p(v) \right\}$$

for any $v \in V$, where $p(v)$ is one of the following continuous seminorms in V

$$\begin{aligned} p(v) &= \left(\int_{\Gamma_0} |v_{|\Gamma}|^2 d\Gamma \right)^{1/2} \\ p(v) &= \left| \int_\Omega v\, d\Omega \right|. \end{aligned}$$

Ehrling inequality. Let X be a Banach space such that $V \subset X \subset H$ with dense and continuous embeddings. Then, for every $\varepsilon > 0$, there exists a positive constant $c_E = c_E(\varepsilon)$ such that

$$\|v\|_X^2 \leq \varepsilon \|\nabla v\|_H^2 + c_E \|v\|_{V^*}^2, \qquad \forall\, v \in V. \tag{2.13}$$

Chapter 3

Variational formulation

Here we introduce the rigorous variational formulations of $\mathbf{P}_k$ with $k > 0$ and $\mathbf{P}_0$ as well as the appropriate assumptions on the data. The related well-posedness issues will be examined in the next Chapters 4 and 5.

We consider two functions ν and μ (memory kernels), which satisfy the assumptions:

$$\begin{array}{llll}
\text{(k1)} & \nu(\sigma),\ \mu(\sigma) \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+) & & \\
\text{(k2)} & \nu(\sigma) \geq 0 & \mu(\sigma) \geq 0 & \forall\, \sigma \in \mathbb{R}^+ \\
\text{(k3)} & \nu'(\sigma) \leq 0 & \mu'(\sigma) \leq 0 & \forall\, \sigma \in \mathbb{R}^+
\end{array}$$

and we set

$$a_0 = \int_0^\infty \nu(\sigma)\, d\sigma \qquad \text{and} \qquad b_0 = \int_0^\infty \mu(\sigma)\, d\sigma$$

in view of (k1)–(k2).

Then we suppose

$$\vartheta_0 \in H \tag{3.1}$$

$$\chi_0 \in V \tag{3.2}$$

$$\eta_0 \in \mathcal{M} \tag{3.3}$$

and

$$\gamma \in C^2(\mathbb{R}) \text{ such that } \gamma',\ \gamma'' \in L^\infty(\mathbb{R}). \tag{3.4}$$

Moreover, we fix $\tau, T \in \mathbb{R}$, $T \geq \tau$ and set $I = [\tau, T]$.

Definition 3.1. Let

$$\lambda \in C^2(\mathbb{R}) \text{ such that } \lambda'' \in L^\infty(\mathbb{R}) \tag{3.5}$$

$$f \in L^1_{\mathrm{loc}}(\mathbb{R}; H) + L^2_{\mathrm{loc}}(\mathbb{R}; V^*). \tag{3.6}$$

A quadruplet $(\vartheta, \chi, w, \eta)$ such that

$$\begin{aligned}
\vartheta &\in C(I;H) \cap L^2(I;V_0)\\
\chi &\in L^\infty(I;V) \cap H^1(I;V^*) \cap L^2(I;W) \subset C(I;H)\\
w &\in L^2(I;V)\\
\eta &\in L^\infty(I;\mathcal{M}) \cap C(I;\mathcal{N})\\
\partial_t(\vartheta + \lambda(\chi)) &\in L^2(I;V_0^*) + L^1(I;H)\\
\partial_t\eta + \partial_s\eta &\in C(I;\mathcal{N}) \cap L^2(I;L^2_\nu(\mathbb{R}^+;H) \cap L^2_\mu(\mathbb{R}^+;V_0))
\end{aligned}$$

is a solution to problem $\mathbf{P}_k$ with $k > 0$ if

$$\langle \partial_t(\vartheta + \lambda(\chi)), v\rangle_{V_0} + (\vartheta, v)_H + k(\nabla\vartheta, \nabla v)_H$$

$$+\nu_0 \int_0^\infty \nu(\sigma)(\eta(\sigma), v)_H d\sigma + \int_0^\infty \mu(\sigma)(\nabla\eta(\sigma), \nabla v)_H d\sigma = \langle f, v\rangle_{V_0}, \qquad \forall\, v \in V_0 \tag{3.7}$$

$$\langle \partial_t\chi, v\rangle_V + (\nabla w, \nabla v)_H = 0, \qquad \forall\, v \in V \tag{3.8}$$

$$\langle v^*, w\rangle_V = \langle v^*, -\Delta\chi + \chi^3 + \gamma'(\chi) - \lambda'(\chi)\vartheta\rangle_V, \qquad \forall\, v^* \in V^* \tag{3.9}$$

$$\int_0^\infty \nu(\sigma)(\partial_t\eta(\sigma) + \partial_s\eta(\sigma), \zeta)_H d\sigma + \int_0^\infty \mu(\sigma)(\partial_t\eta(\sigma) + \partial_s\eta(\sigma), \zeta)_V d\sigma$$

$$= \int_0^\infty \nu(\sigma)(\vartheta, \zeta)_H d\sigma + \int_0^\infty \mu(\sigma)(\vartheta, \zeta)_V d\sigma, \qquad \forall\, \zeta \in \mathcal{M} \tag{3.10}$$

a.e. in I, and

$$\vartheta(\tau) = \vartheta_0 \qquad \text{a.e. in } \Omega \tag{3.11}$$

$$\chi(\tau) = \chi_0 \qquad \text{a.e. in } \Omega \tag{3.12}$$

$$\eta^\tau = \eta_0 \qquad \text{in } V_0^*, \text{ a.e. in } \mathbb{R}^+. \tag{3.13}$$

Observe that, in the hyperbolic case ($k = 0$), assumptions (3.5) and (3.6) need to be strengthened. Indeed, the variational formulation of $\mathbf{P}_0$ reads

Definition 3.2. Let

$$\lambda \in C^2(\mathbb{R}) \text{ such that } \lambda', \lambda'' \in L^\infty(\mathbb{R}) \tag{3.14}$$

$$f \in L^1_{\text{loc}}(\mathbb{R}; H). \tag{3.15}$$

A quadruplet $(\vartheta, \chi, w, \eta)$ such that

$$\begin{aligned}
\vartheta &\in L^\infty(I;H)\\
\chi &\in L^2(I;W) \cap L^\infty(I;V) \cap H^1(I;V^*) \subset C(I;H)\\
w &\in L^2(I;V)\\
\eta &\in L^\infty(I;\mathcal{M})\\
\partial_t(\vartheta + \lambda(\chi)) &\in L^1(I;H) + L^\infty(I;V_0^*)\\
\partial_t\eta + \partial_s\eta &\in L^\infty(I;\mathcal{N})
\end{aligned}$$

is a solution to problem $\mathbf{P}_0$ if

$$\langle \partial_t(\vartheta+\lambda(\chi)), v\rangle_{V_0} + (\vartheta, v)_H$$
$$+\,\nu_0\int_0^\infty \nu(\sigma)(\eta(\sigma), v)_H d\sigma + \int_0^\infty \mu(\sigma)(\nabla\eta(\sigma), \nabla v)_H d\sigma = (f, v)_H, \qquad \forall\, v\in V_0 \tag{3.16}$$

$$\langle \partial_t\chi, v\rangle_V + (\nabla w, \nabla v)_H = 0, \qquad \forall\, v \in V \tag{3.17}$$

$$\langle v^*, w\rangle_V = \langle v^*, -\Delta\chi + \chi^3 + \gamma'(\chi) - \lambda'(\chi)\vartheta\rangle_V, \qquad \forall\, v^*\in V^* \tag{3.18}$$

$$\begin{aligned}
&\int_0^\infty \nu(\sigma)(\partial_t\eta(\sigma)+\partial_s\eta(\sigma),\zeta)_H d\sigma + \int_0^\infty \mu(\sigma)(\partial_t\eta(\sigma)+\partial_s\eta(\sigma),\zeta)_H d\sigma\\
&\quad - \int_0^\infty \mu(\sigma)(\partial_t\eta(\sigma)+\partial_s\eta(\sigma),\Delta\zeta)_H d\sigma\\
&= \int_0^\infty \nu(\sigma)(\vartheta,\zeta)_H d\sigma + \int_0^\infty \mu(\sigma)(\vartheta,\zeta)_H d\sigma\\
&\quad - \int_0^\infty \mu(\sigma)(\vartheta,\Delta\zeta)_H d\sigma, \qquad \forall \zeta\in L^2_\nu(I;H)\cap L^2(I;W_0)
\end{aligned} \tag{3.19}$$

a.e. in I, together with

$$\vartheta(\tau) = \vartheta_0 \qquad \text{in } V_0^* \tag{3.20}$$
$$\chi(\tau) = \chi_0 \qquad \text{a.e. in } \Omega \tag{3.21}$$
$$\eta^\tau = \eta_0 \qquad \text{in } V_0^*, \text{ a.e. in } \mathbb{R}^+. \tag{3.22}$$

Remark 3.3. In the above definitions the operator $-\partial_s$ has to be understood as a linear operator acting on $\mathcal{M}$ with domain

$$\mathcal{D}(-\partial_s) = \Big\{\phi\in H^1_\nu(\mathbb{R}^+;H)\cap H^1_\mu(\mathbb{R}^+;V) : \phi(0)=0\Big\}.$$

The operator A turns out to be the infinitesimal generator of the right-translation semigroup $T_r(t)$, which acts on $\mathcal{M}$ as follows

$$[T_r(t)\phi]\,(s) = \begin{cases} \phi(t-s) & \text{for } s\le t\\ 0 & \text{for } s>t. \end{cases}$$

Remark 3.4. Notice that testing (3.8) with $v=1$ yields

$$\frac{d}{dt}\langle\chi,1\rangle_V = 0, \qquad \text{a.e. in } I.$$

This implies that

$$\langle\chi(t),1\rangle_V = \langle\chi_0,1\rangle_V, \qquad \text{a.e. in } I$$

that is, the mean value of χ in problems $\mathbf{P}_k$ is conserved. This fact will be often used in the sequel.

Remark 3.5. The inclusion

$$H^1(I;V^*)\cap L^\infty(I;V)\subset C(I;H)$$

is a consequence of Lemma A.3 in the Appendix.

Chapter 4

Existence

In this chapter we state and prove existence results for problems $\mathbf{P}_k$. We shall employ a Faedo-Galerkin approximation scheme in order to show existence of solutions to $\mathbf{P}_k$ with $k > 0$. A solution to $\mathbf{P}_0$ will be then obtained performing the *hyperbolic* limit ($k \to 0$) of $\mathbf{P}_k$, as $k \to 0$.

4.1 Main theorems

Theorem 4.1. *Let $k > 0$. Assume that* (k1)–(k3) *and* (3.5)–(3.6) *hold. Then problem $\mathbf{P}_k$ has a solution according to* Definition 3.1.

Theorem 4.2. *Assume that* (k1)–(k3) *and* (3.14)–(3.15) *hold. Then problem $\mathbf{P}_0$ has a solution according to* Definition 3.2.

Remark 4.3. If $k > 0$, the additional hypothesis $\lambda' \in L^\infty(\mathbb{R})$ entails the extra regularity $\chi \in L^2(I; H^3(\Omega))$. Indeed, proceeding by comparison in (3.9), since $\lambda'(\chi)\vartheta \in L^2(I;V)$, we easily recover $\Delta\chi \in L^2(I;V)$.

4.2 Proof of Theorem 4.1

The proof is divided into several steps; we will make use of a Faedo-Galerkin approximating scheme to project problem $\mathbf{P}_k$ on a finite dimensional space, where we can locally solve a Cauchy problem for a system of ordinary differential equations $\mathbf{P}_k^n$, having local solution, by means of standard techniques. Then, we provide some a priori estimates which guarantee that any local solution to problem $\mathbf{P}_k^n$ is actually global on I. The a priori boundedness is also used to extract a subsequence which weakly converges to a certain function, candidate to be the solution to the problem. Due to the presence of non-linearities, before passing to the limit, we will need some extra (strong) convergence, in order to prove that the mentioned weak limit is indeed the solution to the problem $\mathbf{P}_k$.

Faedo-Galerkin approximation. We solve the problem in the case $\Gamma_0 = \emptyset$, so that $V_0 = V$ and $W_0 = W$; this is done for simplifying the notation, but no substantial ingredient is neglected and the argument can be easily adapted to the other cases.

Let $\{v_j\}$ ($j \in \mathbb{N}$) be a sufficiently smooth orthonormal basis of H which is orthogonal in V too. For instance, we can take the set of eigenfunctions of $-\Delta$ with Neumann boundary conditions, normalized in H, that is

$$\begin{aligned} -\Delta v_j &= \alpha_j v_j && \text{in } \Omega \\ \partial_{\mathbf{n}} v_j &= 0 && \text{on } \Gamma \end{aligned}$$

where α_j is the (positive) eigenvalue corresponding to v_j. Next, we take a smooth sequence $\{z_j\}$ forming an orthonormal basis of $\mathcal{M}$.

Let $n \in \mathbb{N}$, and consider the projections P_n and Q_n given by

$$\begin{aligned} P_n &: V \longrightarrow V_n = \mathrm{Span}\{v_1, \dots, v_n\} \subset V \\ Q_n &: \mathcal{M} \longrightarrow \mathcal{M}_n = \mathrm{Span}\{z_1, \dots, z_n\} \subset \mathcal{M}. \end{aligned}$$

Moreover, we fix a decomposition of f, say $f = f^1 + f^2$, and we consider two sequences

$$\{f_n^i\} \subset C(I; H), \qquad i = 1, 2$$

such that

$$\begin{aligned} f_n^1 &\longrightarrow f^1 && \text{in } L^1(I; H) \\ f_n^2 &\longrightarrow f^2 && \text{in } L^2(I; V^*). \end{aligned}$$

We fix n. We are ready to introduce the sequence of approximating problems.

Problem $\mathbf{P}_k^n$. Find $t_n \in (\tau, T]$ and

$$\mathbf{a}_i^n \in C^1([\tau, t_n]; \mathbb{R}^n), \qquad i = 1, 2, 3$$

such that the V_n-valued functions

$$\vartheta_n(t) = \sum_{j=1}^n a_{1j}^n(t) v_j \qquad \text{and} \qquad \chi_n(t) = \sum_{j=1}^n a_{2j}^n(t) v_j$$

and the $\mathcal{M}_n$-valued function

$$\eta_n^t = \sum_{j=1}^n a_{3j}^n(t) z_j$$

fulfill

$$\langle \partial_t(\vartheta_n + \lambda(\chi_n)), v \rangle_V + (\vartheta_n, v)_H + k(\nabla \vartheta_n, \nabla v)_H + \nu_0 \int_0^\infty \nu(\sigma)(\eta_n(\sigma), v)_H d\sigma$$

$$+ \int_0^\infty \mu(\sigma)(\nabla \eta_n(\sigma), \nabla v)_H d\sigma = (f_n^1, v)_H + \langle f_n^2, v \rangle_V, \qquad \forall\, v \in V_n \tag{4.1}$$

$$\langle \partial_t \chi_n, v \rangle_V + (\nabla w_n, \nabla v)_H = 0, \qquad \forall\, v \in V_n \tag{4.2}$$

$$(w_n, v)_H = (-\Delta \chi_n + \chi_n^3 + \gamma'(\chi_n) - \lambda'(\chi_n)\vartheta_n, v)_H, \qquad \forall\, v \in V_n \tag{4.3}$$

$$\langle \partial_t \eta_n, \zeta \rangle_{\mathcal{M}} + \langle \partial_s \eta_n, \zeta \rangle_{\mathcal{M}} = \langle \vartheta_n, \zeta \rangle_{\mathcal{M}}, \qquad \forall\, \zeta \in \mathcal{M}_n \tag{4.4}$$

and

$$\begin{aligned}\vartheta_n(\tau) &= P_n\vartheta_0\\ \chi_n(\tau) &= P_n\chi_0\\ \eta_n^\tau &= Q_n\eta_0\end{aligned}$$

for every $t \in (\tau, t_n)$, where, due to (4.3),

$$w_n = P_n(-\Delta\chi_n + \chi_n^3 + \gamma'(\chi_n) - \lambda'(\chi_n)\vartheta_n).$$

Then

$$\langle -\Delta w_n, v\rangle_V = \langle w_n, -\Delta v\rangle_V = \sum_{j=1}^n \zeta_j\alpha_j\langle w_n, v_j\rangle_V, \qquad \forall\, v \in V_n$$

where ζ_j is the coefficient of the expansion of v in V corresponding to v_j, that is,

$$v = \sum_{j=1}^n \zeta_j v_j.$$

Then the following equations have to be satisfied:

$$\langle\partial_t(\vartheta_n + \lambda(\chi_n)), v\rangle_V + (\vartheta_n, v)_H + k(\nabla\vartheta_n, \nabla v)_H + \nu_0\int_0^\infty \nu(\sigma)(\eta_n(\sigma), v)_H d\sigma$$
$$+ \int_0^\infty \mu(\sigma)(\nabla\eta_n(\sigma), \nabla v)_H d\sigma = (f_n^1, v)_H + \langle f_n^2, v\rangle_V, \qquad \forall\, v \in V_n \tag{4.5}$$

$$\langle\partial_t\chi_n, v\rangle_V + \sum_{j=1}^n \zeta_j\alpha_j\langle -\Delta\chi_n + \chi_n^3 + \gamma'(\chi_n) - \lambda'(\chi_n)\vartheta_n, v_j\rangle_V = 0, \qquad \forall\, v \in V_n \tag{4.6}$$

$$\langle\partial_t\eta_n, \zeta\rangle_{\mathcal{M}} + \langle\partial_s\eta_n, \zeta\rangle_{\mathcal{M}} = \langle\vartheta_n, \zeta\rangle_{\mathcal{M}}, \qquad \forall\, \zeta \in \mathcal{M}_n. \tag{4.7}$$

We now transform (4.5)–(4.7) and the initial conditions of $\mathbf{P}_k^n$ into a Cauchy problem for a non linear system of ordinary differential equations, choosing $v = v_\kappa$, $\kappa = 1, \dots, n$ in (4.5), (4.6) and $\zeta = z_\kappa$ in (4.7).

We may rewrite problem $\mathbf{P}_k^n$ as

$$(\partial_t\mathbf{a}_1^n(t), \partial_t\mathbf{a}_2^n(t), \partial_t\mathbf{a}_3^n(t)) = \mathbf{F}^n(t, \mathbf{a}_1^n(t), \mathbf{a}_2^n(t), \mathbf{a}_3^n(t))$$

where $\mathbf{F}^n : [\tau, t_n] \times \mathbb{R}^{3n} \longrightarrow \mathbb{R}^{3n}$ is continuous with respect to its arguments, uniformly in t, by the continuity of the projections and of the approximations of f. We can then deduce the local existence of a C^1-solution $(\vartheta_n, \chi_n, \eta_n)$.

A priori estimates. The following estimates, in particular, ensure that we can take $t_n = T$ for every n. Through this proof, let c denote a generic constant (depending at most on $|I|$, $|\Omega|$, a_0, b_0, k, but independent of n) which may vary even in the same line. Further dependence of the constants will be pointed out on occurrence.

Let $v = \vartheta_n$ in (4.1) to get

$$\frac{1}{2}\frac{d}{dt}\|\vartheta_n\|_H^2 + \|\vartheta_n\|_H^2 + k\|\nabla\vartheta_n\|_H^2 + \langle\lambda'(\chi_n)\partial_t\chi_n, \vartheta_n\rangle_V + \nu_0\int_0^\infty \nu(\sigma)(\eta_n(\sigma), \vartheta_n)_H d\sigma$$
$$+ \int_0^\infty \mu(\sigma)(\nabla\eta_n(\sigma), \nabla\vartheta_n)_H d\sigma = (f_n^1, \vartheta_n)_H + \langle f_n^2, \vartheta_n\rangle_V. \tag{4.8}$$

Since by (4.2) we have that $\langle\partial_t\chi_n, 1\rangle_V = 0$, we can apply the Riesz map $\mathcal{T}$, defined in (2.7), to the function $\partial_t\chi_n$. Let then $v = \mathcal{T}\partial_t\chi_n$ in (4.2). Using (2.8)–(2.9), we have

$$\|\partial_t\chi_n\|_{V^*}^2 + \langle\partial_t\chi_n, w_n\rangle_V = 0.$$

Consider now equation (4.3) with $v = \partial_t\chi_n$. This yields

$$\|\partial_t\chi_n\|_{V^*}^2 + \frac{1}{2}\frac{d}{dt}\left(\|\nabla\chi_n\|_H^2 + \frac{1}{2}\|\chi_n\|_{L^4(\Omega)}^4 + (\gamma(\chi_n), 2)_H\right) - \langle\partial_t\chi_n, \lambda'(\chi_n)\vartheta_n\rangle_V = 0. \tag{4.9}$$

Then, taking $z = \eta_n$ in (4.4), we deduce

$$\frac{1}{2}\frac{d}{dt}\|\eta_n\|_{\mathcal{M}}^2 \le \int_0^\infty \nu(\sigma)(\eta_n(\sigma), \vartheta_n)_H d\sigma + \int_0^\infty \mu(\sigma)(\eta_n(\sigma), \vartheta_n)_V d\sigma. \tag{4.10}$$

The term $(\partial_s\eta_n, \eta_n)_{\mathcal{M}}$ has been neglected for integration by parts entails

$$\begin{aligned}
&(\partial_s\eta_n, \eta_n)_{\mathcal{M}} \\
&= \frac{1}{2}\left\{\int_0^\infty \nu(\sigma)\frac{d}{d\sigma}\|\eta_n(\sigma)\|_H^2 d\sigma + \int_0^\infty \mu(\sigma)\frac{d}{d\sigma}\|\eta_n(\sigma)\|_V^2 d\sigma\right\} \\
&= -\frac{1}{2}\left\{\int_0^\infty \nu'(\sigma)\|\eta_n(\sigma)\|_H^2 d\sigma + \int_0^\infty \mu'(\sigma)\|\eta_n(\sigma)\|_V^2 d\sigma\right\} \\
&\le 0
\end{aligned}$$

Addition of (4.8)–(4.10) together with the elementary inequality

$$\frac{1}{2}\frac{d}{dt}\|\chi_n\|_H^2 \le \frac{1}{2}\|\chi_n\|_V^2 + \frac{1}{2}\|\partial_t\chi_n\|_{V^*}^2$$

lead to

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\vartheta_n\|_H^2 + \|\chi_n\|_V^2 + \frac{1}{2}\|\chi_n\|_{L^4(\Omega)}^4 + (\gamma(\chi_n), 2)_H + \|\eta_n\|_{\mathcal{M}}^2\right) \\
&\quad + \|\vartheta_n\|_H^2 + k\|\nabla\vartheta_n\|_H^2 + \frac{1}{2}\|\partial_t\chi_n\|_{V^*}^2 \\
&\le 2\int_0^\infty \nu(\sigma)\,|(\eta_n(\sigma), \vartheta_n)_H|\,d\sigma \\
&\quad + \int_0^\infty \mu(\sigma)(\eta_n(\sigma), \vartheta_n)_H d\sigma + \frac{1}{2}\|\chi_n\|_V^2 + (f_n^1, \vartheta_n)_H + \langle f_n^2, \vartheta_n\rangle_V.
\end{aligned}$$

Set now

$$\Phi_n^2(t) = \|\vartheta_n(t)\|_H^2 + \|\chi_n(t)\|_V^2 + \frac{1}{2}\|\chi_n(t)\|_{L^4(\Omega)}^4 + 2\int_\Omega \gamma(\chi_n(t))d\Omega + \|\eta_n^t\|_{\mathcal{M}}^2.$$

By means of repeated applications of Young inequality, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Phi_n^2+\frac{k}{2}\|\nabla\vartheta_n\|_H^2+\frac{1}{2}\|\partial_t\chi_n\|_{V^*}^2\\&\le\frac{1}{2}(a_0+b_0-2)\|\vartheta_n\|_H^2+\frac{1}{2}\|\chi_n\|_V^2+\frac{1}{2}\|\eta_n\|_{\mathcal{M}}^2+\|f_n^1\|_H\|\vartheta_n\|_H+\frac{1}{2k}\|f_n^2\|_{V^*}^2.\end{aligned}$$

Performing an appropriate choice of the constants c_1, c_2, whose values are independent of n, we are led to the inequality

$$\frac{d}{dt}\Phi_n^2+c_1\Big(k\|\vartheta_n\|_V^2+\|\partial_t\chi_n\|_{V^*}^2\Big)\le c_2\Big(\Phi_n^2+\|f_n^1\|_H\Phi_n+\|f_n^2\|_{V^*}^2\Big). \tag{4.11}$$

Invoking Lemma A.5 in Appendix, we finally get

$$\Phi_n^2(t)\le 2e^{c_2|I|}\Phi_n^2(\tau)+c_2^2e^{2c_2|I|}\|f_n^1\|_{L^1(I;H)}^2+2c_2e^{c_2|I|}\|f_n^2\|_{L^2(I;V^*)}^2 \tag{4.12}$$

for every $t\in[\tau,t_n]$.

In particular, we get that $t_n=T$ for every n.

By (4.12), we learn that the sequence $(\vartheta_n,\chi_n,\eta_n)$ is uniformly bounded in norm in the space $H\times V\times\mathcal{M}$ by a constant depending only on the size of initial data and on the source term f. More precisely, we have

$$\begin{aligned}\|\vartheta_n\|_{L^\infty(I;H)\cap L^2(I;V)}&\le c &(4.13)\\ \|\chi_n\|_{L^\infty(I;V)\cap H^1(I;V^*)}&\le c &(4.14)\\ \|\eta_n\|_{L^\infty(I;\mathcal{M})}&\le c. &(4.15)\end{aligned}$$

We can also prove a further regularity of χ_n. Indeed, taking $v=\chi_n$ in (4.2) and $v=-\Delta\chi_n\in V_n$ in (4.3), we deduce

$$\frac{1}{2}\frac{d}{dt}\|\chi_n\|_H^2+\|\Delta\chi_n\|_H^2+3(\chi_n^2\nabla\chi_n,\nabla\chi_n)_H\le\langle\gamma'(\chi_n),\Delta\chi_n\rangle_V-\langle\lambda'(\chi_n)\vartheta_n,\Delta\chi_n\rangle_V$$

and using Young inequality we infer

$$\frac{d}{dt}\|\chi_n\|_H^2+\|\Delta\chi_n\|_H^2\le 3\Big(\|\gamma'(\chi_n)\|_H^2+\|\lambda'(\chi_n)\vartheta_n\|_H^2\Big). \tag{4.16}$$

Observe that

$$\|\lambda'(\chi_n)\vartheta_n\|_{L^2(I;H)}\le c$$

due to the Lipschitz continuity of λ', the boundedness of χ_n in $L^\infty(I;V)$, and the fact that $\vartheta_n\in L^2(I;V)$. Lipschitz continuity of γ', together with the boundedness of χ_n in $L^2(I;H)$, also entails

$$\|\gamma'(\chi_n)\|_{L^2(I;H)}\le c.$$

Then, integration in time on I of equation (4.16) bears

$$\|\chi_n\|_{L^2(I;W)\cap L^\infty(I;V)\cap H^1(I;V^*)}\le c. \tag{4.17}$$

By comparison in (3.9), the relations $\chi_n \in L^\infty(I;V) \cap L^2(I;W)$ and $\chi_n^3 \in L^2(I;V)$ entail

$$\|w_n\|_{L^2(I;H)} \leq c.$$

Consider then equation (3.8), with $v = w_n$, to get

$$\|\nabla w_n\|_H^2 = \langle \partial_t \chi_n, w_n \rangle_V \leq \frac{1}{2}\|\partial_t \chi_n\|_{V^*}^2 + \frac{1}{2}\|w_n\|_V^2.$$

Hence we obtain the further bound

$$\|w_n\|_{L^2(I;V)} \leq c. \tag{4.18}$$

Passage to the limit. Along this section we shall make use of lemmas cited in the Appendix. From (4.13), (4.15) and (4.17), (4.18) we argue that, up to subsequences,

$$\begin{aligned}
\vartheta_n &\longrightarrow \vartheta && \text{weakly}^* \text{ in } L^\infty(I;H) && (4.19)\\
\vartheta_n &\longrightarrow \vartheta && \text{weakly in } L^2(I;V) && (4.20)\\
\chi_n &\longrightarrow \chi && \text{weakly}^* \text{ in } L^\infty(I;V) && (4.21)\\
\chi_n &\longrightarrow \chi && \text{weakly in } L^2(I;W) \cap H^1(I;V^*) && (4.22)\\
\eta_n &\longrightarrow \eta && \text{weakly}^* \text{ in } L^\infty(I;\mathcal{M}) && (4.23)\\
w_n &\longrightarrow w && \text{weakly in } L^2(I;V). && (4.24)
\end{aligned}$$

Exploiting the compactness argument of Lemma A.3, (4.21) and (4.22) imply

$$\chi_n \longrightarrow \chi \quad \text{strongly in } L^2(I;V) \cap C(I;H). \tag{4.25}$$

In particular, up to a subsequence,

$$\chi_n \to \chi \quad \text{and} \quad \chi_n^3 \to \chi^3 \quad \text{a.e. on } \Omega \times I.$$

Moreover, since $\chi \in L^\infty(I;V)$, it is easy to see that

$$\|\chi_n^3\|_{L^2(I\times\Omega)} \leq c.$$

Then, by Lemma A.1 we deduce that

$$\chi_n^3 \longrightarrow \chi^3 \quad \text{weakly in } L^2(I;H).$$

On account of (3.4)–(3.5), thanks to (4.25), we get at once

$$\gamma'(\chi_n) \longrightarrow \gamma'(\chi) \qquad \text{and} \qquad \lambda'(\chi_n) \longrightarrow \lambda'(\chi) \quad \text{strongly in } C(I;H).$$

Furthermore, due to (3.5), we can find a positive constant c such that

$$\begin{aligned}
|\lambda(r)| &\leq c(1+r^2), && \forall\, r \in \mathbb{R} && (4.26)\\
|\lambda(r_1) - \lambda(r_2)| &\leq c(1+|r_1|+|r_2|)|r_1 - r_2|, && \forall\, r_1, r_2 \in \mathbb{R} && (4.27)\\
|\lambda'(r)| &\leq c(1+|r|), && \forall\, r \in \mathbb{R} && (4.28)\\
|\lambda'(r_1) - \lambda'(r_2)| &\leq c|r_1 - r_2|, && \forall\, r_1, r_2 \in \mathbb{R}. && (4.29)
\end{aligned}$$

Therefore, using the continuous embedding $L^p(\Omega) \hookrightarrow V$, for $1 \le p \le 6$, we have the inequalities

$$
\begin{aligned}
\|\lambda(\chi_n) - \lambda(\chi)\|_H^2 &\le c \int_\Omega \big|1 + \chi_n^2 + \chi^2\big| \big|\chi_n - \chi\big|^2 d\Omega \le c \int_\Omega \big|1 + \chi_n^3 + \chi^3\big| \big|\chi_n - \chi\big| d\Omega \\
&\le c \left(\int_\Omega \big|1 + |\chi_n|^3 + |\chi|^3\big|^2 d\Omega \right)^{1/2} \left(\int_\Omega \big|\chi_n - \chi\big|^2 d\Omega \right)^{1/2} \\
&\le c \left(1 + \|\chi_n\|_V^3 + \|\chi\|_V^3\right) \|\chi_n - \chi\|_H
\end{aligned}
\tag{4.30}
$$

and

$$
\begin{aligned}
\|\lambda'(\chi_n) - \lambda'(\chi)\|_{L^4(\Omega)}^4 &\le c \int_\Omega \big||\chi_n| + |\chi|\big|^3 \big|\chi_n - \chi\big| d\Omega \\
&\le c \left(\int_\Omega \big||\chi_n| + |\chi|\big|^6 d\Omega \right)^{1/2} \left(\int_\Omega \big|\chi_n - \chi\big|^2 d\Omega \right)^{1/2} \\
&\le c \left(\|\chi_n\|_V^3 + \|\chi\|_V^3 \right) \|\chi_n - \chi\|_H .
\end{aligned}
\tag{4.31}
$$

Since χ_n and χ are uniformly bounded in $L^\infty(I;V)$, inequalities (4.30)–(4.31), and the strong convergence of χ_n to χ in $C(I;H)$, entail

$$\lambda(\chi_n) \longrightarrow \lambda(\chi) \qquad \text{strongly in } C(I;H) \tag{4.32}$$

$$\lambda'(\chi_n) \longrightarrow \lambda'(\chi) \qquad \text{strongly in } C(I;L^4(\Omega)) \tag{4.33}$$

$$\lambda'(\chi_n)\vartheta_n \longrightarrow \lambda'(\chi)\vartheta \qquad \text{weakly in } L^2(I;L^{4/3}(\Omega)) \tag{4.34}$$

the last being a consequence of (4.20) and (4.33).

We can now pass to the limit in each term of equations (4.2)–(4.3).

Concerning equation (4.4), we may choose $\zeta \in \mathcal{D}(I;\mathcal{D}(\mathbb{R}^+;V))$. Then, following [33], an integration by parts yields

$$\int_\tau^T (\vartheta_n, \zeta)_{\mathcal{M}} \longrightarrow \int_\tau^T (\vartheta, \zeta)_{\mathcal{M}}$$

$$\int_\tau^T (\partial_t \eta_n, \zeta)_{\mathcal{M}} = -\int_\tau^T (\eta_n, \partial_t \zeta)_{\mathcal{M}} \longrightarrow \int_\tau^T (\eta, \partial_t \zeta)_{\mathcal{M}}$$

$$
\begin{aligned}
\int_\tau^T (\partial_s \eta_n, \zeta)_{\mathcal{M}} &= -\int_\tau^T \int_0^\infty \Big(\nu'(\eta_n, \zeta)_H + \mu'(\eta_n, \zeta)_V + \nu(\eta_n, \partial_s \zeta)_H + \mu(\eta_n, \partial_s \zeta)_V \Big) \\
&\longrightarrow -\int_\tau^T \int_0^\infty \Big(\nu'(\eta, \zeta)_H + \mu'(\eta, \zeta)_V + \nu(\eta, \partial_s \zeta)_H + \mu(\eta, \partial_s \zeta)_V \Big).
\end{aligned}
$$

Exploiting the distributional equality

$$
\begin{aligned}
&\int_\tau^T \Big((\eta, \partial_t \zeta)_{\mathcal{M}} + \int_0^\infty \big(\nu'(\eta, \zeta)_H + \mu'(\eta, \zeta)_V + \nu(\eta, \partial_s \zeta)_H + \mu(\eta, \partial_s \zeta)_V \big) \Big) \\
&= -\int_\tau^T (\partial_t \eta + \partial_s \eta, \zeta)_{\mathcal{M}}
\end{aligned}
$$

and using a density argument, we get (3.10).

To pass to the limit in (4.1) it is necessary first to perform an integration of (4.1) over (τ, t); then, passing to the limit, we get

$$\begin{aligned}&(\vartheta(t), v)_H + (\lambda(\chi(t)), v)_H + \int_\tau^t (\vartheta, v)_H + k \int_\tau^t (\nabla\vartheta, \nabla v)_H \\ &+\nu_0 \int_\tau^t\int_0^\infty \nu(\eta, v)_H + \int_\tau^t\int_0^\infty \mu(\nabla\eta, \nabla v)_H \\ &= (\vartheta_0, v)_H + (\lambda(\chi_0), v)_H + \int_\tau^t \big((f^1, v)_H + \langle f^2, v\rangle_V\big)\,.\end{aligned}$$

This equality holds in a distributional sense, and we can write

$$\begin{aligned}&\int_\tau^T (\vartheta(t), \phi(t))_H + \int_\tau^T (\lambda(\chi(t)), \phi(t))_H + \int_\tau^T\int_\tau^t (\vartheta, \phi)_H \\ &+k \int_\tau^T\int_\tau^t (\nabla\vartheta, \nabla\phi)_H + \nu_0 \int_\tau^T\int_\tau^t\int_0^\infty \nu(\eta, \phi)_H + \int_\tau^T\int_\tau^t\int_0^\infty \mu(\nabla\eta, \nabla\phi)_H \\ &= \int_\tau^T (\vartheta_0, \phi(t))_H + \int_\tau^T (\lambda(\chi_0), \phi(t))_H + \int_\tau^T\int_\tau^t \big((f^1, \phi)_H + \langle f^2, \phi\rangle_V\big)\end{aligned}$$

for every $\phi \in \mathcal{D}(I; V)$.

Taking the derivative with respect to t, we have

$$\begin{aligned}&\langle \partial_t(\vartheta + \lambda(\chi)), \phi\rangle_{\mathcal{D}} + \int_\tau^T (\vartheta, \phi)_H + k \int_\tau^T (\nabla\vartheta, \nabla\phi)_H \\ &+\nu_0 \int_\tau^T\int_0^\infty \nu(\eta, \phi)_H + \int_\tau^T\int_0^\infty \mu(\nabla\eta, \nabla\phi)_H \\ &= \int_\tau^T \big((f^1, \phi)_H + \langle f^2, \phi\rangle_V\big)\,.\end{aligned}$$

We deduce by comparison that

$$\partial_t(\vartheta + \lambda(\chi)) \in L^2(I; V^*) + L^1(I; H).$$

Equation (4.20), and the estimate

$$\|\nabla\lambda(\chi)\|_H^2 \le c \int_\Omega \left|1 + \chi^2\right| |\nabla\chi|^2 \, d\Omega \le c\Big(1 + \|\chi\|_V^2\Big) \|\chi\|_W^2$$

lead to

$$\vartheta + \lambda(\chi) \in L^2(I; V).$$

Finally, due to 3.5 Ch. XVIII in [31], we obtain

$$\vartheta + \lambda(\chi) \in C(I; H).$$

Therefore, the relation

$$\vartheta \in C(I; H)$$

is a straightforward consequence, in force of (4.32). □

4.3 Proof of Theorem 4.2

We want to exploit the results developed in the previous proof. Again, for the sake of simplicity, we limit ourselves to treat the case $\Gamma_0 = \emptyset$.

We introduce an auxiliary problem, whose solution is a regularized approximation of problem $\mathbf{P}_0$. Introducing a smoothing, or viscosity effect, represented by $-k\Delta\vartheta$, we solve then

$$\langle \partial_t(\vartheta + \lambda(\chi)), v\rangle_V + (\vartheta, v)_H + k(\nabla\vartheta, v)_H + \nu_0 \int_0^\infty \nu(\sigma)(\eta(\sigma), v)_H d\sigma$$
$$+ \int_0^\infty \mu(\sigma)(\nabla\eta(\sigma), \nabla)_H d\sigma = (f, v)_H, \qquad \forall\, v \in V \text{ a.e. in } I \tag{4.35}$$

where $k > 0$ together with (3.17)–(3.22).

Clearly, problem (4.35), (3.17)–(3.22) admits a solution $(\vartheta_k, \chi_k, w_k, \eta_k)$, according to Theorem 4.1. We search for estimates of the solution uniform with respect to k and point out the explicit dependence of the solution on the parameter.

Setting
$$\Phi_k^2(t) = \|\vartheta_k(t)\|_H^2 + \|\chi_k(t)\|_V^2 + \frac{1}{2}\|\chi_k(t)\|_{L^4(\Omega)}^4 + \|\eta_k^t\|_{\mathcal{M}}^2$$
we easily get (see (3.15) and (4.11))

$$\frac{d}{dt}\Phi_k^2 + \varepsilon\|\nabla\vartheta_k\|_H^2 + \|\partial_t\chi_k\|_{V^*}^2 \le c\Big(\Phi_k^2(t) + \|f\|_H\Phi_k\Big). \tag{4.36}$$

Here and in the sequel, the generic constant c is independent of k. Applying Lemma A.5 in the Appendix to (4.36), we get

$$\|\vartheta_k\|_{L^\infty(I;H)} + \sqrt{k}\|\nabla\vartheta_k\|_{L^2(I;H)} \le c \tag{4.37}$$
$$\|\chi_k\|_{L^\infty(I;V)\cap H^1(I;V^*)} \le c \tag{4.38}$$
$$\|\eta_k\|_{L^\infty(I;\mathcal{M})} \le c. \tag{4.39}$$

We now consider equation (3.17) with $v = \chi_\varepsilon$ and (3.18) with $v = -\Delta\chi_\varepsilon$. This is possible in light of Remark 4.3. We get

$$\frac{1}{2}\frac{d}{dt}\|\chi_k\|_H^2 + \|\Delta\chi_k\|_H^2 + (\nabla\chi_k^3, \nabla\chi_k)_H \le (\gamma'(\chi_k), \Delta\chi_k)_H - (\lambda'(\chi_k)\vartheta_k, \Delta\chi_k)_H.$$

Again, with hypothesis $\lambda' \in L^\infty(\mathbb{R})$, Young inequality gives

$$\frac{d}{dt}\|\chi_k\|_H^2 + \|\Delta\chi_k\|_V^2 \le c\big(1 + \|\chi_k\|_H^2 + \|\vartheta_k\|_H^2\big)$$

and an integration over I, on account of (4.37)–(4.38), leads to

$$\|\chi_k\|_{L^2(I;W)} \le c.$$

By the same argument used in the previous proof, we deduce also

$$\|w_k\|_{L^2(I;V)} \le c. \tag{4.40}$$

Therefore, up to subsequence, we find the following convergences:

$$\begin{aligned}
\vartheta_k &\longrightarrow \vartheta && \text{weakly}^* \text{ in } L^\infty(I;H)\\
\chi_k &\longrightarrow \chi && \text{weakly}^* \text{ in } L^\infty(I;V)\\
\chi_k &\longrightarrow \chi && \text{weakly in } L^2(I;W)\cap H^1(I;V^*)\\
\eta_k &\longrightarrow \eta && \text{weakly}^* \text{ in } L^\infty(I;\mathcal{M})\\
w_k &\longrightarrow w && \text{weakly in } L^2(I;V)
\end{aligned}$$

as $k \to 0$.

Moreover, from (4.37), we also learn that

$$\|k\nabla\vartheta_k\|_{L^2(I;H^3)} \le c\sqrt{k}.$$

This implies that

$$k\nabla\vartheta_k \longrightarrow 0 \quad \text{strongly in } L^2(I;H^3)$$

as $k \to 0$. In particular

$$k(\nabla\vartheta_k,\nabla v)_H \longrightarrow 0, \qquad \forall\, v\in V, \text{ a.e. in } I.$$

Thus, arguing as in the previous proof, we conclude that the quadruplet (ϑ,χ,w,η) solves equations (3.16)–(3.18).

Concerning equation (3.19), we need some extra work. So far, we know that η_ε satisfies

$$(\partial_t\eta_\varepsilon + \partial_s\eta_\varepsilon,\zeta)_{\mathcal{M}} = (\vartheta_\varepsilon,\zeta)_{\mathcal{M}}, \qquad \forall\zeta\in\mathcal{M}.$$

In particular, for every $\zeta\in\mathcal{D}(I;\mathcal{D}(\mathbb{R}^+;W))\subset\mathcal{M}$ we have that

$$\begin{aligned}
&\int_0^\infty \nu(\sigma)(\partial_t\eta_\varepsilon(\sigma)+\partial_s\eta_\varepsilon(\sigma),\zeta)_H d\sigma + \int_0^\infty \mu(\sigma)(\partial_t\eta_\varepsilon(\sigma)+\partial_s\eta_\varepsilon(\sigma),\zeta)_H d\sigma\\
&\quad - \int_0^\infty \mu(\sigma)(\partial_t\eta_\varepsilon(\sigma)+\partial_s\eta_\varepsilon(\sigma),\Delta\zeta)_H d\sigma\\
&= \int_0^\infty \nu(\sigma)(\vartheta_\varepsilon,\zeta)_H d\sigma + \int_0^\infty \mu(\sigma)(\vartheta_\varepsilon,\zeta)_H d\sigma - \int_0^\infty \mu(\sigma)(\vartheta_\varepsilon,\Delta\zeta)_H d\sigma
\end{aligned}$$

Using the weak* convergence of ϑ_ε to ϑ in $L^\infty(I;H)$, together with the convergence

$$\partial_t\eta_\varepsilon + \partial_s\eta_\varepsilon \longrightarrow \partial_t\eta + \partial_s\eta$$

in $\mathcal{D}'(I;\mathcal{D}'(\mathbb{R}^+;H))$, we get the equality

$$\begin{aligned}
&\int_0^\infty \nu(\sigma)(\partial_t\eta(\sigma)+\partial_s\eta(\sigma),\zeta)_H d\sigma + \int_0^\infty \mu(\sigma)(\partial_t\eta(\sigma)+\partial_s\eta(\sigma),\zeta)_H d\sigma\\
&\quad - \int_0^\infty \mu(\sigma)(\partial_t\eta(\sigma)+\partial_s\eta(\sigma),\Delta\zeta)_H d\sigma\\
&= \int_0^\infty \nu(\sigma)(\vartheta,\zeta)_H d\sigma + \int_0^\infty \mu(\sigma)(\vartheta,\zeta)_H d\sigma - \int_0^\infty \mu(\sigma)(\vartheta,\Delta\zeta)_H d\sigma
\end{aligned}$$

for every $\zeta\in\mathcal{D}(I;\mathcal{D}(\mathbb{R}^+;W))\subset\mathcal{M}$. Equation (3.19) is then satisfied, since $\mathcal{D}(I;\mathcal{D}(\mathbb{R}^+;W))$ is dense in $L^2_\nu(I;H)\cap L^2_\mu(I;W)$. □

Chapter 5

Continuous dependence and uniqueness

In this chapter we state and prove continuous dependence estimates for solutions to problems $\mathbf{P}_k$. Uniqueness straightforwardly follows from these results.

As we shall see, some restriction on λ are needed. More precisely, if $k > 0$ then (3.5) is replaced by (3.14); while in the case $k = 0$, λ must be linear, namely

$$\lambda(r) = \lambda_0 r, \qquad \forall\, r \in \mathbb{R}. \tag{5.1}$$

These restrictions seem difficult to remove.

5.1 Main theorems

Theorem 5.1. *Let $k > 0$. Assume (k1)–(k3), (3.4) and (3.14). Let $\{f_i, \vartheta_{0i}, \chi_{0i}, \eta_{0i}\}$, $i = 1, 2$, be two sets of data which satisfy (3.6) and (3.1)–(3.3), respectively, and set*

$$m_{\chi_1} = \int_\Omega \chi_{01} d\Omega \qquad \textit{and} \qquad m_{\chi_2} = \int_\Omega \chi_{02} d\Omega.$$

Denoting by $\{\vartheta_i, \chi_i, w_i, \eta_i\}$ two corresponding solutions of problem $\mathbf{P}_k$, there exists $c > 0$ such that, for any $t \in I$,

$$\begin{aligned}
&\|\vartheta_1(t) - \vartheta_2(t)\|^2_{V^*} + \|\chi_1(t) - \chi_2(t)\|^2_{V^*} + \|\eta_1^t - \eta_2^t\|^2_{\mathcal{N}} \\
&\quad + k\int_\tau^t \|\vartheta_1(y) - \vartheta_2(y)\|^2_H dy + \int_\tau^t \|\chi_1(y) - \chi_2(y)\|^2_V dy \\
&\le c\Big(\|\vartheta_{01} - \vartheta_{02}\|^2_{V^*} + \|\chi_{01} - \chi_{02}\|^2_{V^*} + \|\eta_{01} - \eta_{02}\|^2_{\mathcal{N}} \\
&\quad + |m_{\chi_1} - m_{\chi_2}|^2 + \|f_1 - f_2\|^2_{L^1(I;V^*)} \Big).
\end{aligned} \tag{5.2}$$

In particular, problem $\mathbf{P}_k$ has a unique solution.

Theorem 5.2. *Assume* (k1)–(k3), (3.4) *and* (5.1). *Let* $\{f_i, \vartheta_{0i}, \chi_{0i}, \eta_{0i}\}$, $i = 1, 2$, *be two sets of data which satisfy* (3.15) *and* (3.1)–(3.3), *respectively, Denoting by* $\{\vartheta_i, \chi_i, w_i, \eta_i\}$ *two corresponding solutions of problem* $\mathbf{P}_0$, *then estimate* (5.2) *holds, with* $k = 0$. *In particular, problem* $\mathbf{P}_0$ *has a unique solution.*

Remark 5.3. The constant c appearing in estimate (5.2) depends (increasingly) at most on $|I|$, on the norm of the initial data and on the structural constants of the problem, but is independent of k. Further dependence in the generic constant c appearing in this chapter will be pointed out on occurrence.

When λ is linear, i.e. when (5.1) hold, we obtain a stronger continuous dependence estimate, which will be basic in the sequel (see next chapter).

Theorem 5.4. *Assume* (k1)–(k3), (3.15) *and* (5.1), (3.1)–(3.3) *hold. Then, the unique solution to problem* $\mathbf{P}_k$ *has the following properties*

$$\begin{aligned} \vartheta &\in C(I; H) \\ \chi &\in C(I; V) \\ \eta &\in C(I; \mathcal{M}). \end{aligned}$$

Moreover, for every $t \in I$ *and every* $\tau \in \mathbb{R}$, *the operator*

$$(\vartheta_0, \chi_0, \eta_0) \mapsto (\vartheta(t), \chi(t), \eta^t)$$

is continuous from $H \times V \times \mathcal{M}$ *to itself. Indeed, if* $\{f_i, \vartheta_{0i}, \chi_{0i}, \eta_{0i}\}$, $i = 1, 2$, *are two sets of data and* $\{\vartheta_i, \chi_i, w_i, \eta_i\}$ *the corresponding solutions to problem* $\mathbf{P}_k$, *there exists* $c > 0$ *such that, for any* $t \in I$,

$$\begin{aligned} &\|\vartheta_1(t) - \vartheta_2(t)\|_H^2 + \|\chi_1(t) - \chi_2(t)\|_V^2 + \|\eta_1^t - \eta_2^t\|_{\mathcal{M}}^2 \\ &\quad + k \int_\tau^t \|\vartheta_1(y) - \vartheta_2(y)\|_V^2 dy + \int_\tau^t \|\partial_t \chi_1(y) - \partial_t \chi_2(y)\|_{V^*}^2 dy \\ &\leq c\Big(\|\vartheta_{01} - \vartheta_{02}\|_H^2 + \|\chi_{01} - \chi_{02}\|_V^2 + \|\eta_{01} - \eta_{02}\|_{\mathcal{M}}^2 + \|f_1 - f_2\|_{L^1(I;V^*)}^2 \Big). \end{aligned} \tag{5.3}$$

5.2 Proof of Theorem 5.1

As we did before, we are going to deal with the case $\Gamma_0 = \emptyset$, so that $V_0 = V$ and $W_0 = W$. The case $\Gamma_0 \neq 0$ can be treated with no substantial changes.

Let us introduce the new variables $\omega_i = \vartheta_i + \lambda(\chi_i)$, $i = 1, 2$, and set

$$\begin{aligned} \vartheta &= \vartheta_1 - \vartheta_2 \\ \omega &= \omega_1 - \omega_2 \end{aligned}$$

$$\begin{aligned}
\chi &= \chi_1 - \chi_2 \\
w &= w_1 - w_2 \\
\eta &= \eta_1 - \eta_2 \\
\widehat{\gamma} &= \gamma'(\chi_1) - \gamma'(\chi_2) \\
\widetilde{\lambda} &= \lambda(\chi_1) - \lambda(\chi_2) \\
\widehat{\lambda} &= \lambda'(\chi_1) - \lambda'(\chi_2) \\
f &= f_1 - f_2.
\end{aligned}$$

According to Definition 3.1, the quadruplet $(\vartheta, \chi, w, \eta)$ fulfills the system

$$\langle \partial_t \omega, v\rangle_V + (\omega, v)_H - (\widetilde{\lambda}, v)_H + k(\nabla(\omega - \widetilde{\lambda}), \nabla v)_H + \nu_0 \int_0^\infty \nu(\sigma)(\eta(\sigma), v)_H d\sigma$$

$$+ \int_0^\infty \mu(\sigma)(\nabla\eta(\sigma), \nabla v)_H d\sigma = \langle f, v\rangle_V, \qquad \forall\, v \in V \tag{5.4}$$

$$\langle \partial_t \chi, v\rangle_V + (\nabla w, \nabla v)_H = 0, \qquad \forall\, v \in V \tag{5.5}$$

$$\begin{aligned}\langle v^*, w\rangle_V = \langle v^*, &-\Delta\chi + \chi_1^3 - \chi_2^3 + \widehat{\gamma} \\ &- \lambda'(\chi_1)(\omega_1 - \lambda(\chi_1)) + \lambda'(\chi_2)(\omega_2 - \lambda'(\chi_2))\rangle_V, \qquad \forall\, v^* \in V^*\end{aligned} \tag{5.6}$$

$$(\partial_t \eta + \partial_s \eta, \zeta)_{\mathcal{M}} = (\omega - \widetilde{\lambda}, \zeta)_{\mathcal{M}}, \qquad \forall\, \zeta \in \mathcal{M} \tag{5.7}$$

a.e. in I, and the initial conditions (3.11)–(3.13) where

$$\begin{aligned}
\vartheta_0 &= \vartheta_{01} - \vartheta_{02} \\
\chi_0 &= \chi_{01} - \chi_{02} \\
\eta_0 &= \eta_{01} - \eta_{02}.
\end{aligned}$$

We first observe that $\omega \in L^2(I; V)$. Indeed, $\vartheta \in L^2(I; V)$, and thanks to (4.28) and (4.29), Hölder inequality, and the continuous embedding $V \hookrightarrow L^4(\Omega)$, we have also

$$\begin{aligned}
\|\nabla\widetilde{\lambda}\|_H^2 &= \int_\Omega |\lambda'(\chi_1)\nabla\chi_1 - \lambda'(\chi_2)\nabla\chi_2|^2 \, d\Omega \\
&= \int_\Omega \Big(|\lambda'(\chi_1)|^2 \, |\nabla\chi|^2 + |\widehat{\lambda}|^2 |\nabla\chi_2|^2 \Big) d\Omega \\
&\leq c \int_\Omega \Big((1 + |\chi_1|^2)|\nabla\chi|^2 + |\chi|^2 |\nabla\chi_2|^2 \Big) d\Omega \\
&\leq c\Big((1 + \|\chi_1\|^2_{L^4(\Omega)}) \|\nabla\chi\|^2_{L^4(\Omega)} + \|\chi\|^2_{L^4(\Omega)} \|\nabla\chi_2\|^2_{L^4(\Omega)} \Big) \\
&\leq c\Big(\, (1 + \|\chi_1\|_V^2) \, \|\chi\|_W^2 + \|\chi\|_V^2 \|\chi_2\|_W^2 \Big)
\end{aligned}$$

so that $\nabla\widetilde{\lambda} \in L^2(I; H)$.

Then, we can take $v = \mathcal{R}\omega$ in equation (5.4), and recalling (2.11)–(2.12) we deduce

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\omega\|_{V^*}^2 + \|\omega\|_{V^*}^2 + k\|\omega\|_H^2 + \int_0^\infty \mu(\sigma)(\eta(\sigma),\omega)_H d\sigma \\
&\le \nu_0 \int_0^\infty \nu(\sigma)(\eta(\sigma),\omega)_{V^*} d\sigma + \int_0^\infty \mu(\sigma)(\eta(\sigma),\omega)_{V^*} d\sigma + k\|\omega\|_{V^*}^2 \\
&\quad + (f,\omega)_{V^*} + (k-1)(\widetilde{\lambda},\omega)_{V^*} + k(\widetilde{\lambda},\omega)_H.
\end{aligned}$$

By means of Young inequality, we easily get

$$\begin{aligned}
&\frac{d}{dt}\|\omega\|_{V^*}^2 + k\|\omega\|_H^2 + 2\int_0^\infty \mu(\sigma)(\eta(\sigma),\omega)_H d\sigma \\
&\le \|\eta\|_{\mathcal{N}}^2 + c(\|\omega\|_{V^*}^2 + \|\chi\|_H^2) + 2\|f\|_{V^*}\|\omega\|_{V^*}.
\end{aligned} \tag{5.8}$$

Since χ is conserved (cf. Remark 3.4), we have

$$m_\chi = \int_\Omega \chi(t) d\Omega = \int_\Omega (\chi_1(t) - \chi_2(t)) d\Omega = \chi_{01} - \chi_{02}.$$

Hence, $\chi - m_\chi$ is a function in V with null mean value, and we can then choose $v = \mathcal{T}(\chi - m_\chi)$ in equation (5.5), to obtain

$$\langle \partial_t \chi, \mathcal{T}(\chi - m_\chi)\rangle_V + (\nabla w, \nabla \mathcal{T}(\chi - m_\chi))_H = 0$$

which, in force of (2.8)–(2.9) and of the fact that χ is conserved, reads

$$\frac{1}{2}\frac{d}{dt}\|\chi - m_\chi\|_{V^*}^2 + (w, \chi - m_\chi)_H = 0. \tag{5.9}$$

Moreover, equation (5.6) with $v^* = \chi - m_\chi$ entails

$$\begin{aligned}
(\chi - m_\chi, w)_H \;=\; & \|\nabla\chi\|_H^2 + (\chi_1^3 - \chi_2^3, \chi - m_\chi)_H + (\widehat{\gamma}, \chi - m_\chi)_H \\
& - (\lambda'(\chi_2)(\omega - \widetilde{\lambda}), \chi - m_\chi)_H - ((\omega_1 - \lambda(\chi_1))\widehat{\lambda}, \chi - m_\chi)_H.
\end{aligned} \tag{5.10}$$

Collecting (5.9)–(5.10) we end up with

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\chi - m_\chi\|_{V^*}^2 + \|\nabla\chi\|_H^2 \;\le\; & -(\chi_1^3 - \chi_2^3, \chi - m_\chi)_H - (\widehat{\gamma}, \chi - m_\chi)_H \\
& + (\lambda'(\chi_2)(\omega - \widetilde{\lambda}), \chi - m_\chi)_H \\
& + ((\omega_1 - \lambda(\chi_1))\widehat{\lambda}, \chi - m_\chi)_H.
\end{aligned} \tag{5.11}$$

We now analyze each term appearing on the right-hand side of (5.11). We recall that $\chi_i \in L^\infty(I;V)$. Then, by Hölder inequality and the continuous embedding $L^4(\Omega) \hookrightarrow V$, we have

$$
\begin{aligned}
|m_\chi| \int_\Omega (\chi_1^3 - \chi_2^3) d\Omega &\le c|m_\chi| \int \chi(\chi_1^2 + \chi_2^2) d\Omega \\
&\le c|m_\chi| \|\chi\|_H \Big(\|\chi_1\|^2_{L^4(\Omega)} + \|\chi_2\|^2_{L^4(\Omega)} \Big) \\
&\le c|m_\chi| \|\chi\|_H \Big(\|\chi_1\|^2_V + \|\chi_2\|^2_V \Big) \\
&\le c\Big(m_\chi^2 + \|\chi\|^2_H \Big) \\
&\le c\Big(m_\chi^2 + \|\chi - m_\chi\|^2_H \Big). \qquad (5.12)
\end{aligned}
$$

Making a repeated use of (3.4), (3.5), (3.14), Young and Hölder inequalities, and the continuous embeddings $V \hookrightarrow L^6(\Omega)$, we see that, for every $\varepsilon > 0$, we find $c = c(\varepsilon) > 0$ such that

$$
\begin{aligned}
-(\chi_1^3 - \chi_2^3, \chi)_H &\le c\|\chi\|^2_H + \varepsilon \int_\Omega (\chi_1^4 + \chi_2^4)\chi^2 d\Omega \\
&\le c\|\chi\|^2_H + \varepsilon \Big(\|\chi_1\|^4_{L^6(\Omega)} + \|\chi_2\|^4_{L^6(\Omega)} \Big) \|\chi\|^2_{L^6(\Omega)} \\
&\le c\|\chi\|^2_H + \varepsilon \|\chi\|^2_V \qquad (5.13)
\end{aligned}
$$

$$
-(\widehat{\gamma}, \chi - m_\chi)_H + (\lambda'(\chi_2)(\omega + \widetilde{\lambda}), \chi - m_\chi)_H \le \varepsilon \|\omega\|^2_H + c\Big(m_\chi^2 + \|\chi - m_\chi\|^2_H \Big) \qquad (5.14)
$$

$$
((\omega_1 - \lambda(\chi_1))\widehat{\lambda}, \chi - m_\chi)_H \le c\Big(1 + \|\omega_1\|_H + \|\chi_1\|_H\Big)\Big(m_\chi^2 + \|\chi - m_\chi\|^2_{L^4(\Omega)} \Big). \qquad (5.15)
$$

Here we exploited the a priori bound on χ_1 and χ_2 in $L^\infty(I; V)$.

Observing that

$$
\|\chi - m_\chi\|^2_H \le \|\chi - m_\chi\|^2_{L^4(\Omega)} \le c\|\chi - m_\chi\|^2_{V^*} + \varepsilon \|\nabla \chi\|^2_H
$$

in force of Ehrling inequality, and taking (5.12)–(5.15) into account, addition of equations (5.8)–(5.11) leads to

$$
\begin{aligned}
&\frac{d}{dt}\Big(\|\omega\|^2_{V^*} + \|\chi - m_\chi\|^2_{V^*} \Big) + k\|\omega\|^2_H + \|\nabla\chi\|^2_H + 2\int_0^\infty \mu(\sigma)(\eta(\sigma), \omega)_H d\sigma \\
&\le c\Big(\|\omega\|^2_{V^*} + \|\chi - m_\chi\|^2_{V^*} + \|\eta\|^2_{\mathcal{N}} + m_\chi^2 \Big) + 2\|f\|_{V^*}\|\omega\|_{V^*}. \qquad (5.16)
\end{aligned}
$$

provided we set ε small enough.

We finally focus on equation (5.7). In order to proceed, we need some regularization with respect to the variable s. Let $\rho_\varepsilon \in \mathcal{D}((0,\varepsilon), \mathbb{R}^+)$, with $\|\rho_\varepsilon\|_{L^1(\mathbb{R}^+)} = 1$, and introduce the approximation

$$
\eta^t_\varepsilon(s) = (\eta^t * \rho_\varepsilon)(s) = \int_0^s \eta^t(s-y)\rho_\varepsilon(y) dy.
$$

Notice that $\eta_\varepsilon^t(0) = 0$. Thanks to the regularizing properties of the convolution, we have that $\partial_s \eta_\varepsilon^t \in \mathcal{N}$ and

$$h_\varepsilon(t) = \|\eta_\varepsilon^t - \eta^t\|_{\mathcal{N}} \longrightarrow 0 \qquad \text{as } \varepsilon \to 0, \quad \forall\, t \in I. \tag{5.17}$$

Moreover, being $\eta \in L^\infty(I;\mathcal{M}) \subset L^\infty(I;\mathcal{N})$, we know that $h_\varepsilon(t)$ is uniformly bounded in I.

Integration by parts, and the fact that $\eta_\varepsilon(0) = 0$, shows (cf. (4.10))

$$(\partial_t \eta_\varepsilon + \partial_s \eta_\varepsilon, \eta_\varepsilon)_{\mathcal{N}} = \frac{1}{2}\frac{d}{dt}\|\eta_\varepsilon\|_{\mathcal{N}}^2 + (\partial_s \eta_\varepsilon, \eta_\varepsilon)_{\mathcal{N}} \geq \frac{1}{2}\frac{d}{dt}\|\eta_\varepsilon\|_{\mathcal{N}}^2.$$

Take then the regularized version of (5.7), namely

$$\partial_t \eta_\varepsilon + \partial_s \eta_\varepsilon = (\omega - \widetilde{\lambda}) * \rho_\varepsilon = \omega - \widetilde{\lambda}$$

and multiply both sides by $2\eta_\varepsilon$ in $\mathcal{N}$. This yields

$$\frac{d}{dt}\|\eta_\varepsilon\|_{\mathcal{N}}^2 \leq 2(\omega - \widetilde{\lambda}, \eta_\varepsilon)_{\mathcal{N}} \leq 2(\omega - \widetilde{\lambda}, \eta)_{\mathcal{N}} + k_\varepsilon(t) \tag{5.18}$$

where we defined

$$k_\varepsilon(t) = 2h_\varepsilon(t) \sup_{t\in I} \|(\omega - \widetilde{\lambda})(t)\|_{\mathcal{N}}.$$

It should be noted that $\lambda(\chi) \in C(I;H)$. Hence we cannot repeat the same argument just replacing the space $\mathcal{N}$ with $\mathcal{M}$. This is relevant in view of Theorem 5.4.

Adding now (5.16)–(5.18), we get to the inequality

$$\begin{aligned}
&\frac{d}{dt}\Big(\|\omega\|_{V^*}^2 + \|\chi\|_{V^*}^2 + \|\eta_\varepsilon\|_{\mathcal{N}}^2\Big) + k\|\omega\|_H^2 + \|\nabla\chi\|_H^2 \\
&\leq c\Big(\|\omega\|_{V^*}^2 + \|\chi\|_{V^*}^2 + \|\eta_\varepsilon\|_{\mathcal{N}}^2\Big) + cm_\chi^2 + 2\|f\|_{V^*}\|\omega\|_{V^*} + k_\varepsilon(t).
\end{aligned} \tag{5.19}$$

Setting

$$\begin{aligned}
\Phi^2(t) &= \|\omega(t)\|_{V^*}^2 + \|\chi(t)\|_{V^*}^2 + \|\eta^t\|_{\mathcal{N}}^2 \\
\Phi_\varepsilon^2(t) &= \|\omega(t)\|_{V^*}^2 + \|\chi(t)\|_{V^*}^2 + \|\eta_\varepsilon^t\|_{\mathcal{N}}^2
\end{aligned}$$

we can rewrite (5.19) as

$$\frac{d}{dt}\Phi_\varepsilon^2(t) \leq c\Phi_\varepsilon^2(t) + c\|f(t)\|_{V^*}\Phi_\varepsilon(t) + cm_\chi^2 + k_\varepsilon(t). \tag{5.20}$$

It is clear that $\Phi_\varepsilon^2(t) \to \Phi^2(t)$ as $\varepsilon \to 0$, for every $t \in I$.

Applying Lemma A.5 to (5.20), we obtain the bound

$$\Phi_\varepsilon^2(t) \leq c\Phi_\varepsilon^2(\tau) + c\|f\|_{L^1(I;V^*)}^2 + cm_\chi^2 + \int_\tau^T k_\varepsilon(y)\,dy, \qquad \forall\, t \in I.$$

Letting $\varepsilon \to 0$, and applying the dominated convergence theorem, we conclude that

$$\Phi^2(t) \leq c\Big(\Phi^2(\tau) + m_\chi^2 + \|f\|_{L^1(I;V^*)}^2\Big), \qquad \forall\, t \in I.$$

A further integration over (τ, t) of equation (5.19) concludes the proof. □

5.3 Proof of Theorem 5.2

As usual, we proceed supposing $\Gamma_0 = \emptyset$ and taking $\nu_0 = 1$.
Let us set

$$\begin{aligned} \omega &= \vartheta_1 - \vartheta_2 + \lambda_0(\chi_1 - \chi_2) \\ \chi &= \chi_1 - \chi_2 \\ \eta &= \eta_1 - \eta_2 \end{aligned}$$

and

$$\begin{aligned} \omega_0 &= \vartheta_{01} - \vartheta_{02} + \lambda_0(\chi_{01} - \chi_{02}) \\ \chi_0 &= \chi_{01} - \chi_{02} \\ \eta_0 &= \eta_{01} - \eta_{02} \\ f &= f_1 - f_2 \\ m_\chi &= m_{\chi_1} - m_{\chi_2}. \end{aligned}$$

Then the triplet (ω, χ, η) fulfills the system

$$\langle \omega_t, v\rangle_V + (\omega, v)_H - \lambda_0(\chi, v)_H + \int_0^\infty \nu(\sigma)(\eta(\sigma), v)_H d\sigma + \int_0^\infty \mu(\sigma)(\nabla\eta(\sigma), \nabla v)_H d\sigma = (f, v)_H, \qquad \forall\, v \in V \tag{5.21}$$

$$\langle \partial_t \chi, v\rangle_V + (\nabla w, \nabla v)_H = 0, \qquad \forall\, v \in V \tag{5.22}$$

$$\langle w, v\rangle_V = \langle -\Delta\chi + \chi_1^3 - \chi_2^3 + (\lambda_0^2 - 1)\chi - \lambda_0\omega, v\rangle_V, \qquad \forall\, v \in V \tag{5.23}$$

$$(\partial_t \eta + \partial_s \eta, \zeta)_{\mathcal{M}} = (\omega - \widetilde{\lambda}, \zeta)_{\mathcal{M}}, \qquad \forall\, \zeta \in \mathcal{M} \tag{5.24}$$

a.e. in I, and the initial conditions (3.20)–(3.22).

Let $\mathcal{R} : V^* \longrightarrow V$ be the Riesz map such that $\mathcal{R}^{-1} = -\Delta + \mathbb{I}$ and choose $v = \mathcal{R}\omega$ in (5.21). Then we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\omega\|_{V^*}^2 &= -\|\omega\|_{V^*}^2 + l(\chi, \omega)_{V^*} - \int_0^\infty \nu(\sigma)(\eta(\sigma), \omega)_{V^*} d\sigma - \int_0^\infty \mu(\sigma)(\sigma)(\eta(\sigma), \omega)_H d\sigma \\ &\quad + \int_0^\infty \mu(\sigma)(\sigma)(\eta(\sigma), \omega)_{V^*} d\sigma + (f, \omega)_{V^*} \\ &\leq \frac{1}{2}(a_0 + b_0 - 1)\|\omega\|_{V^*}^2 \\ &\quad + \frac{\lambda_0^2}{8}\|\chi\|_{V^*}^2 + \frac{1}{2}\|\eta\|_{\mathcal{N}}^2 - \int_0^\infty \mu(\sigma)(\sigma)(\eta(\sigma), \omega)_H d\sigma + \|f\|_{V^*}\|\omega\|_{V^*}. \end{aligned} \tag{5.25}$$

Take now $v = \mathcal{T}(\chi - m_\chi)$ in (5.5), and $v = \chi - m_\chi$ in (5.23), to get

$$\frac{d}{dt}\|\chi - m_\chi\|_{V^*}^2 + 2\langle \chi - m_\chi, -\Delta\chi + \chi_1^3 - \chi_2^3 + (\lambda_0^2 - 1)\chi - \lambda_0\omega\rangle_V = 0. \tag{5.26}$$

Since $\chi - m_\chi \in H$, we can study separately each term of the right-hand side of the above equation.

We recall that $\chi_i \in L^\infty(I;V)$. Then, by Hölder inequality and the continuous embedding $L^4(\Omega) \hookrightarrow V$, we have

$$\begin{aligned} |m_\chi| \int_\Omega (\chi_1^3 - \chi_2^3) d\Omega &\leq c|m_\chi| \int \chi(\chi_1^2 + \chi_2^2) d\Omega \\ &\leq c|m_\chi| \|\chi\|_H \Big(\|\chi_1\|^2_{L^4(\Omega)} + \|\chi_2\|^2_{L^4(\Omega)} \Big) \\ &\leq c|m_\chi| \|\chi\|_H \Big(\|\chi_1\|^2_V + \|\chi_2\|^2_V \Big) \\ &\leq c\Big(m_\chi^2 + \|\chi\|^2_H \Big) \\ &\leq c\Big(m_\chi^2 + \|\chi - m_\chi\|^2_H \Big). \end{aligned}$$

Integrating by parts equation (5.26), and taking advantage of the relation

$$\int_\Omega (\chi - m_\chi)\chi d\Omega = \|\chi - m_\chi\|^2_H$$

we have

$$\begin{aligned} &\frac{d}{dt}\|\chi - m_\chi\|^2_{V^*} + 2\|\nabla(\chi - m_\chi)\|^2_H + 2(\chi_1^3 - \chi_2^3, \chi)_H \\ &\leq 2|m_\chi| \int_\Omega (\chi_1^3 - \chi_2^3) d\Omega + 2(\lambda_0^2 + 1)\|\chi - m_\chi\|^2_H + 2\lambda_0 \langle \omega, \chi - m_\chi \rangle_V \\ &\leq \frac{1}{2}\|\chi - m_\chi\|^2_V + 2\lambda_0^2 \|\omega\|^2_{V^*} + c\Big(m_\chi^2 + \|\chi - m_\chi\|^2_H \Big); \end{aligned}$$

Choosing $\varepsilon > 0$, Ehrling inequality entails

$$\|\chi - m_\chi\|^2_H \leq \varepsilon \|\nabla(\chi - m_\chi)\|^2_H + c\|\chi - m_\chi\|^2_{V^*}$$

where $c = c_E(\varepsilon)$. Finally, we deduce

$$\frac{d}{dt}\|\chi - m_\chi\|^2_{V^*} + \|\nabla(\chi - m_\chi)\|^2_H \leq c\left(\|\chi - m_\chi\|^2_{V^*} + m_\chi^2 \right) + 2\lambda_0^2 \|\omega\|^2_{V^*}. \tag{5.27}$$

Proceeding as in proof of Theorem 5.1, we introduce the approximation

$$\eta^t_\varepsilon(\sigma) = (\eta^t * \rho_\varepsilon)(\sigma) = \int_0^\sigma \eta^t(\sigma - y)\rho_\varepsilon(y) dy$$

which satisfies

$$\frac{d}{dt}\|\eta_\varepsilon\|^2_{\mathcal{N}} \leq 2(\omega - \lambda_0 \chi, \eta_\varepsilon)_{\mathcal{N}} \leq 2(\omega - \lambda_0\chi, \eta)_{\mathcal{N}} + ch_\varepsilon(t) \tag{5.28}$$

where the constant c equals in fact $2\sup_{t \in I} \|(\omega - \lambda_0\chi)(t)\|_{\mathcal{N}}$ and the function $h_\varepsilon(t)$ is introduced as in (5.17).

Adding (5.25), (5.27) and (5.28) together yields

$$\frac{d}{dt}\Big(\|\omega\|^2_{V^*}+\|\chi-m_\chi\|^2_{V^*}+\|\eta_\varepsilon\|^2_{\mathcal{N}}\Big)+\|\nabla(\chi-m_\chi)\|^2_H$$
$$\leq c\Big(\|\omega\|^2_{V^*}+\|\chi-m_\chi\|^2_{V^*}+\|\eta_\varepsilon\|^2_{\mathcal{N}}\Big)+c(m_\chi^2+h_\varepsilon^t)+2\|f\|_{V^*}\|\omega\|_{V^*}. \tag{5.29}$$

Finally, set

$$\Phi_\varepsilon^2(t)=\|\omega(t)\|^2_{V^*}+\|\chi(t)-m_\chi\|^2_{V^*}+\|\eta_\varepsilon\|^2_{\mathcal{N}}$$

and

$$\Phi^2(t)=\|\omega(t)\|^2_{V^*}+\|\chi(t)-m_\chi\|^2_{V^*}+\|\eta^t\|^2_{\mathcal{N}}.$$

It is clear that $\Phi_\varepsilon^2(t)\to\Phi^2(t)$ as $\varepsilon\to 0$ for every $t\in I$. Then Lemma A.5 implies

$$\Phi_\varepsilon^2(t)\leq c\left(\Phi_\varepsilon^2(\tau)+\int_\tau^t(h_\varepsilon(s)+m_\chi^2)ds+\left(\int_\tau^t\|f(s)\|_{V^*}ds\right)^2\right).$$

We recall that the function h_ε is uniformly bounded; then, letting $\varepsilon\to 0$, and applying the dominated convergence theorem, we conclude that

$$\Phi^2(t)\leq c\Big(\Phi^2(\tau)+m_\chi^2+\|f\|^2_{L^1(I;V^*)}\Big).$$

This gives

$$\|\omega_1(t)-\omega_2(t)\|^2_{V^*}+\|\chi_1(t)-\chi_2(t)\|^2_{V^*}+\|\eta_1^t-\eta_2^t\|^2_{\mathcal{N}}$$
$$\leq c\Big(\|\omega_{01}-\omega_{02}\|^2_{V^*}+\|\chi_{01}-\chi_{02}\|^2_{V^*}$$
$$+\|\eta_{01}-\eta_{02}\|^2_{\mathcal{N}}+\|f_1-f_2\|^2_{L^1(I;V^*)}+|m_{\chi_1}-m_{\chi_2}|^2\Big).$$

An integration of (5.29) over (τ,T), letting $\varepsilon\to 0$, and the previous inequality lead to the boundedness of the term $\|\nabla(\chi-m_\chi)\|_{L^2(I;H)}$. This argument completes the proof. □

5.4 Proof of Theorem 5.4

We work in a Faedo-Galerkin approximating scheme. With reference to the proof of Theorem 4.1, let $(\vartheta_m,\chi_m,\eta_m)$ and $(\vartheta_n,\chi_n,\eta_n)$ be two solutions to problems $\mathbf{P}_k^m$ and $\mathbf{P}_k^n$, corresponding to the initial data $(P_m\vartheta_{01},P_m\chi_{01},Q_m\eta_{01})$ and $(P_n\vartheta_{02},P_n\chi_{02},Q_n\eta_{02})$, respectively. We first notice that

$$\begin{aligned}\vartheta_m,\vartheta_n &\in C(I;H)\\ \chi_m,\chi_n &\in C(I;V)\\ \eta_m,\eta_n &\in C(I;\mathcal{M}).\end{aligned}$$

Setting

$$\begin{aligned} \vartheta_{mn} &= \vartheta_m - \vartheta_n \\ \chi_{mn} &= \chi_m - \chi_n \\ \eta_{mn} &= \eta_m - \eta_n \end{aligned}$$

the following equations are satisfied

$$\langle \partial_t \vartheta_{mn}, v \rangle_V + \lambda_0 \langle \partial_t \chi_{mn}, v \rangle_V + (\vartheta_{mn}, v)_H + k(\nabla \vartheta_{mn}, \nabla v)_H$$

$$+\nu_0 \int_0^\infty \nu(\sigma)(\eta_{mn}(\sigma), v)_H d\sigma + \int_0^\infty \mu(\sigma)(\nabla \eta_{mn}(\sigma), \nabla v)_H d\sigma = 0, \qquad \forall\, v \in V_m \cap V_n, \quad (5.30)$$

$$\langle \partial_t \chi_{mn}, v + (\nabla w_{mn}, \nabla v)_H, \qquad \forall\, v \in V_m \cap V_n, \quad (5.31)$$

$$\langle v, w_{mn} \rangle_V = \langle v, -\Delta \chi_{mn} + \chi_m^3 - \chi_n^3 + \gamma'(\chi_m) - \gamma'(\chi_n) - \lambda_0 \vartheta_{mn} \rangle_V, \qquad \forall\, v \in V^* \quad (5.32)$$

$$(\partial_t \eta_{mn} + \partial_s \eta_{mn}, \zeta)_{\mathcal{M}} = (\vartheta_{mn}, \zeta)_{\mathcal{M}}, \qquad \forall\, \zeta \in \mathcal{M}_m \cap \mathcal{M}_n \quad (5.33)$$

with initial data

$$\begin{aligned} \vartheta_{mn}(\tau) &= P_m \vartheta_{01} - P_n \vartheta_{02} \\ \chi_{mn}(\tau) &= P_m \chi_{01} - P_n \chi_{02} \\ \eta_{mn}^\tau &= Q_m \eta_{01} - Q_n \eta_{02}. \end{aligned}$$

For sake of simplicity, we have supposed here that the forcing term is null; but we could have forced the system by $f \in L^1(I; H) + L^2(I; V^*)$ if $k > 0$, or $f \in L^1(I; H)$ if $k = 0$, as we did in the proof of Theorem 4.1.

Take $v = \mathcal{T} \partial_t \chi_{mn}$ in (5.31); then, by identity (2.9),

$$\|\partial_t \chi_{mn}\|_{V^*}^2 + \langle \partial_t \chi_{mn}, w_{mn} \rangle_V = 0.$$

Then take $v = \partial_t \chi_{mn}$ in (5.32). This, together with the above equality, yields

$$\begin{aligned} &\|\partial_t \chi_{mn}\|_{V^*}^2 + \frac{1}{2} \frac{d}{dt} \|\nabla \chi_{mn}\|_H^2 - \lambda_0 \langle \partial_t \chi_{mn}, \vartheta_{mn} \rangle_V \\ &= -\langle \partial_t \chi_{mn}, \chi_m^3 - \chi_n^3 + \gamma'(\chi_m) - \gamma'(\chi_n) \rangle_V. \end{aligned} \quad (5.34)$$

Our aim is to find a bound for the right-hand side of (5.34). Exploiting the Lipschitz continuity of γ', we have

$$\|\nabla(\gamma'(\chi_m) - \gamma'(\chi_n))\|_H^2 \le c\Big(\|\nabla \chi_{mn}\|_H^2 + \|\chi_{mn}(\nabla \chi_{mn})\|_H^2 \Big) \le c\psi - \|\chi_{mn}\|_V^2 \quad (5.35)$$

where

$$\psi(t) = 1 + \|\chi_n(t)\|_W^2 + \|\chi_m(t)\|_W^2.$$

note that $\psi \in L^1(I)$.

Hölder inequality and the continuous embedding $V \hookrightarrow L^6(\Omega)$, entails

$$
\begin{aligned}
&\|\nabla(\chi_m^3 - \chi_n^3)\|_H^2 \\
&\quad \le c \int_\Omega |\chi_m^2 \nabla\chi_m - \chi_n^2 \nabla\chi_n|^2 d\Omega \\
&\quad \le c \left(\int_\Omega |\chi_m^2 - \chi_n^2|^2 |\nabla\chi_m|^2 d\Omega + \int_\Omega |\chi_n|^4 \nabla\chi_{mn}|^2 d\Omega \right) \\
&\quad \le c \left(\int_\Omega (\chi_m^2 + \chi_n^2)|\nabla\chi_m|^2 \chi_{mn}^2 d\Omega + \left(\int_\Omega \chi_m^6 d\Omega \right)^{2/3} \left(\int_\Omega |\nabla\chi_{mn}|^6 d\Omega \right)^{1/3} \right) \\
&\quad \le c \left(\left(\int_\Omega (|\chi_m|^3 + |\chi_n|^3)|\nabla\chi_m|^3 d\Omega \right)^{2/3} \left(\int_\Omega \chi_{mn}^6 d\Omega \right)^{1/3} + \|\chi_{mn}\|_W^2 \right) \\
&\quad \le c \left(\|\chi_{mn}\|_V^2 \left(\int_\Omega (|\chi_m|^3 + |\chi_n|^3)^2 d\Omega \right)^{1/3} \left(\int_\Omega |\nabla\chi_m|^6 d\Omega \right)^{1/3} + \|\chi_{mn}\|_W^2 \right) \\
&\quad \le c \left(\|\chi_{mn}\|_W^2 + \psi \|\chi\|_V^2 \right).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\langle \partial_t \chi_{mn}, \chi_m^3 - \chi_n^3 + \gamma'(\chi_m) - \gamma'(\chi_n) \rangle_V \\
&\le \varepsilon \|\partial_t \chi_{mn}\|_{V^*}^2 + c\Big(\|\gamma'(\chi_m) - \gamma'(\chi_n)\|_V^2 + \|\chi_m^3 - \chi_n^3\|_V^2 \Big) \\
&\le \varepsilon \|\partial_t \chi_{mn}\|_{V^*}^2 + c\psi \|\chi_{mn}\|_V^2 + c\|\chi_{mn}\|_W^2
\end{aligned}
$$

and equation (5.34) can be rewritten as

$$
\frac{1}{2}\frac{d}{dt}\|\nabla\chi_{mn}\|_H^2 + \frac{1}{2}\|\partial_t\chi_{mn}\|_{V^*}^2 - \lambda_0 \langle \partial_t\chi_{mn}, \vartheta_{mn} \rangle_V \le c\psi\|\chi_{mn}\|_V^2 + c\|\chi_{mn}\|_W^2. \tag{5.36}
$$

Then, take equation (5.30) with $v = \vartheta_{mn}$, and equation (5.33), with $\zeta = \eta_{mn}$, to get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\vartheta_{mn}\|_H^2 + \lambda_0 \langle \partial_t\chi_{mn}, \vartheta_{mn} \rangle_V + \|\vartheta_{mn}\|_H^2 + k\|\nabla\vartheta_{mn}\|_H^2 \\
&+ \nu_0 \int_0^\infty \nu(\sigma)(\eta_{mn}(\sigma), \vartheta_{mn})_H d\sigma + \int_0^\infty \mu(\sigma)(\nabla\eta_{mn}(\sigma), \nabla\vartheta_{mn})_H d\sigma = 0
\end{aligned} \tag{5.37}
$$

and

$$
\frac{1}{2}\frac{d}{dt}\|\eta_{mn}\|_{\mathcal{M}}^2 \le (\vartheta_{mn}, \eta_{mn})_{\mathcal{M}}. \tag{5.38}
$$

Adding (5.36)–(5.38), and exploiting the cancellation of the term $\langle \partial_t\chi_{mn}, \vartheta_{mn} \rangle_V$, we end up with

$$
\begin{aligned}
&\frac{d}{dt}\Big(\|\vartheta_{mn}\|_H^2 + \|\nabla\chi_{mn}\|_H^2 + \|\eta_{mn}\|_{\mathcal{M}}^2 \Big) + \|\partial_t\chi_{mn}\|_{V^*}^2 + 2k\|\nabla\vartheta_{mn}\|_H^2 \\
&\le c\psi\Big(\|\vartheta_{mn}\|_H^2 + \|\nabla\chi_{mn}\|_H^2 + \|\eta_{mn}\|_{\mathcal{M}}^2 \Big) + c\|\chi_{mn}\|_W^2.
\end{aligned} \tag{5.39}
$$

Finally, set $v = \chi_{mn}$ in (5.31) and $v = -\Delta\chi_{mn}$ in (5.32). This bears

$$\frac{d}{dt}\|\chi_{mn}\|_H^2 + \|\Delta\chi_{mn}\|_H^2 \le c\Big(\|\chi_m n\|_V^2 + \|\vartheta_{mn}\|_H^2\Big). \tag{5.40}$$

Adding (5.39) and κ-times (5.40) (for κ big enough), and setting

$$\Phi_{mn}^2(t) = \|\vartheta_{mn}(t)\|_H^2 + \kappa\|\chi_{mn}(t)\|_H^2 + \|\nabla\chi_{mn}(t)\|_H^2 + \|\eta_{mn}^t\|_{\mathcal{M}}^2$$

we obtain the differential inequality

$$\frac{d}{dt}\Phi_{mn}^2 + \|\partial_t\chi_{mn}\|_{V^*}^2 + 2k\|\nabla\vartheta_{mn}\|_H^2 \le c\psi\Phi_{mn}^2. \tag{5.41}$$

Since $\psi \in L^1(I)$, Lemma A.5 implies

$$\Phi_{mn}^2(t) \le c\Phi_{mn}^2(\tau), \qquad \forall\, t \in I \tag{5.42}$$

where c is a constant depending on $\|\psi\|_{L^1(I)}$, which can be estimated by the a priori bounds on the approximating solutions.

Now, let $m, n \to \infty$, we get for ant $t \in I$

$$\begin{aligned}&\|\vartheta_1(t) - \vartheta_2(t)\|_H^2 + \|\chi_1(t) - \chi_2(t)\|_V^2 + \|\eta_1^t - \eta_2^t\|_{\mathcal{M}}^2\\ &\le c\Big(\|\vartheta_{01} - \vartheta_{02}\|_H^2 + \|\chi_{01} - \chi_{02}\|_V^2 + \|\eta_{01} - \eta_{02}\|_{\mathcal{M}}^2\Big).\end{aligned}$$

Integration over (τ, t) in (5.41), the previous inequality, and a passage to the limit yield

$$\begin{aligned}&k\int_\tau^t \|\vartheta_1(y) - \vartheta_2(y)\|_V^2 dy + \int_\tau^t \|\partial_t\chi_1(y) - \partial_t\chi_2(y)\|_{V^*}^2 dy\\ &\le c\Big(\|\vartheta_{01} - \vartheta_{02}\|_H^2 + \|\chi_{01} - \chi_{02}\|_V^2 + \|\eta_{01} - \eta_{02}\|_{\mathcal{M}}^2\Big)\end{aligned}$$

hence, (5.3) is proved.

Let us choose now

$$\begin{aligned}\vartheta_{01} &= \vartheta_{02}\\ \chi_{01} &= \chi_{02}\\ \eta_{01} &= \eta_{02}.\end{aligned}$$

It is clear that

$$\Phi_{mn}^2(\tau) \longrightarrow 0, \qquad m, n \to \infty$$

and (5.42) then implies that $\{(\vartheta_m, \chi_m, \eta_m)\}$ is a Cauchy sequence in a complete Banach space. We conclude that the triplet (ϑ, χ, η), solution to problem $\mathbf{P}_k$, satisfy

$$\begin{aligned}\vartheta &\in C(I; H)\\ \chi &\in C(I; V)\\ \eta &\in C(I; \mathcal{M})\end{aligned}$$

for any $k \ge 0$. □

Chapter 6

Uniformly absorbing sets

The study of infinite dimensional nonlinear dynamical systems is usually a very hard problem. In fact, it is not easy to predict whether a system will evolve towards rest or towards a sequence of bifurcations leading to periodic, quasi periodic or fully chaotic states. This fact suggests that the correct approach to the problem should be more *geometric*; the goal is then to describe the evolution of any ball of initial data. Indeed, even a chaotic behavior of a certain dynamical system could be described, in some sense, by the existence of a complicated attractor to which trajectories converge as $t \to \infty$.

The first step in proving the existence of an attractor is the existence of absorbing sets; that is, sets into which all the orbits corresponding to different initial data eventually enter. The existence of such sets is also a strong evidence of the dissipative nature of the system.

6.1 Dissipative processes and their absorbing sets

We recall some basic definitions (cf., e.g., [42, 56, 57]).

Let (X, d) be a metric space.

Definition 6.1. A two parameter family of operators

$$U(t,\tau) : X \longrightarrow X, \qquad \tau \in \mathbb{R},\ t \geq \tau$$

is a *(strongly continuous) process*, or a *dynamical system*, on X if the following properties are satisfied:

$$\begin{array}{lll}
\text{(i)} & U(\tau,\tau) = \mathbb{I} & \forall\, \tau \in \mathbb{R} \\
\text{(ii)} & U(t,\tau) = U(t,s)U(s,\tau) & \forall\, \tau \in \mathbb{R},\ t \geq s \geq \tau \\
\text{(iii)} & U(t,\tau)z_0 \in C([\tau,+\infty);\mathcal{H}) & \forall\, \tau \in \mathbb{R},\ \forall\, z_0 \in X \\
\text{(iv)} & U(t,\tau) \in C(X;X) & \forall\, \tau \in \mathbb{R},\ t \geq \tau.
\end{array}$$

A process is the natural extension of the concept of semigroup. Indeed, if the process is translation-invariant, that is

$$U(t+s,\tau+s) = U(t,\tau), \qquad \forall\, s \in \mathbb{R}$$

then $\{U(t,0), t \geq 0\}$ is easily seen to be a semigroup.

Sometimes it is useful to highlight the dependence of the process on a functional symbol $f \in \mathcal{F}$, where $\mathcal{F}$, the *symbol space*, is a complete metric space, and consider a family of processes

$$\{U_f(t,\tau), f \in \mathcal{F}\}$$

rather than a single process. This symbol commonly models an external force.

We suppose that a certain semigroup $\{T(s), s \geq 0\}$ acts on the symbol space $\mathcal{F}$, so that the following *translation identity* holds

$$U_f(t+s, \tau+s) = U_{T(s)f}(t,\tau) \tag{6.1}$$

for every $f \in \mathcal{F}$, $t \geq \tau$, $\tau \in \mathbb{R}$, $s \geq 0$. Then, it is possible to define a semigroup $\{\Sigma(t), t \geq 0\}$ acting on the (*extended phase*) space $X \times \mathcal{F}$ by

$$\Sigma(t)(z,f) = (U_f(t,0)z, T(t)f), \qquad t \geq 0, \quad (z,f) \in X \times \mathcal{F}. \tag{6.2}$$

Indeed,

$$\begin{aligned}
\Sigma(s)\Sigma(t)(z,f) &= \Sigma(s)(U_f(t,0)z, T(t)f) \\
&= (U_{T(t)f}(s,0)U_f(t,0)z, T(s)T(t)f) \\
&= (U_{T(t)f}(s,0)U_{T(t)f}(0,-t)z, T(s+t)f) \\
&= (U_{T(t)f}(s,-t)z, T(s+t)f) \\
&= (U_f(s+t,0)z, T(s+t)f) \\
&= \Sigma(s+t)(z,f)
\end{aligned}$$

and the other semigroup properties are immediate to prove.

We consider the projections Π_1 and Π_2 from $X \times \mathcal{F}$ onto X and $\mathcal{F}$, namely

$$\begin{aligned}
\Pi_1(z,f) &= z \\
\Pi_2(z,f) &= f.
\end{aligned}$$

Definition 6.2. A set $\mathcal{B}_0 \subset X$ is said to be *uniformly absorbing* (with respect to $f \in \mathcal{F}$) for the family $\{U_f(t,\tau),\ f \in \mathcal{F}\}$ if for any bounded set $\mathcal{B} \subset X$ there exists $t_{\mathcal{B}} \geq 0$ such that

$$\bigcup_{f \in \mathcal{F}} U_f(t,\tau)\mathcal{B} \subset \mathcal{B}_0$$

for every $\tau \in \mathbb{R}$ and every $t \geq \tau + t_{\mathcal{B}}$.

6.2 Existence of a process

We introduce the product Hilbert space

$$\mathcal{H} = H \times V \times \mathcal{M}.$$

Since we are working with a phase-field system which conserves the phase variable, we also introduce the subset $\mathcal{H}_\alpha$ of $\mathcal{H}$ defined by

$$\mathcal{H}_\alpha = \left\{ (\vartheta, \chi, \eta) \in \mathcal{H} : \left| \int_\Omega \chi d\Omega \right| \leq \alpha \right\}$$

for some fixed $\alpha \geq 0$. Note that $\mathcal{H}_\alpha$ is a complete metric space with respect to the metric induced by the norm of $\mathcal{H}$.

We agree to denote by

$$U_f^k(t,\tau)(\vartheta_0, \chi_0, \eta_0) = (\vartheta(t), \chi(t), \eta^t) \tag{6.3}$$

the solution to $\mathbf{P}_k$ at time t, with initial data $(\vartheta_0, \chi_0, \eta_0)$ assumed at time $\tau \leq t$, and forcing term f.

We take

$$(\vartheta_0, \chi_0, \eta_0) \in \mathcal{H}_\alpha \tag{6.4}$$

and f belonging to the space

$$L^1_{\text{loc}}(\mathbb{R}; H) + L^2_{\text{loc}}(\mathbb{R}; V^*) \qquad \text{if } k > 0 \tag{6.5}$$

or

$$L^1_{\text{loc}}(\mathbb{R}; H) \qquad \text{if } k = 0. \tag{6.6}$$

As a consequence of Theorems 4.1, 4.2, 5.1 and 5.2 we can state the following result.

Proposition 6.3. *Assume* (k1)–(k3) *and* (3.1)–(3.4) *hold. Let* (3.6) *and* (3.14) *hold, if* $k > 0$, *or* (3.15) *and* (5.1) *hold, if* $k = 0$. *The two parameter family* $\{U_f^k(t,\tau)\}$, *acting on* $\mathcal{H}_\alpha$ *and defined in* (6.3), *fulfills* (i) *and* (ii) *of* Definition (6.1).

Furthermore, the contracting estimates given in Theorem 5.4 ensure that the continuity properties (iii) and (iv) are also satisfied.

Proposition 6.4. *Assume* (k1)–(k3), (3.1)–(3.4) *and* (5.1) *hold. Let* (3.6) *holds if* $k > 0$, *or* (3.15) *holds if* $k = 0$. *Thus,* $U_f^k(\tau, t)$, *defined in* (6.3), *is a process on the space* $\mathcal{H}$, *for every* $k \geq 0$. *Moreover, if* (6.4) *holds, since* χ *is conserved, the restriction of* $U_f^k(t,\tau)$ *on* $\mathcal{H}_\alpha$ *is still a process on* $\mathcal{H}_\alpha$.

6.3 Basic assumptions

The asymptotic behavior of solutions to problems $\mathbf{P}_k$, $k \geq 0$, strongly depends on the long time behavior of the memory kernels, and, in particular, on the choice of the parameter ν_0, i.e., on a (cf. Introduction in [33]). To prove existence of an absorbing set, we will consider first the case $\nu_0 = 1$. In this case we require that the memory kernels ν and μ satisfy, besides (k1)–(k3), the following *exponential decay* condition

$$\text{(k4)} \qquad \nu'(\sigma) + \delta\nu(\sigma) \leq 0 \qquad \mu'(\sigma) + \delta\mu(\sigma) \leq 0 \qquad \forall\, \sigma \in \mathbb{R}^+$$

for some $\delta > 0$. We remark that the exponential decay of the memory kernels is the standard assumption even in the stability analysis for linear integrodifferential equations or systems (see, e.g., [48]).

To deal with the case $\nu_0 = -1$, we need to assume $|\Gamma_0| > 0$ in order to take advantage of Poincaré inequality. Furthermore, we require ν to be suitably dominated by μ; that is,

$$\text{(k5)} \qquad \nu(\sigma) \leq \frac{\delta_0}{8a_0 C_P^2}\mu(\sigma) \qquad \mu'(\sigma) + \delta\mu(\sigma) \leq 0 \qquad \forall\, \sigma \in \mathbb{R}^+$$

for some $0 < \delta_0 < \delta$. Again, we refer to [33] for some comments on (k5).

Remark 6.5. Note that, thanks to Poincaré inequality, we can endow $L^2_\mu(\mathbb{R}^+; V_0)$ with the norm

$$\|\eta\|^2_{L^2_\mu(\mathbb{R}^+;V_0)} = \int_0^\infty \mu(\sigma)\|\nabla\eta(\sigma)\|^2_H d\sigma.$$

As a consequence of (k5), the inclusion $L^2_\mu(\mathbb{R}^+; V_0) \subset L^2_\nu(\mathbb{R}^+; H)$ holds. Indeed

$$\int_0^\infty \nu(\sigma)\|\eta(\sigma)\|^2_H d\sigma \leq \frac{\delta_0}{C_P^2 a_0}\int_0^\infty \nu(\sigma)\|\eta(\sigma)\|^2_H d\sigma \leq \frac{\delta_0}{a_0}\int_0^\infty \mu(\sigma)\|\nabla\eta(\sigma)\|^2_H d\sigma.$$

Hence, if (k5) holds, we have $\mathcal{M} \equiv L^2_\mu(\mathbb{R}^+; V_0)$.

6.4 Main theorems

Here we show that the two parameter family of operators $\{U_f(t,\tau)\}$ defined by (6.3) satisfies a uniform in time estimate, provided that Proposition 6.3 and either (k4) or (k5) hold. This estimate is also uniform with respect to the symbol f running in a suitable bounded set. Moreover it yields the existence of a (uniform) absorbing set for the family of processes $\{U_f(t,\tau)\}$ when λ is linear (cf. Proposition 6.4).

Theorem 6.6. *Let* (k1)–(k3) *and* (3.14) *hold. Assume* $\mathcal{F} \subset \mathcal{T}^1(H)$ *be a bounded set, and denote*

$$F = \sup_{f\in\mathcal{F}} \|f\|_{\mathcal{T}^1(H)}.$$

If (6.4) *holds and* $f \in \mathcal{F}$, *denote by* $(\vartheta(t), \chi(t), \eta(t)) \in \mathcal{H}_\alpha$ *the corresponding solution to problem* $\mathbf{P}_k$. *In addition suppose that either* (k4) *holds, if* $\nu_0 = 1$, *or* $|\Gamma_0| > 0$ *and* (k5) *holds, if* $\nu_0 = -1$. *Then, there exists* $R_0 > 0$, *depending only on* F *such that, given any* $R > 0$ *and initial data which satisfy* $\|(\vartheta_0, \chi_0, \eta_0)\|_{\mathcal{H}} \leq R$ *and any* $\tau \in \mathbb{R}$, *there exists* $t_R = t_R(F)$ *such that*

$$\sup_{f\in\mathcal{F}} \|(\vartheta(t), \chi(t), \eta^t)\|_{\mathcal{H}} \leq R_0 \tag{6.7}$$

for almost every $t \geq \tau + t_R$.

The assumption on f can be slightly generalized if $k > 0$. Indeed, we have

Proposition 6.7. *Let $k > 0$ and let* (k1)–(k3) *and* (3.14) *hold. Assume $\mathcal{F}$ bounded set in $\mathcal{T}^1(H) + \mathcal{T}^2(V^*)$. If* (6.4) *holds and $f \in \mathcal{F}$, denote by $(\vartheta(t), \chi(t), \eta(t)) \in \mathcal{H}_\alpha$ the corresponding solution to problem $\mathbf{P}_k$. In addition suppose that either* (k4) *holds if $\nu_0 = 1$ or $|\Gamma_0| > 0$ and* (k5) *holds if $\nu_0 = -1$. Then,* Theorem 6.6 *still holds, the constant R_0 being dependent only on* $\sup_{f\in\mathcal{F}} \|f\|_{\mathcal{T}^1(H)+\mathcal{T}^2(V^*)}$.

We can now state the following basic result, which is a direct consequence of Theorem 6.6 and Proposition 6.7.

Theorem 6.8. *Let λ satisfies* (5.1) *and let* (k1)–(k3) *hold. Assume that $\mathcal{F}$ is a bounded set in $\mathcal{T}^1(H) + \mathcal{T}^2(V^*)$ if $k > 0$, or that $\mathcal{F}$ is a bounded set in $\mathcal{T}^1(H)$ if $k = 0$. In addition suppose that either* (k4) *holds, if $\nu_0 = 1$, or $|\Gamma_0| > 0$ and* (k5) *holds, if $\nu_0 = -1$. If* (6.4) *holds and $f \in \mathcal{F}$, then the family of processes $U_f^k(t,\tau)$ defined in* (6.3) *has a bounded uniform absorbing set $\mathcal{B}_0$. Furthermore, owing to the continuity property of processes, relation* (6.7) *actually holds for every $t \geq \tau + t_R$.*

Remark 6.9. Any ball of $\mathcal{H}_\alpha$ with radius strictly greater of R_0, can be chosen as a uniform absorbing set $\mathcal{B}_0$ for the family of processes $\{U_f(t,\tau)\}$.

When λ is linear, it is also possible to obtain further uniform estimates for the solutions to problem $\mathbf{P}_k$, which will be helpful in the following chapter. Indeed, we have

Corollary 6.10. *Under the hypotheses of* Theorem 6.8*, there exists a positive constant K depending (increasingly) only on F, R, and $t - \tau$, such that*

$$\sup_{f\in\mathcal{F}} \int_\tau^t \Big(\|\partial_t \chi(y)\|_{V^*}^2 + k\|\nabla\vartheta(y)\|_H^2 + \|\Delta\chi(y)\|_H^2 + k\|\nabla\Delta\chi(y)\|_H^2 \Big) dy \leq K. \tag{6.8}$$

6.5 Proof of Theorem 6.6

We first analyze the case $\nu_0 = 1$; for the sake of simplicity, we consider problem $\mathbf{P}_k$, $k \geq 0$, when $\Gamma_0 = \emptyset$, but no substantial changes in the proof are needed in the general case. Then we have

$$\begin{aligned}
&\langle \partial_t(\vartheta + \lambda(\chi)), v\rangle_V + (\vartheta, v)_H + k(\nabla\vartheta, \nabla v)_H \\
&\quad + \int_0^\infty \nu(\sigma)(\eta(\sigma), v)_H d\sigma + \int_0^\infty \mu(\sigma)(\nabla\eta(\sigma), \nabla v)_H d\sigma = (f, v)_H, \qquad \forall\, v \in V &(6.9)\\
&\langle \partial_t\chi, v\rangle_V + (\nabla w, \nabla v)_H = 0,. \qquad \forall\, v \in V &(6.10)\\
&\langle v^*, w\rangle_V = \langle v^*, -\Delta\chi + \chi^3 + \gamma'(\chi) - \lambda'(\chi)\vartheta\rangle_V, \qquad \forall\, v^* \in V^* &(6.11)\\
&(\partial_t\eta + \partial_s\eta, \zeta)_{\mathcal{M}} = (\vartheta, \zeta)_{\mathcal{M}}, \qquad \forall\, \zeta \in \mathcal{M} &(6.12)
\end{aligned}$$

and

$$\begin{aligned}
\vartheta(\tau) &= \vartheta_0\\
\chi(\tau) &= \chi_0\\
\eta^\tau &= \eta_0.
\end{aligned}$$

We perform some formal a priori estimates, which can be made rigorous in a suitable regularizing scheme. Take first equation (6.9) with $v = \vartheta$. This gives

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\vartheta\|_H^2 + \|\vartheta\|_H^2 + k\|\nabla\vartheta\|_H^2 + \langle\lambda'(\chi)\partial_t\chi, \vartheta\rangle_V \\ &\quad + \int_0^\infty \nu(\sigma)(\eta(\sigma), \vartheta)_H d\sigma + \int_0^\infty \mu(\sigma)(\nabla\eta(\sigma), \nabla\vartheta)_H d\sigma \\ &\leq \|f\|_H\|\vartheta\|_H. \end{aligned} \tag{6.13}$$

Recalling (4.9), take (6.10) with $v = \mathcal{T}\partial_t\chi$, and (6.11) with $v = \partial_t\chi$, to get

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla\chi\|_H^2 + \frac{1}{2}\|\chi\|_{L^4(\Omega)}^4 + 2\int_\Omega \gamma(\chi)d\Omega\right) + \|\partial_t\chi\|_{V^*}^2 - \langle\partial_t\chi, \lambda'(\chi)\vartheta\rangle_V = 0. \tag{6.14}$$

Introduce the functional

$$\rho_{\mathcal{M}}(\eta) = \int_0^\infty \nu(\sigma)\|\eta(\sigma)\|_H^2 d\sigma + \int_0^\infty \mu(\sigma)\|\nabla\eta(\sigma)\|_H^2 d\sigma$$

and consider (3.10) in the strong sense, that is a.e. in $I \times \Omega \times \mathbb{R}^+$. Multiply then (3.10) by $\nu\eta$ and the gradient of (3.10) by $\mu\nabla\eta$; integrate over $\Omega \times \mathbb{R}^+$, and add the results to obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\rho_{\mathcal{M}}(\eta) = &-\frac{1}{2}\int_0^\infty \nu(\sigma)\frac{d}{dt}\|\eta(\sigma\|_H^2 d\sigma - \frac{1}{2}\int_0^\infty \mu(\sigma)\frac{d}{dt}\|\nabla\eta(\sigma\|_H^2 d\sigma \\ &+ \int_0^\infty \nu(\sigma)(\eta(\sigma), \vartheta)_H d\sigma + \int_0^\infty \mu(\sigma)(\nabla\eta(\sigma), \nabla\vartheta)_H d\sigma. \end{aligned}$$

Using (k4), with an integration by parts, we get

$$\frac{1}{2}\frac{d}{dt}\rho_{\mathcal{M}}(\eta) + \frac{\delta}{2}\rho_{\mathcal{M}}(\eta) \leq \int_0^\infty \nu(\sigma)(\eta(\sigma), \vartheta)_H d\sigma + \int_0^\infty \mu(\sigma)(\nabla\eta(\sigma), \nabla\vartheta)_H d\sigma. \tag{6.15}$$

Addition of (6.13)–(6.15) entails

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\vartheta\|_H^2 + \|\nabla\chi\|_H^2 + \frac{1}{2}\|\chi\|_{L^4(\Omega)}^4 + 2\int_\Omega \gamma(\chi)d\Omega + \rho_{\mathcal{M}}(\eta)\right) \\ &\quad + \|\vartheta\|_H^2 + k\|\nabla\vartheta\|_H^2 + \|\partial_t\chi\|_{V^*}^2 + \frac{\delta}{2}\rho_{\mathcal{M}}(\eta) \\ &\leq \|f\|_H\|\vartheta\|_H. \end{aligned} \tag{6.16}$$

Now, taking $v = \mathcal{T}(\chi - m_\chi)$ in (3.8) we have

$$\frac{1}{2}\frac{d}{dt}\|\chi - m_\chi\|_{V^*}^2 + \langle\chi - m_\chi, w\rangle_V = 0$$

and setting $v = \chi - m_\chi$ in (3.9), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\chi - m_\chi\|_{V^*}^2 + (\chi - m_\chi, -\Delta\chi + \chi^3 + \gamma'(\chi) - \lambda'(\chi)\vartheta)_H = 0.$$

Writing separately each term appearing in the scalar product of the above equation (recall that by divergence theorem $\Delta\chi$ has null average), we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\chi - m_\chi\|_{V^*}^2 + \|\nabla\chi\|_H^2 + \int_\Omega \chi^4 d\Omega \\ &\leq (m_\chi, \chi^3 + \gamma'(\chi) - \lambda'(\chi)\vartheta)_H - (\chi, \gamma'(\chi) - \lambda'(\chi)\vartheta)_H.\end{aligned}$$

We denote by Λ_λ and by Λ_γ the Lipschitz constants of the functions λ and γ. By the properties of polynomials, and since there exists a positive constant $c = c(|\Omega|, \Lambda_\gamma)$ such that

$$\int_\Omega \gamma(\chi) d\Omega \leq \int_\Omega \chi^4 d\Omega + c$$

we have

$$\frac{1}{2}\frac{d}{dt}\|\chi - m_\chi\|_{V^*}^2 + \|\nabla\chi\|_H^2 + \frac{1}{2}\int_\Omega \chi^4 d\Omega + \frac{1}{2}\int_\Omega \gamma(\chi) d\Omega \leq \frac{1}{4}\|\vartheta\|_H^2 + c \tag{6.17}$$

where $c = c(\alpha, |\Omega|, \Lambda_\lambda, \Lambda_\gamma)$.

Consider equation (6.10) with $v = \chi$, and equation (6.11) with $v^* = -\Delta\chi$. By means of Young inequality we get

$$\frac{1}{2}\frac{d}{dt}\|\chi\|_H^2 + \|\Delta\chi\|_H^2 + (\nabla\chi, \nabla\chi^3)_H \leq \frac{1}{2}\|\Delta\chi\|_H^2 + \frac{1}{2}\left(\|\gamma'(\chi)\|_H^2 + \|\lambda'(\chi)\vartheta\|_H^2\right).$$

By the Lipschitz continuity of γ', and (3.14),

$$\|\gamma'(\chi)\|_H^2 + \|\lambda'(\chi)\vartheta\|_H^2 \leq c\left(\|\chi\|_V^2 + \|\vartheta\|_H^2\right)$$

hence, we are led to

$$\frac{1}{2}\frac{d}{dt}\|\chi\|_H^2 + \frac{1}{2}\|\Delta\chi\|_H^2 + (\nabla\chi, \nabla\chi^3)_H \leq c\left(\|\chi\|_V^2 + \|\vartheta\|_H^2\right). \tag{6.18}$$

At this point, let $\kappa > 0$ be a small constant. Add the elementary inequality

$$\kappa\|\chi - m_\chi\|_{V^*}^2 \leq \kappa\|\chi\|_{L^4(\Omega)}^4 + c(\alpha)$$

to (6.16), (6.17) and (6.18) multiplied by κ. Then, setting

$$\begin{aligned}\Phi^2(t) = \|\vartheta(t)\|_H^2 + \|\chi(t) - m_\chi\|_{V^*}^2 + \kappa\|\chi(t)\|_H^2 + \|\nabla\chi(t)\|_H^2 \\ + \frac{1}{2}\|\chi(t)\|_{L^4(\Omega)}^4 + 2\int_\Omega \gamma(\chi(t)) d\Omega + \rho_{\mathcal{M}}(\eta^t)\end{aligned}$$

there exists $\varepsilon > 0$ small enough such that

$$\frac{d}{dt}\Phi^2 + 2\varepsilon\Phi^2 + k\|\nabla\vartheta\|_H^2 + \|\partial_t\chi\|_{V^*}^2 + \kappa\|\Delta\chi\|_H^2 \leq 2\|f\|_H\|\vartheta\|_H + C \tag{6.19}$$

where C, once α is fixed, depends only on the structural constants of the problem, but is independent of k.

Applying Lemma A.6 to the differential inequality (6.19), we get the estimate

$$\Phi^2(t) \le C_0(R)e^{-2\varepsilon(t-\tau)} + C_1(F) \tag{6.20}$$

where

$$C_0(R) = 2\Phi^2(\tau)$$

and

$$C_1(F) = \frac{C}{\varepsilon} + \frac{4e^{2\varepsilon F}}{(1-e^{-\varepsilon})^2}.$$

To conclude the case $\nu_0 = 1$, we are left to find an estimate for the term

$$\int_0^\infty \mu(\sigma)\|\eta(\sigma)\|_H^2 d\sigma.$$

Consider then equation (6.12) with $v = \eta$. We have

$$\frac{1}{2}\left(\frac{d}{dt}\|\eta\|_{\mathcal{M}}^2 + \int_0^\infty \nu(\sigma)\frac{d}{d\sigma}\|\eta(\sigma\|_H^2 d\sigma + \int_0^\infty \mu(\sigma)\frac{d}{d\sigma}\|\eta(\sigma)\|_V^2 d\sigma\right) = (\vartheta, \eta)_{\mathcal{M}}.$$

Integration by parts, and the assumption (k4) on the kernels yield

$$\frac{1}{2}\frac{d}{dt}\|\eta\|_{\mathcal{M}}^2 + \frac{\delta}{2}\|\eta\|_{\mathcal{M}}^2 \le (\vartheta, \eta)_{\mathcal{M}}. \tag{6.21}$$

Denote

$$\begin{aligned}\Psi^2(t) = {} & \|\vartheta(t)\|_H^2 + \|\chi(t) - m_\chi\|_{V^*}^2 + \kappa\|\chi(t)\|_H^2 + \|\nabla\chi(t)\|_H^2 \\ & + \frac{1}{2}\|\chi(t)\|_{L^4(\Omega)}^4 + 2\int_\Omega \gamma(\chi(t))d\Omega + \|\eta^t\|_{\mathcal{M}}^2.\end{aligned}$$

Repeating the same argument leading to (6.19) with $\|\eta\|_{\mathcal{M}}^2$ in place of $\rho_{\mathcal{M}}(\eta)$, we obtain

$$\frac{d}{dt}\Psi^2 + 2\varepsilon\Psi^2 \le 2\|f\|_H\|\vartheta\|_H + 2\int_0^\infty \mu(\sigma)(\eta(\sigma), \vartheta)_H d\sigma + C_1 \tag{6.22}$$

for almost every $t \ge \tau$. On the other hand,

$$\begin{aligned}2\int_0^\infty \mu(\sigma)(\eta(\sigma), \vartheta)_H d\sigma & \le \varepsilon\|\eta\|_{\mathcal{M}}^2 + \frac{b_0^2}{\varepsilon}\|\vartheta\|_H^2 \\ & \le \varepsilon\Psi^2(t) + \frac{b_0^2}{\varepsilon}\Phi^2(t).\end{aligned}$$

Taking into account estimate (6.20), inequality (6.22) becomes

$$\frac{d}{dt}\Psi^2(t) + \varepsilon\Psi^2(t) \le 2\|f\|_H\Psi(t) + \frac{b_0^2}{\varepsilon}\left(C_0(R)e^{-2\varepsilon(t-\tau)} + C_1(F)\right) + C.$$

A further application of Lemma A.6 yields

$$\begin{aligned}\Psi^2(t) &\leq 2\Psi^2(\tau)e^{-\varepsilon(t-\tau)} + \frac{b_0^2 C_0(R)}{\varepsilon}\int_\tau^t e^{-2\varepsilon(s-\tau)}e^{-\varepsilon(t-s)}ds + \widehat{C}_1(F) \\ &\leq \widehat{C}_0(R)e^{-\varepsilon(t-\tau)} + \widehat{C}_1(F) \end{aligned} \tag{6.23}$$

where we have set

$$\widehat{C}_0(R) = 2\Psi^2(\tau) + \frac{b_0^2 C_0(R)}{\varepsilon^2}$$

and

$$\widehat{C}_1(F) = \frac{2}{\varepsilon}\left(\frac{b_0^2 C_1(F)}{\varepsilon} + C\right) + \frac{4e^{\varepsilon}F}{(1-e^{-\varepsilon/2})^2}.$$

Now, if we choose

$$R_0 = 2\widehat{C}_1(F)$$

then, (6.7) is satisfied with

$$t_R(F) = \max\left\{0, \frac{1}{\varepsilon}\ln\frac{\widehat{C}_0(R)}{\widehat{C}_1(F)}\right\}.$$

Consider now the case $\nu_0 = -1$, together with $|\Gamma_0| > 0$ and hypothesis (k5). In light of Remark 6.5, we work in $\mathcal{M} = L^2_\mu(\mathbb{R}^+; V_0)$.

We argue as in the former case, the only difference being that here we use $\|\eta\|_{\mathcal{M}}$ instead of $\rho_{\mathcal{M}}(\eta)$, namely we consider equation (6.12) in a strong sense, take its gradient, and multiply it by $\mu(\sigma)\nabla\eta(\sigma)$. Integration over $\Omega\times\mathbb{R}^+$ gives

$$\frac{1}{2}\frac{d}{dt}\int_0^\infty \mu(\sigma)\|\nabla\eta(\sigma)\|_H^2 d\sigma + \frac{1}{2}\int_0^\infty \mu(\sigma)\frac{d}{ds}\|\nabla\eta(\sigma)\|_H^2 d\sigma = \int_0^\infty \mu(\sigma)(\nabla\vartheta, \nabla\eta(\sigma))_H d\sigma.$$

Integration by parts, and (k5) lead to

$$\frac{1}{2}\frac{d}{dt}\|\eta\|_{\mathcal{M}}^2 + \frac{\delta}{2}\|\eta\|_{\mathcal{M}}^2 \leq \int_0^\infty \mu(\sigma)(\nabla\vartheta, \nabla\eta(\sigma))_H d\sigma. \tag{6.24}$$

On the other hand, the following analog to (6.13) holds

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\vartheta\|_H^2 + \|\vartheta\|_H^2 + k\|\nabla\vartheta\|_H^2 + \langle\lambda'(\chi)\partial_t\chi, \vartheta\rangle_V + \int_0^\infty \mu(\sigma)(\nabla\eta(\sigma), \nabla\vartheta)_H d\sigma \\ &\leq \int_0^\infty \nu(\sigma)(\eta(\sigma), \vartheta)_H d\sigma + \|f\|_H\|\vartheta\|_H. \end{aligned} \tag{6.25}$$

Using Young inequality and (k5), we have

$$\begin{aligned} 2\int_0^\infty \nu(\sigma)(\eta(\sigma),\vartheta)_H d\sigma &\leq \frac{1}{4}\|\vartheta\|_H^2 + 4a_0\int_0^\infty \nu(\sigma)\|\eta(\sigma)\|_H^2 d\sigma \\ &\leq \frac{1}{4}\|\vartheta\|_H^2 + \frac{\delta_0}{2C_P^2}\int_0^\infty \mu(\sigma)\|\eta(\sigma)\|_H^2 d\sigma \\ &\leq \frac{1}{4}\|\vartheta\|_H^2 + \frac{\delta_0}{2}\|\eta\|_{\mathcal{M}}^2. \end{aligned}$$

Taking the above inequality into account, adding (6.24) and (6.25), we then get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|\vartheta\|_H^2 + \|\eta\|_{\mathcal{M}}^2\Big) + \frac{3}{4}\|\vartheta\|_H^2 + k\|\nabla\vartheta\|_H^2 \\
&\quad + \langle\lambda'(\chi)\partial_t\chi, \vartheta\rangle_V + \frac{\delta-\delta_0}{2}\|\eta\|_{\mathcal{M}}^2 \\
&\leq \|f\|_H\|\vartheta\|_H. \qquad (6.26)
\end{aligned}$$

We repeat the same steps leading to (6.19), using equation (6.26) instead of (6.15) and of (6.13). Setting

$$\begin{aligned}
\Phi^2(t) &= \|\vartheta(t)\|_H^2 + \|\chi(t) - m_\chi\|_{V^*}^2 + \kappa\|\chi(t)\|_H^2 + \|\nabla\chi(t)\|_H^2 \\
&\quad + \frac{1}{2}\|\chi(t)\|_{L^4(\Omega)}^4 + 2\int_\Omega \gamma(\chi(t))d\Omega + \|\eta^t\|_{\mathcal{M}}^2
\end{aligned}$$

and choosing ε small enough, we obtain

$$\begin{aligned}
&\frac{d}{dt}\Phi^2 + \varepsilon\Phi^2 + k\|\nabla\vartheta\|_H^2 + \|\partial_t\chi\|_{V^*}^2 + \kappa\|\Delta\chi\|_H^2 \\
&\leq 2\|f\|_H\|\vartheta\|_H + C
\end{aligned}$$

where C is again independent of initial data. The proof follows from an application of Lemma A.6. □

6.6 Proof of Corollary 6.10

We consider here the case $\nu_0 = 1$. The case $\nu_0 = -1$ can also be proved with the same techniques used in the proof of Theorem 6.6.

Let $0 \leq t - \tau \leq \rho$. Adding equations (6.16) and (6.18),

$$\frac{d}{dt}\widehat{\Phi}^2 + k\|\nabla\vartheta\|_H^2 + \|\partial_t\chi\|_{V^*}^2 + \|\Delta\chi\|_H^2 \leq c\widehat{\Phi}^2 + 2\|f\|_H\widehat{\Phi} \qquad (6.27)$$

where

$$\widehat{\Phi}^2(t) = \|\vartheta(t)\|_H^2 + \|\nabla\chi(t)\|_H^2 + \frac{1}{2}\|\chi(t)\|_{L^4(\Omega)}^4 + 2\int_\Omega \gamma(\chi(t))d\Omega + \rho_{\mathcal{M}}(\eta^t).$$

Notice that $\widehat{\Phi}^2(t) \leq \Phi^2(\tau)$ for every $t \geq \tau$. Integration of (6.27) over (τ, t) yields

$$\begin{aligned}
&\int_\tau^t \Big(\|\partial_t\chi(y)\|_{V^*}^2 + k\|\nabla\vartheta(y)\|_H^2 + \|\Delta\chi(y)\|_H^2\Big)dy \\
&\leq 2\Phi^2(\tau) + c\int_\tau^t \Phi^2(y)dy + 2\int_\tau^t \|f(y)\|\Phi(y)dy \\
&\leq C(F, R, \rho) \qquad (6.28)
\end{aligned}$$

thanks to estimate (6.20).

Equation (6.10) with $v^* = -\Delta\chi$ and equation (6.11) with $v = \Delta^2\chi$ give

$$\frac{1}{2}\frac{d}{dt}\|\nabla\chi\|_H^2 - (\nabla w, \nabla\Delta\chi)_H = 0$$

and

$$-(\nabla\Delta\chi, \nabla w)_H = \langle \Delta^2\chi, -\Delta\chi + \chi^3 + \gamma'(\chi) - \lambda_0\vartheta\rangle_V.$$

Exploiting the continuous injection $V \hookrightarrow L^6(\Omega)$, and Hölder and Young inequalities, we obtain the estimate

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla\chi\|_H^2 + \frac{1}{2}\|\nabla\Delta\chi\|_H^2 \le \frac{1}{2}\left(\|\nabla\chi^3\|_H^2 + \|\nabla\gamma'(\chi)\|_H^2 + \lambda_0^2\|\nabla\vartheta\|_H^2\right)\\
&\le c\left(\|\chi\|_{L^6(\Omega)}^4\|\chi\|_W^2 + \|\nabla\chi\|_H^2 + \|\nabla\vartheta\|_H^2\right)\\
&\le c\left(1 + \|\Delta\chi\|_H^2 + \|\nabla\vartheta\|_H^2\right). \qquad (6.29)
\end{aligned}$$

The last inequality is a consequence of the V-bound of the second component of the solution; the constant c is a function of R.

Multiply (6.29) by k and integrate over (τ, t), then estimates (6.20) and (6.28) complete the proof. $\square$

Chapter 7

The uniform attractor for the parabolic case

> It is impossible to study the properties of a single mathematical trajectory. The physicist knows only bundles of trajectories, corresponding to slightly different initial conditions.
>
> (Leon Brillouin)

Longtime behavior of a single solution to a well-posed initial value problem is often described by the so-called ω-limit set, i.e., a set to which the orbit converges as $t \to \infty$. In general the existence of such sets, enjoying special properties as, e.g., invariance, has to be proved. In this chapter we show that such sets exist for $\mathbf{P}_k$, and that they are part of a bigger compact invariant set $\mathcal{A}$ which attracts *all* the orbits.

7.1 Dissipative processes and their attractors

We first recall some basic definitions and properties (see, e.g., [42, 56, 57]).

Definition 7.1. A set $\mathcal{K}$ is said to be *uniformly attracting* for the family $\{U_f(t,\tau), f \in \mathcal{F}\}$ if, for any $\tau \in \mathbb{R}$ and any bounded set $\mathcal{B}$ in X,

$$\lim_{t \longrightarrow \infty} \left[\sup_{f \in \mathcal{F}} \delta_X(U_f(t,\tau)\mathcal{B}, \mathcal{K}) \right] = 0$$

where

$$\delta_X(\mathcal{B}_1, \mathcal{B}_2) = \sup_{z_1 \in \mathcal{B}_1} \inf_{z_2 \in \mathcal{B}_2} d(z_1, z_2)$$

is the Hausdorff semidistance of two sets $\mathcal{B}_1$ and $\mathcal{B}_2$ in X.

Definition 7.2. A family of processes that possesses a uniformly attracting compact set is said to be *uniformly asymptotically compact.*

Definition 7.3. A closed set $\mathcal{A} \subset X$ is said to be a *uniform attractor* for the family $\{U_f(t,\tau), f \in \mathcal{F}\}$ if it is at the same time uniformly attracting and contained in every closed uniformly attracting set.

As a direct consequence of Definition 7.3, the uniform attractor of a family of processes (if it exists) is unique. This minimality property replaces the fully invariance requested to attractors of semigroups (see Theorem 5.2.5 in [42]).

If X is a complete metric space, we introduce the concept of bounded complete trajectory of a process (see, for instance, [13, 14]).

Definition 7.4. A curve $z(t) \in X$, $t \in \mathbb{R}$, is a *complete trajectory* of a process $U(t,\tau)$ if the following identity holds

$$U(t,\tau)z(\tau) = z(t), \qquad \forall\, t \geq \tau,\ \tau \in \mathbb{R}.$$

Definition 7.5. The *kernel* $\mathcal{U}$ of the process $U(t,\tau)$ consists of all bounded complete trajectories of the process.

Definition 7.6. The set

$$\mathcal{U}(s) = \Big\{ z(s) : z(\cdot) \in \mathcal{U} \Big\}$$

is the *kernel section* at time $t = s$, $s \in \mathbb{R}$.

We denote by $\mathcal{U}_f(0)$ the kernel section at time $t = 0$ of the kernel $\mathcal{U}_f$ of the process $U_f(t,\tau)$ with the symbol f. We recall that a semigroup which possesses a compact absorbing set has a *global* attractor (see [4, 42, 56]). Though the assumption $\mathcal{F}$ is compact is physically acceptable and not very strict, asking for a compact absorbing set is a strong restriction on the class of dynamical system under consideration. We weaken the request demanding asymptotic compactness of the set of trajectories. Indeed, we shall exploit the following fundamental result, which appears as Theorem 3.2 in [14] (see also [13]).

Theorem 7.7. *Assume the family of processes $\{U_f(t,\tau), f \in \mathcal{F}\}$ acting on X be uniformly asymptotically compact and $X \times \mathcal{F}$-continuous for every $\tau \in \mathbb{R}$ and $t \geq \tau$. Also let $\mathcal{F}$ be a compact metric space and $\{T(t), t \geq 0\}$ be a continuous semigroup on $\mathcal{F}$, satisfying the translation identity* (6.1)*. Then the semigroup $\{\Sigma(t), t \geq 0\}$, acting on $X \times \mathcal{F}$ and defined by* (6.2)*, possesses a compact attractor $\widetilde{\mathcal{A}}$, which is fully invariant with respect to $\Sigma(t)$, i.e.*

$$\Sigma(t)\widetilde{\mathcal{A}} = \widetilde{\mathcal{A}}$$

for every $t \geq 0$. Moreover, the following properties hold

1. *$\Pi_1\widetilde{\mathcal{A}} = \mathcal{A}$ is the uniform attractor of the family of processes $\{U_f(t,\tau), f \in \mathcal{F}\}$;*

2. *$\Pi_2\widetilde{\mathcal{A}} = \mathcal{A}_{\mathcal{F}}$ is the global attractor of $T(t)$;*

3. *$\widetilde{\mathcal{A}} = \displaystyle\bigcup_{f \in \mathcal{F}} \mathcal{U}_f(0) \times \{f\}$.*

The existence of a global attractor of $T(t)$ acting on the compact space $\mathcal{F}$ easily follows from standard arguments (see [56]).

We can also show sufficient conditions to have a connected attractor.

Theorem 7.8. *In the hypothesis of* Theorem 7.7, *if $\mathcal{F}$ is connected and $\{U_f(t,\tau), f \in \mathcal{F}\}$ has a connected uniform absorbing set, then the attractor $\mathcal{A}$ for the family $\{U_f(t,\tau), f \in \mathcal{F}\}$ is connected.*

Proof. The set $\mathcal{A} \times \mathcal{A}_{\mathcal{F}}$ is the attractor of the semigroup $\Sigma(t)$ on the complete metric space $X \times \mathcal{F}$. If $\mathcal{B}_0$ is a bounded connected absorbing set for $U_f(t,\tau)$, then $\mathcal{B}_0 \times \mathcal{F}$ is a bounded connected absorbing set for $\Sigma(t)$. Hence (see Proposition 5.2.7 in [42]) the attractor $\mathcal{A} \times \mathcal{F}$ of $\Sigma(t)$, as well as its projection on X, are connected. □

7.2 The solution decomposition

In the sequel, we consider a translation compact function

$$g \in L^1_{\text{loc}}(\mathbb{R}; H). \tag{7.1}$$

Then, the hull $\mathrm{H}(g)$ is a compact metric space, on which it is naturally defined a strongly continuous semigroup, namely the translation semigroup $T(t)$

$$T(t)f = f^t, \qquad \forall\, f \in \mathrm{H}(g).$$

This semigroup satisfies the translation equality (6.1). The space $\mathrm{H}(g)$ is also compact by construction. In particular the global attractor of $\mathrm{H}(g)$ coincide with $\mathrm{H}(g)$ itself.

We require some more smoothness on the function γ (cf. (3.4))

$$\gamma \in C^3(\mathbb{R}) \text{ such that } \gamma',\ \gamma'',\ \gamma''' \in L^\infty(\mathbb{R}). \tag{7.2}$$

We study the asymptotic behavior of the family of processes

$$\{U^k_f(t,\tau), f \in \mathrm{H}(g)\}$$

acting on $\mathcal{H}_\alpha$ (for any given $\alpha > 0$), which is generated by the solution to problem $\mathbf{P}_k$ with $k > 0$. To exploit Theorem 7.7, and to prove that $\{U^k_f(t,\tau), f \in \mathrm{H}(g)\}$ is uniformly asymptotically compact we need to decompose the solution to $\mathbf{P}_k$ in three parts: the first vanishing at infinity, and the two others belonging to a compact set.

We write down (formally) $\mathbf{P}_k$ in a slightly different form; that is,

$$\begin{aligned}
\partial_t \vartheta &= -\vartheta - \lambda_0 \partial_t \chi + k\Delta\vartheta - \nu_0 \int_0^\infty \nu(\sigma)\eta(\sigma)d\sigma + \int_0^\infty \mu(\sigma)\Delta\eta(\sigma)d\sigma + f \\
\partial_t \chi &= \Delta(-\Delta\chi + \chi^3 + \gamma'(\chi) - \lambda_0\vartheta) \\
\partial_t \eta &= -\partial_s \eta + \vartheta
\end{aligned}$$

with

$$\begin{aligned}
\vartheta(\tau) &= \vartheta_0 \\
\chi(\tau) &= \chi_0 \\
\eta^\tau &= \eta_0.
\end{aligned}$$

Furthermore, we introduce the triplets

$$\begin{aligned} z_0 &= (\vartheta_0, \chi_0, \eta_0) \\ z(t) = U_f^k(t,\tau)z_0 &= (\vartheta(t), \chi(t), \eta(t)). \end{aligned}$$

We let

$$z_0 \in \mathcal{B}_0 \subset \mathcal{H}_\alpha$$

where $\mathcal{B}_0$ is a bounded, uniform absorbing set, whose existence is guaranteed by Theorem 6.8.

Observe that $z(t) \in \mathcal{H}_0$ whenever $z_0 \in \mathcal{H}_0$, where

$$\mathcal{H}_0 = H \times V_A \times \mathcal{M}.$$

We now decompose the solution to problem $\mathbf{P}_k$ in the following way:

$$z = z_L + z_E + z_N$$

where z_L, z_E, z_N are the solutions to the systems

$$\begin{aligned}
\partial_t \vartheta_L &= -\vartheta_L - \lambda_0 \partial_t \chi_L + k\Delta\vartheta_L - \nu_0 \int_0^\infty \nu(\sigma)\eta_L(\sigma)d\sigma + \int_0^\infty \mu(\sigma)\Delta\eta_L(\sigma)d\sigma && (7.3)\\
\partial_t \chi_L &= \Delta(-\Delta\chi_L - \lambda_0\vartheta_L) && (7.4)\\
\partial_t \eta_L &= -\partial_s \eta_L + \vartheta_L && (7.5)\\
z_L(\tau) &= (\vartheta_0, \chi_0 - m_{\chi_0}, \eta_0) && (7.6)
\end{aligned}$$

$$\begin{aligned}
\partial_t \vartheta_E &= -\vartheta_E - \lambda_0 \partial_t \chi_E + k\Delta\vartheta_E - \nu_0 \int_0^\infty \nu(\sigma)\eta_E(\sigma)d\sigma + \int_0^\infty \mu(\sigma)\Delta\eta_E(\sigma)d\sigma + f && (7.7)\\
\partial_t \chi_E &= \Delta(-\Delta\chi_E - \lambda_0\vartheta_E) && (7.8)\\
\partial_t \eta_E &= -\partial_s \eta_E + \vartheta_E && (7.9)\\
z_E(\tau) &= (0,0,0) && (7.10)
\end{aligned}$$

and

$$\begin{aligned}
\partial_t \vartheta_N &= -\vartheta_N - \lambda_0 \partial_t \chi_N + k\Delta\vartheta_N - \nu_0 \int_0^\infty \nu(\sigma)\eta_N(\sigma)d\sigma + \int_0^\infty \mu(\sigma)\Delta\eta_N(\sigma)d\sigma && (7.11)\\
\partial_t \chi_N &= \Delta(-\Delta\chi_N + \chi^3 + \gamma'(\chi) - \lambda_0\vartheta_N) && (7.12)\\
\partial_t \eta_N &= -\partial_s \eta_N + \vartheta_N && (7.13)\\
z_N(\tau) &= (0, m_{\chi_0}, 0). && (7.14)
\end{aligned}$$

Along the lines of Theorems 4.1 and 4.2, one can easily prove that systems (7.3)–(7.6), (7.7)–(7.10), (7.11)–(7.14) admit unique solutions which belong to the spaces $C(\mathcal{H}_0, \mathcal{H}_0)$, $C(\mathcal{H}_0, \mathcal{H}_0)$, $C(\mathcal{H}_\alpha, \mathcal{H}_\alpha)$, respectively.

Consider first the linear and autonomous system (7.3)–(7.5). It is not difficult to realize that there exists a strongly continuous semigroup of bounded linear operators

$$S(t) : \mathcal{H}_0 \longrightarrow \mathcal{H}_0$$

such that for every $t \geq \tau$

$$z_L(t) = S(t-\tau) z_L(\tau). \tag{7.15}$$

Consequently, the solution $z_E(t)$ to (7.7)–(7.10) is given by the Duhamel integral

$$z_E(t) = \int_\tau^t S(t-y)(f(y), 0, 0)dy = \int_0^{t-\tau} S(t-\tau-y)(f^\tau(y), 0, 0)dy$$

for any $t \geq \tau$.

7.3 Some technical lemmas

We state some results, which resume the basic properties of z_L, z_E, z_N.

Lemma 7.9. *The semigroup $S(t)$ given by* (7.15) *has an exponential decay; that is,*

$$\|S(t)z\|_{\mathcal{H}} \leq M e^{-\varepsilon t} \|z\|_{\mathcal{H}}, \qquad \forall\, z \in \mathcal{H}_0 \tag{7.16}$$

for some $M \geq 1$ and $\varepsilon > 0$.

Lemma 7.10. *There exists a relatively compact set $\mathcal{K}_E \subset \mathcal{H}_0$ such that*

$$\bigcup_{f \in \mathrm{H}(g)} \bigcup_{z_0 \in \mathcal{B}_0} \bigcup_{\tau \in \mathbb{R}} \bigcup_{t \geq \tau} z_E(t) \subset \mathcal{K}_E.$$

Lemma 7.11. *Let* (7.2) *hold. Then there exists a positive constant C depending only on $\sup_{z_0 \in \mathcal{B}_0} \|z_0\|_{\mathcal{H}}$ and on g such that*

$$\|z_N(t)\|_{\mathcal{V}} \leq C$$

for every $t \geq \tau$ and every $\tau \in \mathbb{R}$.

It is not immediate, after Lemma 7.11, that

$$\bigcup_{f \in \mathrm{H}(g)} \bigcup_{z_0 \in \mathcal{B}_0} \bigcup_{\tau \in \mathbb{R}} \bigcup_{t \geq \tau} \eta_N^t$$

is compact in $\mathcal{M}$, since the injection $\mathcal{L} \hookrightarrow \mathcal{M}$ is not. This is a property relying on the structure of the solution η. Indeed we have

Lemma 7.12. *The set*

$$\mathcal{E} = \bigcup_{f \in \mathrm{H}(g)} \bigcup_{z_0 \in \mathcal{B}_0} \bigcup_{\tau \in \mathbb{R}} \bigcup_{t \geq \tau} \eta_N^t \subset \mathcal{M}$$

is relatively compact.

Denote the ball of radius C in $V\times W$ by $\mathcal{B}_C$. The compact embedding $V\times W \hookrightarrow H\times V$, Lemma 7.11 and Lemma 7.12 ensure that the set

$$\mathcal{K}_N = \mathcal{B}_C \times \mathcal{E} \subset \mathcal{H}_\alpha$$

is relatively compact.

Define

$$\mathcal{K} = \mathcal{K}_E + \mathcal{K}_N \subset \mathcal{H}_\alpha. \tag{7.17}$$

By continuity, collecting Lemma 7.10, 7.11, 7.12, we have that $\mathcal{K}$ is relatively compact.

Corollary 7.13. *There exists a relatively compact set $\mathcal{K}\subset\mathcal{H}_\alpha$ such that*

$$\bigcup_{f\in \mathrm{H}(g)} \bigcup_{z_0\in\mathcal{B}_0} \bigcup_{\tau\in\mathbb{R}} \bigcup_{t\geq\tau} \big(z_E(t)+z_N(t)\big) \subset \mathcal{K}.$$

7.4 Main theorems

The natural consequence of the behavior of the solutions $z_L(t)$, $z_E(t)$, $z_N(t)$ is the existence of a compact attracting set. Indeed, the set $\overline{\mathcal{K}}$, where $\mathcal{K}$ is defined in Corollary 7.13, is uniformly attracting for the family of processes $\{U_f^k(t,\tau), f\in \mathrm{H}(g)\}$.

Theorem 7.14. *When $k>0$, the family*

$$\{U_f^k(t,\tau), f\in \mathrm{H}(g)\}$$

is uniformly asymptotically compact.

Moreover, Theorem 5.4 ensures that the map

$$U_\bullet^k(t,\tau):\mathcal{H}_\alpha\times \mathrm{H}(g)\longrightarrow \mathcal{H}_\alpha$$

is continuous.

We can now exploit Theorem 7.7 to state the main theorem of this chapter. Indeed, the semigroup $\Sigma(t)$ acting on the extended phase space has an attractor, whose projection $\mathcal{A}$ on $\mathcal{H}$ is the attractor of the family of processes (6.3) and whose projection on $\mathrm{H}(g)$ coincide with $\mathrm{H}(g)$ itself. The property that $\mathcal{A}$ is connected comes from Theorem 7.8.

Theorem 7.15. *Let* (k1)–(k3), (5.1) *and* (7.2) *hold, and let g be a translation compact function in $L^1_{\mathrm{loc}}(\mathbb{R};H)$. Assume that either* (k4) *holds with $\nu_0=1$, or* (k5) *and $|\Gamma|>0$ hold with $\nu_0=-1$. Then the family of processes defined by* (6.3), *for $k>0$, has a compact and connected uniform attractor given by*

$$\mathcal{A}=\mathcal{A}_\alpha^k=\Big\{\mathcal{U}_f(0),\ \textit{for any } f\in \mathrm{H}(g)\Big\}.$$

Remark 7.16. Note that $\mathcal{A}$ depends both on k and on α. For the sake of simplicity, in the sequel we shall omit such dependence since k and α are supposed to be fixed.

When we consider

$$g \in L^2_{\text{loc}}(\mathbb{R}^+; H) \tag{7.18}$$

instead of (7.1), every result in this section obviously follows. In addiction, we can state some more regularity about the attractor $\mathcal{A}$. Indeed, we can proceed with a double decomposition of the solution $z = z_L + z_N$, where the external source is included in the nonlinear part, and related estimates are well treated. It follows that

$$\mathcal{K} \equiv \mathcal{K}_E$$

then the properties of the attractor $\mathcal{A}$ are the same properties of $\mathcal{K}_E$. In particular, vectors belonging to $\mathcal{A}$ are $\mathcal{V}$-uniformly bounded.

Proposition 7.17. *Let the same hypothesis of* Theorem 7.15 *hold, but assume g to be a translation compact function in $L^2_{\text{loc}}(\mathbb{R}; H)$. Then the family of processes defined by* (6.3), *for $k > 0$, has a compact, connected and $\mathcal{V}$-bounded uniform attractor $\mathcal{A}$. Moreover, the third component of $\mathcal{A}$ is bounded in $H^1_\nu(\mathbb{R}^+; H) \cap H^1_\mu(\mathbb{R}^+; H)$.*

7.5 Proof of Lemma 7.9

All proofs in this chapter are carried out in the case $\nu_0 = 1$. The case $\nu_0 = -1$ follows on account of Remark 6.5.

We notice first that the second component of the solution has null medium average; then, if we choose

$$p(v) = \int_\Omega v d\Omega$$

in the Poincaré inequality, we have

$$\|\nabla \chi\|_H \leq \|\chi\|_V \leq (1 + c_P)\|\nabla \chi\|_H, \qquad \forall\, \chi \in V_A$$

then, along this proof, we exploit this equivalence.

Take the product in H of equation (7.3) with ϑ_L:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\vartheta_L\|^2_H + \|\vartheta_L\|^2_H + k\|\nabla\vartheta_L\|^2_H + \lambda_0(\partial_t\chi_L, \vartheta_L)_H \\
&+ \int_0^\infty \nu(\sigma)(\eta_L(\sigma), \vartheta_L)_H d\sigma + \int_0^\infty \mu(\sigma)(\nabla\eta_L(\sigma), \vartheta_L) d\sigma = 0,
\end{aligned} \tag{7.19}$$

and consider the product in H of equation (7.4) with $\mathcal{T}\partial_t\chi_L$:

$$\frac{1}{2}\frac{d}{dt}\|\nabla\chi_L\|^2_H + \|\partial_t\chi_L\|^2_{V^*} - \lambda_0(\partial_t\chi_L, \vartheta_L)_H = 0. \tag{7.20}$$

Consider again the product in H of equation (7.4) with $\mathcal{T}\chi_L$:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\chi_L\|^2_H + \|\nabla\chi_L\|^2_H &= \lambda_0(\chi_L, \vartheta_L) \\
&\leq \frac{\lambda_0}{2}\|\chi_L\|^2_H + \frac{\lambda_0}{2}\|\vartheta_L\|^2_H
\end{aligned} \tag{7.21}$$

add (7.19), (7.20), (6.15) (which holds also for η_L and ϑ_L) to (7.21) multiplied by a suitably small constant $\kappa > 0$. We can then find $\varepsilon > 0$ such that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\vartheta_L\|_H^2 + \kappa\|\chi_L\|_H^2 + \|\chi_L\|_V^2 + \rho_{\mathcal{M}}(\eta_L)\right) \\
&+ 2\varepsilon\left(\|\vartheta_L\|_H^2 + \kappa\|\chi_L\|_H^2 + \|\chi_L\|_V^2 + \rho_{\mathcal{M}}(\eta_L)\right) \\
&+ \frac{k}{2}\|\nabla\vartheta_L\|_H^2 + \frac{1}{2}\|\partial_t\chi_L\|_{V^*}^2 \leq 0.
\end{aligned} \tag{7.22}$$

Apply Lemma A.6 to the previous equation with

$$\Phi_L^2(t) = \|\vartheta_L(t)\|_H^2 + \kappa\|\chi_L(t)\|_H^2 + \|\chi_L(t)\|_V^2 + \rho_{\mathcal{M}}(\eta_L^t)$$

then we have

$$\Phi_L^2(t) \leq 2\Phi_L^2(\tau)e^{-2\varepsilon(t-\tau)} \tag{7.23}$$

for all $t \geq \tau$.

Set now

$$\Psi_L^2(t) = \|\vartheta_L(t)\|_H^2 + \kappa\|\chi_L(t)\|_H^2 + \|\chi_L(t)\|_V^2 + \|\eta_L^t\|_{\mathcal{M}}^2$$

and repeat the same argument leading to (7.22) with $\|\eta_L\|_{\mathcal{M}}^2$ in place of $\rho_{\mathcal{M}}(\eta_L)$. We obtain

$$\frac{d}{dt}\Psi_L^2 + \frac{3}{2}\varepsilon\Psi_L^2 \leq 2\int_0^\infty \mu(\sigma)(\eta_L(\sigma), \vartheta)_H d\sigma \tag{7.24}$$

On the other hand,

$$\begin{aligned}
2\int_0^\infty \mu(\sigma)(\eta_L(\sigma), \vartheta_L)_H d\sigma &\leq \frac{\varepsilon}{2}\|\eta_L\|_{\mathcal{M}}^2 + \frac{2b_0^2}{\varepsilon}\|\vartheta_L\|_H^2 \\
&\leq \frac{\varepsilon}{2}\Psi_L^2(t) + \frac{2b_0^2}{\varepsilon}\Phi_L^2(t),
\end{aligned}$$

then, taking into account estimate (7.23), equation (7.24) becomes

$$\frac{d}{dt}\Psi_L^2(t) + \varepsilon\Psi_L^2(t) \leq \frac{4b_0^2}{\varepsilon}\Phi_L^2(\tau)e^{-2\varepsilon(t-\tau)};$$

a further application of Lemma A.6 yields

$$\begin{aligned}
\Psi_L^2(t) &\leq 2\Psi_L^2(\tau)e^{-\varepsilon(t-\tau)} + \frac{4b_0^2}{\varepsilon}\Phi_L^2(\tau)\int_\tau^t e^{-2\varepsilon(s-\tau)}e^{-\varepsilon(t-s)}ds \\
&\leq \left(2 + \frac{2b_0^2}{\varepsilon^2}\Phi_L^2(\tau)\right)\Psi_L^2(\tau)e^{-\varepsilon(t-\tau)}
\end{aligned} \tag{7.25}$$

and the exponential decay of the solution follows. □

7.6 Proof of Lemma 7.10

This proof goes exactly as in Lemma 4.3 of [37]. We report it here for the reader's convenience.

Denote

$$\mathcal{K}_T = \left\{ \int_\tau^t S(t-y)(f(y),0,0)dy, t \in [\tau, T], f \in \mathrm{H}(g) \right\}$$

and let

$$\mathcal{K}_\infty = \bigcup_{T \geq 0} \mathcal{K}_T.$$

Clearly $z_E(t) \in \mathcal{K}_\infty$, for any $f \in \mathrm{H}(g)$ and any $t \geq \tau$.

We first show that $\mathcal{K}_T$ is compact for any $T \geq 0$, then we shall prove that $\mathcal{K}_\infty$ is relatively compact. The set $\mathcal{K}_E$ is thus the norm-closure of $\mathcal{K}_\infty$.

Let $\{u_n\}$ be a sequence of elements of $\mathcal{K}_T$; that is,

$$u_n = \int_0^{t_n} S(t_n - y)(f_n(y), 0, 0)dy.$$

The compactness of $[0,T]$ and $\mathrm{H}(g)$ ensures that, up to a subsequence, there exists $t \in [0,T]$ and $f \in \mathrm{H}(g)$ such that

$$t_n \longrightarrow t \qquad \text{and} \qquad f_n \longrightarrow f$$

as $n \to \infty$, in $\mathbb{R}$ and in $\mathrm{H}(g)$, respectively.

We can suppose, without loss of generality, that $0 < t_n \leq t$. Let

$$u = \int_0^t S(t-y)(f(y),0,0)dy$$

we get

$$\begin{aligned} \|u_n - u\|_{\mathcal{H}} &= \left\| \int_\tau^t S(t-y)(f(y),0,0)dy - \int_\tau^{t_n} S(t_n-y)(f_n(y),0,0)dy \right\|_{\mathcal{H}} \\ &\leq \int_{t_n}^t \|S(t-y)(f(y),0,0)\|_{\mathcal{H}}\, dy + \int_0^{t_n} \|S(t-y)(f(y)-f_n(y),0,0)\|_{\mathcal{H}}\, dy \\ &\quad + \int_\tau^t I_{[0,t_n]} \|S(t-y)f_n(y) - S(t_n-y)f_n(y),0,0)\|_{\mathcal{H}}\, dy \end{aligned} \tag{7.26}$$

where $I_{[0,t_n]}$ is the characteristic function of the specified interval.

It is immediate to see that the first two terms of (7.26) vanish as $n \to \infty$. Indeed estimate (7.16) yields

$$\begin{aligned} &\int_{t_n}^t \|S(t-y)(f(y),0,0)\|_{\mathcal{H}}\, dy + \int_0^{t_n} \|S(t-y)(f(y)-f_n(y),0,0)\|_{\mathcal{H}}\, dy \\ &\leq M \left(\int_{t_n}^t \|(f(y),0,0)\|_{\mathcal{H}}\, dy + \int_0^{t_n} \|(f(y)-f_n(y),0,0)\|_{\mathcal{H}}\, dy \right) \end{aligned}$$

Since $S(t)$ is a strongly continuous semigroup of linear operators for any fixed $y \in [0,t]$, there holds

$$\lim_{n\to\infty} I_{[0,t_n]} \|S(t-y)(f(y),0,0) - S(t_n - y)(f(y),0,0)\|_{\mathcal{H}} = 0.$$

Also, from (7.16),

$$\|S(t-y)(f(y),0,0) - S(t_n - y)(f(y),0,0)\|_{\mathcal{H}} \leq M\|f\|_H.$$

Hence by the dominated convergence theorem, we get that the third term in (7.26) vanishes as well as $n \to \infty$.

Now we show that $\mathcal{K}_\infty$ is relatively compact. Recalling (2.5), estimate (7.16) entails for $Q > T > 0$

$$\begin{aligned}
& \left\| \int_0^Q S(Q-y)(f(y),0,0)dy - \int_0^T S(T-y)(f^{Q-T}(y),0,0)dy \right\|_{\mathcal{H}} \\
= & \left\| \int_0^{Q-T} S(Q-y)(f(y),0,0)dy \right\|_{\mathcal{H}} \leq M \int_0^{Q-T} e^{-\varepsilon(Q-y)} \|f(y)\|_H dy \\
\leq & \ M e^{-\varepsilon T} \frac{e^{\varepsilon}}{\varepsilon} \|g\|_{\mathcal{T}^1(H)}
\end{aligned}$$

where the last inequality is obtained arguing as in Remark A.7.

Select now $\varepsilon_0 > 0$ and let $T_0 > 0$ such that

$$M e^{-\varepsilon T_0} \frac{e^{\varepsilon}}{\varepsilon} \|g\|_{\mathcal{T}^1(H)} \leq \frac{\varepsilon_0}{2}.$$

Due to the compactness of $\mathcal{K}_{T_0}$ one can find $k_1, \ldots, k_n \in \mathcal{K}_{T_0}$ such that

$$\mathcal{K}_{T_0} \subset \bigcup_{j=1}^{n} \left\{ u \in \mathcal{H} : \|u - k_j\|_{\mathcal{H}} < \frac{\varepsilon_0}{2} \right\}.$$

Thus, if $k_\infty \in \mathcal{K}_\infty$, there exist $k_0 \in \mathcal{K}_{T_0}$ and $i \in \{1, \ldots, n\}$ such that

$$\|k_\infty - k_0\|_{\mathcal{H}} < \frac{\varepsilon_0}{2} \qquad \text{and} \qquad \|k_0 - k_i\|_{\mathcal{H}} < \frac{\varepsilon_0}{2}.$$

Therefore we conclude

$$\mathcal{K}_\infty \subset \bigcup_{j=0}^{n} \left\{ u \in \mathcal{H} : \|u - k_j\|_{\mathcal{H}} < \varepsilon_0 \right\}$$

hence $\mathcal{K}_\infty$ is relatively compact in $\mathcal{H}$. $\square$

7.7 Proof of Lemma 7.11

We first note that $\|z_N(\tau)\|_{\mathcal{V}} = 0$; furthermore, after Theorem 6.6 and Lemmas 7.9, 7.10, it is apparent that

$$\sup_{t\geq\tau,\tau\in\mathbb{R}} \|z(t)\|_{\mathcal{H}} \leq c \qquad \text{and} \qquad \sup_{t\geq\tau,\tau\in\mathbb{R}} \|z_N(t)\|_{\mathcal{H}} \leq c. \tag{7.27}$$

In this proof c is a generic positive constant which may even vary within the same formula, and depends only on $\mathcal{B}_0$ and on g, besides the structural constants of the problem.

We first focus on some inequalities related to the second component of the solution z. Observe that Hölder inequality and (7.27) imply

$$\begin{aligned} \|\Delta\chi\|_H^4 &= \left(\int_\Omega \Delta\chi\Delta\chi d\Omega\right)^2 = \left(\int_\Omega \nabla\chi\cdot\nabla\Delta\chi d\Omega\right)^2 \\ &\leq \left(\int_\Omega |\nabla\chi|^2 d\Omega\right)\left(\int_\Omega |\nabla\Delta\chi|^2 d\Omega\right) \\ &\leq c\int_\Omega |\nabla\Delta\chi|^2 d\Omega \end{aligned} \tag{7.28}$$

for every $t \geq \tau$.

Using again Hölder inequality and the compact injection $V \hookrightarrow L^6(\Omega)$, we obtain

$$\begin{aligned} \|\Delta\chi^3\|_H^2 &\leq c\left(\int_\Omega \chi^2|\nabla\chi\cdot\nabla\chi|^2 d\Omega + \int_\Omega \chi^4|\Delta\chi|^2 d\Omega\right) \\ &\leq c\left(\int_\Omega \chi^6 d\Omega\right)^{1/3}\left(\int_\Omega |\nabla\chi|^6 d\Omega\right)^{2/3} + c\left(\int_\Omega \chi^6 d\Omega\right)^{2/3}\left(\int_\Omega |\Delta\chi|^6 d\Omega\right)^{1/3} \\ &\leq c\|\chi\|_V^2\left(\int_\Omega |\nabla\chi|^6 d\Omega\right)^{2/3} + c\|\chi\|_V^4\left(\int_\Omega |\Delta\chi|^6 d\Omega\right)^{1/3} \end{aligned}$$

and taking into account equation (7.27) and (7.28) into account, we deduce

$$\begin{aligned} \|\Delta\chi^3\|_H^2 &\leq c\left(\|\nabla\chi\|_V^4 + \|\Delta\chi\|_V^2\right) \\ &\leq c\left(1 + \|\Delta\chi\|_H^2 + \|\Delta\chi\|_H^4 + \|\nabla\Delta\chi\|_H^2\right) \\ &\leq c\left(1 + \|\Delta\chi\|_H^2 + \|\nabla\Delta\chi\|_H^2\right). \end{aligned} \tag{7.29}$$

Arguing similarly and taking advantage of (7.2), we infer

$$\begin{aligned} \|\Delta\gamma'(\chi)\|_H^2 &\leq c\left(\int_\Omega |\gamma'''(\chi)|^2|\nabla\chi|^4 d\Omega + \int_\Omega |\gamma''(\chi)|^2|\Delta\chi|^2 d\Omega\right) \\ &\leq c\left(\int_\Omega |\nabla\chi|^4 d\Omega + \int_\Omega |\Delta\chi|^2 d\Omega\right). \end{aligned} \tag{7.30}$$

Then combining (7.30) with (7.29)

$$\|\Delta\chi^3\|_H^2 + \|\gamma'(\chi)\|_H^2 \le c\left(1 + \|\Delta\chi\|_H^2 + \|\nabla\Delta\chi\|_H^2\right). \tag{7.31}$$

We multiply equation (7.10) by $-\Delta\vartheta_N$ in H and equation (7.11) by $\partial_t\chi_N$. We obtain, respectively,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla\vartheta_N\|_H^2 + \|\nabla\vartheta_N\|_H^2 + k\|\Delta\vartheta_N\|_H^2 - \lambda_0(\partial_t\chi_N, \Delta\vartheta_N)_H\\ &+ \int_0^\infty \nu(\sigma)(\nabla\eta_N(\sigma), \nabla\vartheta_N)_H d\sigma + \int_0^\infty \mu(\sigma)(\Delta\eta_N(\sigma), \Delta\vartheta_N)_H d\sigma = 0\end{aligned} \tag{7.32}$$

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\Delta\chi_N\|_H^2 + \|\partial_t\chi_N\|_H^2\\ &-(\Delta(\chi^3 + \gamma'(\chi)), \partial_t\chi_N)_H + \lambda_0(\Delta\vartheta_N, \partial_t\chi_N)_H = 0.\end{aligned} \tag{7.33}$$

Consider (7.33) and note that Young inequality implies

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\Delta\chi_N\|_H^2 + \frac{1}{2}\|\partial_t\chi_N\|_H^2 + \lambda_0(\Delta\vartheta_N, \partial_t\chi_N)_H\\ &\le \frac{1}{2}\left(\|\Delta\chi^3\|_H^2 + \|\gamma'(\chi)\|_H^2\right).\end{aligned} \tag{7.34}$$

Consider again (7.11) multiplied by χ_N in H. We have

$$\frac{1}{2}\frac{d}{dt}\|\chi_N\|_H^2 + \frac{1}{2}\|\Delta\chi_N\|_H^2 \le \frac{1}{2}\left(\|\vartheta_N\|_H^2 + \|\chi^3\|_H^2 + \|\gamma'(\chi)\|_H^2\right)$$

then, on account of (7.27)

$$\frac{1}{2}\frac{d}{dt}\|\chi_N\|_H^2 + \frac{1}{2}\|\chi_N\|_H^2 + \frac{1}{2}\|\Delta\chi_N\|_H^2 \le c. \tag{7.35}$$

Take now the inner product in $\mathcal{L}$ of (7.13) and η_N, and integrate by parts using (k4), to get

$$\frac{1}{2}\frac{d}{dt}\|\eta_N\|_{\mathcal{L}}^2 + \frac{\delta}{2}\|\eta_N\|_{\mathcal{L}}^2 \le (\eta_N, \vartheta_N)_{\mathcal{L}}.$$

Then, using (7.27) and Young inequality, we deduce

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\eta_N\|_{\mathcal{L}}^2 + \frac{\delta}{2}\|\eta_N\|_{\mathcal{L}}^2\\ &\le \int_0^\infty \nu(\sigma)(\eta_N(\sigma), \vartheta_N)_H d\sigma + \int_0^\infty \nu(\sigma)(\nabla\eta_N(\sigma), \nabla\vartheta_N)_H d\sigma\\ &\quad + \int_0^\infty \mu(\sigma)(\eta_N(\sigma), \vartheta_N)_V d\sigma + \int_0^\infty \mu(\sigma)(\Delta\eta_N(\sigma), \Delta\vartheta_N)_H d\sigma\\ &\le c + \frac{1}{2}\|\nabla\vartheta_N\|_H^2\\ &\quad + \int_0^\infty \nu(\sigma)(\nabla\eta_N(\sigma), \nabla\vartheta_N)_H d\sigma + \int_0^\infty \mu(\sigma)(\Delta\eta_N(\sigma), \Delta\vartheta_N)_H d\sigma.\end{aligned} \tag{7.36}$$

Summation of (7.32)–(7.36) with the help of (7.31) bears

$$\frac{d}{dt}\Phi^2(t) + \varepsilon\Phi^2(t) + k\|\Delta\vartheta_N\|_H^2 + \|\partial_t\chi_N\|_H^2$$
$$\leq c\left(1 + \|\Delta\chi\|_H^2 + \|\nabla\Delta\chi\|_H^2\right) \tag{7.37}$$

where we have set

$$\Phi^2(t) = \|\nabla\vartheta_N\|_H^2 + \|\chi_N\|_H^2 + \|\Delta\chi_N\|_H^2 + \|\eta_N\|_{\mathcal{L}}^2.$$

Observe that $\|\Phi^2(\tau)\|_{\mathcal{H}} = 0$. Hence, applying Lemma A.6 to inequality (7.37), and recalling Remark A.7, we obtain

$$\Phi^2(t) \leq \frac{ce^{\varepsilon}}{\varepsilon}\Big(1 + \sup_{r\in\mathbb{R}}\int_r^{r+1}\|\Delta\chi(s)\|_H^2 ds + \sup_{r\in\mathbb{R}}\int_r^{r+1}\|\nabla\Delta\chi(s)\|_H^2 ds\Big).$$

recalling then Corollary 6.10, we derive the bound

$$\Phi^2(t) \leq c \tag{7.38}$$

for every $t \geq \tau$ and every $\tau \in \mathbb{R}$. The conclusion follows combining (7.27) and (7.38). □

7.8 Proof of Lemma 7.12

Here we report the proof of Lemma 7.7 in [32] for the sake of convenience.

Lemma 7.11 implies that $\mathcal{E}$ is bounded in $\mathcal{L}$. Observe that (7.13) and (7.14) imply that we can compute η_N explicitly by observing that

$$\eta_N^t(s) = \begin{cases} \displaystyle\int_0^s \vartheta_N(t-y)dy & 0 < s \leq t-\tau \\ \displaystyle\int_0^{t-\tau} \vartheta_N(t-y)dy & s > t-\tau. \end{cases} \tag{7.39}$$

Differentiating (7.39) with respect to s yields

$$\eta_N^t(s) = \begin{cases} \vartheta_N(t-s) & 0 < s \leq t-\tau \\ 0 & s > t-\tau \end{cases} \tag{7.40}$$

therefore (7.27) and (7.40) give at once

$$\int_0^\infty \mu(s)\|\partial_s\eta_N^t(s)\|_H^2 ds = \int_0^{t-\tau}\mu(s)\|\vartheta_N(t-s)\|_H^2 ds \leq c$$

and, analogously,

$$\int_0^\infty \nu(s)\|\partial_s\eta_N^t(s)\|_H^2 ds \leq c.$$

We conclude that $\mathcal{E}$ is bounded in

$$L^2_\nu(\mathbb{R}^+;V) \cap H^1_\nu(\mathbb{R}^+;H) \cap L^2_\mu(\mathbb{R}^+;W) \cap H^1_\mu(\mathbb{R}^+;H). \tag{7.41}$$

Finally, from (7.39) we infer

$$\|\eta^t_N(s)\|_V \leq \begin{cases} \displaystyle\int_{t-s}^t \|\vartheta_N(y)\|_V dy & \tau \leq t-s < t \\ \displaystyle\int_\tau^t \|\vartheta_N(y)\|_V dy & t-s < \tau. \end{cases} \tag{7.42}$$

thus, from Lemma 7.11 and (7.42) it follows

$$\|\eta_N(s)\|^2_V \leq (c_1 + c_2 s)^2 \in L^1_\nu(\mathbb{R}^+) \cap L^1_\mu(\mathbb{R}^+)$$

in force of the exponential decays of ν and μ. Finally, Lemma A.4 yields the thesis. □

7.9 Proof of Theorem 7.14

The proof follows collecting the results of technical lemmas above. We are left to show that the closure of the set $\mathcal{K}$ given by (7.17) is uniformly attracting. Let $\mathcal{B} \subset \mathcal{H}_\alpha$ be any bounded set, $\mathcal{B}_0 \subset \mathcal{H}_\alpha$ be the uniformly absorbing set defined by Theorem 6.8, and let $t_\mathcal{B}$ be as in Definition 6.2. We recall that R_0 is a constant characterizing $\mathcal{B}_0$ (see Remark 6.9).

The following relation hold

$$\begin{aligned} &U_f(t + t_\mathcal{B} + \tau, \tau)\mathcal{B} \\ &= U_f(t + t_\mathcal{B} + \tau, t_\mathcal{B} + \tau) U_f(t_\mathcal{B} + \tau, \tau)\mathcal{B} = \\ &\subset U_f(t + t_\mathcal{B} + \tau, t_\mathcal{B})\mathcal{B}_0 \end{aligned}$$

for every $t > 0$. Then, from Lemma 7.9 and Corollary 7.13

$$\sup_{f \in \mathrm{H}(g)} \delta_\mathcal{H}(U_f(t,\tau)\mathcal{B}, \overline{\mathcal{K}}) \leq M R_0 e^{\varepsilon(t_\mathcal{B}+\tau)} e^{-\varepsilon t}$$

for any $t \geq t_\mathcal{B} + \tau$. Hence, Definition 7.1 is satisfied. □

7.10 Proof of Proposition 7.17

We sketch the double decomposition of the solution $z = z_L + z_N$. Instead of (7.7)-(7.10) and (7.11)-(7.14), we have

$$\partial_t \vartheta_N = -\vartheta_N - \lambda_0 \partial_t \chi_N + k\Delta\vartheta_N - \nu_0 \int_0^\infty \nu(\sigma)\eta_N(\sigma)d\sigma + \int_0^\infty \mu(\sigma)\Delta\eta_N(\sigma)d\sigma + f \tag{7.43}$$

$$\begin{aligned} \partial_t \chi_N &= \Delta(-\Delta\chi_N + \chi^3 + \gamma'(\chi) - \lambda_0 \vartheta_N) \\ \partial_t \eta_N &= -\partial_s \eta_N + \vartheta_N \\ z_N(\tau) &= (0, m_{\chi_0}, 0). \end{aligned}$$

We can proceed as in proof of Lemma 7.11, the only difference being that we use equation (7.43) instead of (7.11). When we multiply (7.43) by $-\Delta\vartheta_N$ in H, we exploit Young inequality to get

$$\frac{1}{2}\frac{d}{dt}\|\nabla\vartheta_N\|_H^2 + \|\nabla\vartheta_N\|_H^2 + k\|\Delta\vartheta_N\|_H^2 - \lambda_0(\partial_t\chi_N, \Delta\vartheta_N)_H$$

$$+ \int_0^\infty \nu(\sigma)(\nabla\eta_N(\sigma), \nabla\vartheta_N)_H d\sigma + \int_0^\infty \mu(\sigma)(\Delta\eta_N(\sigma), \Delta\vartheta_N)_H d\sigma \leq \varepsilon\|\Delta\vartheta_n\|_H^2 + \frac{1}{4\varepsilon}\|f\|_H^2$$

then the main statement follows along the lines of the proof of Lemma 7.11. The last assertion of the proposition is proved by (7.41).

Chapter 8

Dimension of the attractor

In this chapter, we prove that the uniform attractor of the family of processes $\{U_f^k(t,\tau)\}$, with $k > 0$ has finite fractal and Hausdorff dimensions. This fact is quite important from a practical view point (e.g., in numerical simulations), since it implies that the long term system dynamics can be described by a finite number of parameters.

8.1 Hausdorff and fractal dimensions

We need to recall some preliminary definitions which can be found, for instance, in [56]. Let $\mathcal{X}$ be a subset of the metric space X.

Definition 8.1. The *Hausdorff dimension* of $\mathcal{X}$ is

$$\dim_{\mathrm{H}}\mathcal{X} = \sup\left\{ d > 0 \quad : \quad \sup_{\varepsilon > 0} \inf_{C_\varepsilon} \sum_{i \in J} r_i^d < +\infty \right\}$$

where $C_\varepsilon = \{\mathcal{B}_i(r_i)\}_{i \subset J}$ is a covering of $\mathcal{X}$ of balls of radii $r_i < \varepsilon$.

Definition 8.2. The *fractal dimension* of $\mathcal{X}$ is

$$\dim_{\mathrm{F}}\mathcal{X} = \sup\left\{ d > 0 \quad : \quad \limsup_{\varepsilon \searrow 0} \varepsilon^d n_{\mathcal{X}}(\varepsilon) < +\infty \right\},$$

where $n_{\mathcal{X}}(\varepsilon)$ is the minimum number of balls of radii ε which is necessary to cover $\mathcal{X}$.

It comes straightly from definitions that

$$\dim_{\mathrm{H}}\mathcal{X} \leq \dim_{\mathrm{F}}\mathcal{X}. \tag{8.1}$$

Let now X be a Hilbert space, and $\mathcal{L}(X)$ be the Banach space of bounded linear operators from X into X, whose norm is defined as

$$\|L\|_{\mathcal{L}(X)} = \sup_{x \in X} \frac{\|Lx\|_X}{\|x\|_X}.$$

Definition 8.3. A (nonlinear) continuous map U from a subset $\mathcal{X} \subset X$ into $\mathcal{X}$ is said to be *uniformly quasi-differentiable* on $\mathcal{X}$ if for any $\zeta_0 \in \mathcal{X}$ there exists $U'(\zeta_0) \in \mathcal{L}(X)$ such that

$$\|U\zeta_0 - U\zeta_0^* - U'(\zeta_0)(\zeta_0 - \zeta_0^*)\|_X \leq \sigma(\|\zeta_0 - \zeta_0^*\|_X)\|\zeta_0 - \zeta_0^*\|_X \qquad \forall \zeta_0^* \in \mathcal{X}$$

where $\sigma : \mathbb{R} \longrightarrow \mathbb{R}^+$ is independent of ζ_0 and $\sigma(y) = o(1)$ as $y \longrightarrow 0^+$

This definition does not require uniqueness for the operator $U'(\zeta_0)$.

Definition 8.4. Let $M \in \mathcal{L}(X)$; for any $m \geq 0$ the *m-dimensional trace* of M is

$$\mathrm{Tr}_m M = \sup_Q \sum_{i=1}^{m} (M\zeta_i, \zeta_i)_X$$

where the supremum ranges over all possible orthogonal projections Q in X on the m-dimensional space QX belonging to the domain of M, and $\{\zeta_1, ..., \zeta_m\}$ is an orthonormal basis of QX.

We recall the following result, which subsumes Theorem 3.3, in Chapter V and Section 3.4 in [56].

Theorem 8.5. *Let $\mathcal{X} \subset X$ be a compact fully invariant set for the strongly continuous semigroup $U(t)$. Assume also that $U(t)$ is uniformly quasi-differentiable on $\mathcal{X}$ for all $t \geq 0$, and*

$$\sup_{\zeta_0 \in \mathcal{X}} \|U'(t, \zeta_0)\|_{\mathcal{L}(X)} < +\infty \quad \forall t \geq 0$$

where $U'(t, \zeta_0)$ is the quasi-differential of $U(t)$ at ζ_0. It is also assumed that $U'(t, \zeta_0)$ is generated by the equation in variation

$$\begin{aligned} Z_t &= M(\zeta)Z \\ Z(0) &= Z_0 \end{aligned}$$

that is, $U'(t, \zeta_0)Z_0 = Z(t)$, with $\zeta(t) = U(t)\zeta_0$, where $M(\zeta)$ is a linear operator on X, for every fixed $\zeta \in \mathcal{X}$. Introducing the number q_m by the formula

$$q_m = \liminf_{T \longrightarrow \infty} \sup_{\zeta_0 \in \mathcal{X}} \left\{ \frac{1}{T} \int_0^T \mathrm{Tr}_m M(U(t)\zeta_0) dt \right\}$$

if there exists $m \geq 2$ such that $q_m < 0$, then

$$\begin{aligned} \dim_{\mathrm{H}} \mathcal{X} &\leq m \\ \dim_{\mathrm{F}} \mathcal{X} &\leq m \max_{1 \leq i \leq m-1} \left(1 + \frac{\max\{q_i, 0\}}{|q_m|} \right). \end{aligned}$$

We recall that if a set $\mathcal{X}$ has Hausdorff dimension less or equal N, then almost all projections of dimension $2N + 1$ are injective on $\mathcal{X}$ (see [49]). Hence, the flow on $\mathcal{X}$ can be parametrized by at most $2N + 1$ parameters belonging to a linear space. Thus, the system dynamics can be fully described by a finite number of parameters.

We finally need to recall a technical result (see [56] for details).

Lemma 8.6. *There exist a positive constant λ such that, for any given m vectors in V $\{v_1 \cdots v_m\}$ and m vectors $\{w_1 \cdots w_m\}$ in W, which are orthonormal in H and in V respectively; it follows that*

$$\sum_{i=1\cdots m} \|v_i\|_V^2 \geq \lambda m^{5/3} \qquad \text{and} \qquad \sum_{i=1\cdots m} \|w_i\|_W^2 \geq \lambda m^{5/3}.$$

8.2 Basic setting

We consider problem $\mathbf{P}_k$ with $k > 0$ and we suppose

$$f \in H \tag{8.2}$$

and

$$\gamma \in C^4(\mathbb{R}) \text{ such that } \gamma',\ \gamma'',\ \gamma''' \in L^\infty(\mathbb{R}). \tag{8.3}$$

Thus, we are dealing with an autonomous dynamical system, and $U_f(t,0)$ reduces to a strongly continuous semigroup acting on $\mathcal{H}_\alpha$.

If we set $\omega = \vartheta + \lambda_0 \chi$ and introduce the triplets

$$\begin{aligned} \zeta_0 &= (\omega_0, \chi_0, \eta_0) \\ \zeta(t) &= (\omega(t), \chi(t), \eta^t) \end{aligned}$$

and the operators

$$L = \begin{pmatrix} -\mathbb{I} + k\Delta & \lambda_0 \mathbb{I} - k\lambda_0\Delta & -\nu_0 \int_0^\infty \nu(\sigma)\mathbb{I} + \int_0^\infty \mu(\sigma)\Delta \\ -\lambda_0\Delta & -\Delta^2 + (\lambda_0^2 - 1)\Delta & 0 \\ \mathbb{I} & -\lambda_0\mathbb{I} & -\partial_s \end{pmatrix}$$

$$N = N(\zeta) = \begin{pmatrix} f \\ \Delta\chi^3 + \Delta\gamma'(\chi) \\ 0 \end{pmatrix} \tag{8.4}$$

we can formally rewrite our problem in this way

$$\begin{aligned} &\partial_t \zeta(t) = L\zeta(t) + N(\zeta(t)) \\ &\zeta(0) = \zeta_0 \\ &\quad + \text{boundary conditions.} \end{aligned} \tag{8.5}$$

The system (8.5) generate a strongly continuous semigroup $U(t)$ on $\mathcal{H}_\alpha$ which is related to $U_f(t,0)$

$$U(t)\zeta_0 = U_f(t,0) z_0 \tag{8.6}$$

whenever $\omega_0 = \vartheta_0 + \lambda_0\chi_0$. Therefore, all the previous results hold for $U(t)$ as well. In particular the set

$$\widehat{\mathcal{A}} = \{(\vartheta_0 + \lambda_0\chi_0, \chi_0, \eta_0) \text{ such that } (\vartheta_0, \chi_0, \eta_0) \in \mathcal{A}\} \tag{8.7}$$

is the global attractor of $U(t)$ on $\mathcal{H}_\alpha$, where $\mathcal{A}$ is the attractor described in the previous chapter.

To prove that the global attractor $\mathcal{A}$ has finite fractal and Hausdorff dimension it will be enough to prove $\dim_{\mathrm{F}}\widehat{\mathcal{A}} < +\infty$. Indeed, let

$$\tilde{\zeta} = (\tilde{\omega}, \tilde{\chi}, \tilde{\eta}) \in \mathcal{H}$$

be fixed, then let

$$\tilde{z} = (\tilde{\omega} - \lambda_0\tilde{\chi}, \tilde{\chi}, \tilde{\eta}).$$

If $r > 0$ each

$$z = (\vartheta, \chi, \eta) \in \mathcal{B}(\tilde{z}, r)$$

determines a

$$\zeta = (\vartheta + \lambda_0\chi, \chi, \eta) \in \mathcal{B}(\tilde{\zeta}, cr)$$

for some $c > 0$. It follows that

$$n_{\widehat{\mathcal{A}}}(cr) \le n_{\mathcal{A}}(r)$$

which entails

$$\dim_{\mathrm{F}}\mathcal{A} \le \dim_{\mathrm{F}}\widehat{\mathcal{A}}.$$

8.3 Main theorems

Theorem 8.7. *Let* (k1)-(k3), (5.1), (8.2) *and* (8.3) *hold. Then the semigroup $U(t)$ acting on $\mathcal{H}$ and defined in* (8.6) *is uniformly quasi-differentiable on $\mathcal{A}$ for every $t \ge 0$, and the quasi-differential $U'(t,\zeta_0)$ at the point $\zeta_0 = (\omega_0, \chi_0, \eta_0) \in \mathcal{A}$ satisfies the equation in variation*

$$\begin{aligned} &Z_t = LZ + N'(\zeta)Z \\ &Z(0) = Z_0 \end{aligned} \tag{8.8}$$

where

$$\begin{aligned} U'(t,\zeta_0)Z_0 = Z(t) &= (\Theta(t), X(t), Y^t) \\ \zeta(t) &= (\omega(t), \chi(t), \eta^t) = U(t)\zeta_0 \end{aligned}$$

and

$$N'(\zeta)Z = \begin{pmatrix} 0 \\ 3\Delta(\chi^2 X) + \Delta(\gamma''(\chi)X) \\ 0 \end{pmatrix}$$

being N' the Fréchet differential of N. Furthermore

$$\sup_{\zeta_0 \in \mathcal{A}} \|U'(t,\zeta_0)\|_{\mathcal{L}(\mathcal{H})} < +\infty \quad \forall t \ge 0.$$

Theorem 8.7 allows to use Theorem 8.5 to conclude the following result.

Theorem 8.8. *The attractor $\mathcal{A}$ of the semigroup $U(t)$ has finite fractal (and Hausdorff) dimension.*

8.4 Proof of Theorem 8.7

First, we point out that $\mathcal{A}$ is fully invariant, that is

$$U(t)\mathcal{A} = \mathcal{A}, \qquad \forall\, t \geq 0$$

(see Theorem 5.3.1 in [42]).

We consider two solutions ζ^* and ζ to problem (8.5) with initial data respectively ζ_0^*, $\zeta_0 \in \mathcal{A}$. Then, we have

$$\zeta^*(t), \zeta(t) \in \mathcal{A}, \qquad \forall\, t \geq 0.$$

Their difference

$$\tilde{\zeta} = (\tilde{\omega}, \tilde{\chi}, \tilde{\eta}) = \zeta - \zeta^*$$

satisfies the problem

$$\begin{aligned} &\partial_t \tilde{\zeta} = L\tilde{\zeta} + N(\zeta) - N(\zeta^*) \\ &\tilde{\zeta}(0) = \zeta_0 - \zeta_0^*. \end{aligned}$$

Set T, fixed in $\mathbb{R}^+$. Theorem 5.4 entails

$$\|\tilde{\zeta}(t)\|^2_{\mathcal{H}} \leq c\|\tilde{\zeta}_0\|^2_{\mathcal{H}} \quad \forall t \in [0,T]$$

where c is a constant depending on T (as almost every constant in what follows).

By means of Theorem 4.1 and 5.4, we argue that problem (8.8) has a unique solution $Z \in C(\mathbb{R}^+, \mathcal{H})$. Take $Z(0) = \zeta_0 - \zeta_0^*$. Our aim is to show that $Z(t) = U'(t,\zeta)(\zeta_0 - \zeta_0^*)$.

We set

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \zeta - \zeta^* - Z = \tilde{\zeta} - Z$$

then we rewrite problem (8.8) as

$$\begin{aligned} &\partial_t \xi = L\xi + N'(\zeta)\xi + (N(\zeta) - N(\zeta^*) - N'(\zeta)\tilde{\zeta}) \\ &\xi(0) = 0. \end{aligned} \tag{8.9}$$

Multiply now equation (8.9) in the space $\mathcal{H}$ by the vector

$$v = \begin{pmatrix} \kappa\xi_1 \\ \xi_2 \\ \kappa\xi_3 \end{pmatrix}$$

where κ is a constant to be chosen later. We get

$$\frac{1}{2}\frac{d}{dt}\left(\kappa\|\xi_1\|^2_H + \|\xi_2\|^2_V + \kappa\|\xi_3\|^2_{\mathcal{M}}\right) = (L\xi, v)_{\mathcal{H}} + (N'(\zeta)\xi, v)_{\mathcal{H}} + (h(\chi,\chi^*), \xi_2)_V \tag{8.10}$$

where we have set

$$h(\chi,\chi^*) = \Delta\left[\chi^3 + \gamma'(\chi) - (\chi^*)^3 - \gamma'(\chi^*) - 3(\chi^2 + \gamma''(\chi))(\chi - \chi^*)\right].$$

Using Young inequality repeatedly, we obtain

$$\begin{aligned}(L\xi, v)_{\mathcal{H}} &= \kappa\Big[-\|\xi_1\|_H^2 - k\|\nabla\xi_1\|_H^2 + \lambda_0(\xi_2 - k\Delta\xi_2, \xi_1)_H \\ &\quad - \nu_0\int_0^\infty \nu(\sigma)(\xi_3(\sigma), \xi_1)_H d\sigma + \int_0^\infty \mu(\sigma)(\Delta\xi_3(\sigma), \xi_1)_H d\sigma\Big] \\ &\quad - \lambda_0(\Delta\xi_1, \xi_2)_V - \|\Delta\xi_2\|_V^2 + \lambda_0^2(\Delta\xi_2, \xi_2)_V \\ &\quad + \kappa\Big[(\xi_3, \xi_1)_{\mathcal{M}} - \lambda_0(\xi_3, \xi_2)_{\mathcal{M}} - (\partial_\sigma\xi_3, \xi_3)_{\mathcal{M}}\Big] \\ &\le \Big(\lambda_0^2 + \frac{\lambda_0^2}{2\varepsilon} + \frac{\kappa^2\lambda_0^2 k^2}{4\varepsilon}\Big)\Big(\|\xi_1\|_H^2 + \|\xi_2\|_V^2\Big) - k\kappa\|\nabla\xi_1\|_H^2 + 2\varepsilon\|\nabla\xi_1\|_H^2 \\ &\quad + \frac{\kappa^2 k^2\lambda_0^2}{4\varepsilon}\|\nabla\xi_2\|_H^2 + \varepsilon\|\nabla\Delta\xi_2\|_H^2 + \frac{\lambda_0^2}{4\varepsilon}\|\nabla\xi_1\|_H^2 - \|\Delta\xi_2\|_V^2 + \varepsilon\|\nabla\Delta\xi_2\|_H^2 \\ &\quad + \kappa\Big[\nu_0\int_0^\infty \nu(\sigma)(\xi_3(\sigma), \xi_1)_H d\sigma + \int_0^\infty \mu(\sigma)(\Delta\xi_3(\sigma), \xi_1) d\sigma \\ &\quad + (\xi_1 - \lambda_0\xi_2, \xi_3)_{\mathcal{M}} - (\partial_s\xi_3, \xi_3)_{\mathcal{M}}\Big].\end{aligned}$$

Now, we proceed in the case $\nu_0 = 1$, then using (k4). We recall that the other case $\nu_0 = -1$, $|\Gamma_0| > 0$ and hypothesis (k5) is done exploiting the fact that $\mathcal{M} = L^2_\mu(\mathbb{R}^+; V_0)$ (see Remark 6.5). We obtain

$$\begin{aligned}(L\xi, v)_{\mathcal{H}} &\le -\Big(\kappa k - 2\varepsilon + \frac{\lambda_0^2}{4\varepsilon}\Big)\|\nabla\xi_1\|_H^2 - \Big(1 - 2\varepsilon\Big)\|\nabla\Delta\xi_2\|_H^2 - \Big(\frac{\kappa\delta}{2} - 2\varepsilon\Big)\|\xi_3\|_{\mathcal{M}}^2 \\ &\quad + c\Big(\|\xi_1\|_H^2 + \|\xi_2\|_V^2\Big) \end{aligned} \tag{8.11}$$

where

$$c = \kappa\lambda_0 + \frac{\lambda_0^2 + \kappa^2\lambda_0^2(a_0 + b_0 + k^2) + \lambda_0^4}{2\varepsilon}.$$

We note that (8.2) obviously implies (7.18); hence, we shall exploit the uniform $\mathcal{V}$-bound of every vector lying in the attractor (see Proposition 7.17); in particular its second component is bounded in W. We use this fundamental fact, together with Hölder inequality, (7.2) and the continuous embeddings $W \hookrightarrow C(\Omega) \hookrightarrow V \hookrightarrow L^4(\Omega)$, when we estimate the second term of the right-hand-side of (8.10). Indeed,

$$\begin{aligned}(N'(\zeta)\xi, v)_{\mathcal{H}} &= 3\left(\Delta(\chi^2\xi_2), \xi_2\right)_V + (\Delta(\gamma''(\chi)\xi_2), \xi_2)_V \\ &\le \varepsilon\|\Delta\xi_2\|_V^2 + c(\varepsilon)\|\chi^2\xi_2 + \gamma''(\chi)\xi_2\|_V^2 \\ &\le \varepsilon\|\Delta\xi_2\|_V^2 + c(\varepsilon)\Big(\|\xi_2\|_H^2 + \|\nabla(\chi^2\xi_2 + \gamma''(\chi)\xi_2)\|_H^2\Big) \\ &\le \varepsilon\|\Delta\xi_2\|_V^2 + c(\varepsilon)\Big(\|\xi_2\|_H^2 + \|\chi\xi_2\nabla\chi + \chi^2\nabla\xi_2 + \xi_2\nabla\chi + \nabla\xi_2\|_H^2\Big)\end{aligned}$$

$$
\begin{aligned}
&\leq \varepsilon\|\Delta\xi_2\|_V^2 + c(\varepsilon)\Big(\|\xi_2\|_H^2 + \|\xi_2\nabla\chi + \nabla\xi_2\|_H^2\Big) \\
&\leq \varepsilon\|\Delta\xi_2\|_V^2 + c(\varepsilon)\Big(\|\xi_2\|_V^2 + \|\xi_2\|_{L^4(\Omega)}^2\|\nabla\chi\|_{L^4(\Omega)}^2\Big) \\
&\leq \varepsilon\|\Delta\xi_2\|_V^2 + c(\varepsilon)\Big(\|\xi_2\|_V^2 + \|\xi_2\|_V^2\|\nabla\chi\|_W^2\Big) \\
&\leq \varepsilon\|\Delta\xi_2\|_V^2 + c(\varepsilon)\|\xi_2\|_V^2. \qquad (8.12)
\end{aligned}
$$

Finally, the last term in (8.10) can be estimated as follows:

$$
\begin{aligned}
(h(\chi,\chi^*),\xi_2)_V &= \Big(\chi^3 - (\chi^*)^3 - 3\chi^2(\chi-\chi^*) + \gamma'(\chi) - \gamma'(\chi^*) - \gamma''(\chi)(\chi-\chi^*), \Delta\xi_2\Big)_V \\
&\leq \varepsilon\|\Delta\xi_2\|_V^2 + \frac{1}{4\varepsilon}\|(\chi-\chi^*)^2(\chi^*+2\chi)\|_V^2 \\
&\quad + \frac{1}{4\varepsilon}\|\gamma'(\chi) - \gamma'(\chi^*) - \gamma''(\chi)(\chi-\chi^*)\|_V^2. \qquad (8.13)
\end{aligned}
$$

On the other hand χ and χ^* are in the projection on the second component of $\mathcal{A}$.

By Hölder inequality, the embedding $W \hookrightarrow V \hookrightarrow L^6(\Omega)$ and the fact that $\chi, \chi^* \in L^\infty(\mathbb{R}^+; V)$, we get

$$
\begin{aligned}
&\|\chi^3 - (\chi^*)^3 - 2\chi^2(\chi-\chi^*)\|_V^2 \\
&= \|(\chi-\chi^*)^2(\chi^*+2\chi)\|_V^2 \\
&\leq c\int_\Omega \big|\nabla[(\chi-\chi^*)^2(\chi+\chi^*)]\big|^2 \\
&\leq c\int_\Omega |\chi-\chi^*|^2|\nabla(\chi-\chi^*)|^2|\chi+\chi^*|^2 + c\int_\Omega |\chi-\chi^*|^4|\nabla(\chi+\chi^*)|^2 \\
&\leq c\|\chi-\chi^*\|_{L^6(\Omega)}^2\|\nabla(\chi-\chi^*)\|_{L^6(\Omega)}^2\|\chi+\chi^*\|_{L^6(\Omega)}^2 + c\|\chi-\chi^*\|_{L^6(\Omega)}^4\|\nabla(\chi+2\chi^*)\|_{L^6(\Omega)}^2 \\
&\leq c\|\chi-\chi^*\|_V^2\|\chi-\chi^*\|_W^2\|\chi+\chi^*\|_V^2 + c\|\chi-\chi^*\|_V^4\|\chi+\chi^*\|_W^2 \\
&\leq \|\chi-\chi^*\|_W^2\sigma(\|\chi-\chi^*\|_V^2) + \|\chi+\chi^*\|_W^2\|\chi-\chi^*\|_V^2\sigma(\|\chi-\chi^*\|_V^2) \qquad (8.14)
\end{aligned}
$$

where σ is independent of χ and $\sigma(y) = o(1)$ as $y \to 0^+$.

It is not difficult to see that, on account of (8.3), an estimate like (8.14) holds for the last term in (8.13) as well.

Hence, taking into account the estimate (5.3), we have

$$
(h(\chi,\chi^*),\xi_2)_V \leq \varepsilon\ |\Delta\xi_2\|_V^2 + \|\chi-\chi^*\|_W^2\sigma(\|\tilde\zeta_0\|_{\mathcal{H}}^2) + \|\chi+\chi^*\|_W^2\|\tilde\zeta_0\|_{\mathcal{H}}^2\sigma(\|\tilde\zeta_0\|_{\mathcal{H}}^2). \qquad (8.15)
$$

Considering (8.11)-(8.15) with equation (8.10), we can always find κ large enough and ε small enough to have $\delta_0 > 0$ such that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\kappa\|\xi_1\|_H^2 + \|\xi_2\|_V^2 + \kappa\|\xi_3\|_{\mathcal{M}}^2\Big) \\
&\leq -\delta_0\Big(\kappa\|\xi_1\|_V^2 + \|\xi_2\|_W^2 + \kappa\|\xi_3\|_{\mathcal{M}}^2\Big) \\
&\quad + c\left(\|\xi_1\|_H^2 + \|\xi_2\|_V^2\right) + \|\chi-\chi^*\|_W^2\sigma(\|\tilde\zeta_0\|_{\mathcal{H}}^2) + \|\chi+\chi^*\|_W^2\|\tilde\zeta_0\|_{\mathcal{H}}^2\sigma(\|\tilde\zeta_0\|_{\mathcal{H}}^2). \qquad (8.16)
\end{aligned}
$$

Setting

$$\phi_\kappa^2(s) = \kappa\|\xi_1(s)\|_H^2 + \|\xi_2(s)\|_V^2 + \kappa\|\xi_3(s)\|_{\mathcal{M}}^2$$

and integrating (8.16) with respect to s over $[0,T]$, we obtain

$$\begin{aligned}\phi_\kappa^2(T) \ \leq \ & c\int_0^T \left(\kappa\|\xi_1(s)\|_H^2 + \|\xi_2(s)\|_V^2\right) ds \\ & + \sigma(\|\tilde{\zeta}_0\|_{\mathcal{H}}^2)\int_0^T \|\chi(s) - \chi^*(s)\|_W^2 ds + \|\tilde{\zeta}_0\|_{\mathcal{H}}^2\sigma(\|\tilde{\zeta}_0\|_{\mathcal{H}}^2)\int_0^T \|\chi(s) + \chi^*(s)\|_W^2 ds\end{aligned}$$

and Gronwall lemma and (6.8) yield

$$\phi_\kappa^2(T) \leq c\|\tilde{\zeta}_0\|_{\mathcal{H}}^2\sigma(\|\tilde{\zeta}_0\|_{\mathcal{H}}^2).$$

Since $\|\xi(T)\|_{\mathcal{H}}^2 \leq c\phi_\kappa^2(T)$, we conclude that, for every $T \in \mathbb{R}^+$,

$$\lim_{\|\tilde{\zeta}_0\|_{\mathcal{H}} \longrightarrow 0} \frac{\|\xi(T)\|_{\mathcal{H}}}{\|\tilde{\zeta}_0\|_{\mathcal{H}}} = 0$$

which entails the required quasi-differentiability.

Multiplying the system in (8.8) by Z in $\mathcal{H}$, and performing similar calculations, we obtain the estimate

$$\|Z(T)\|_{\mathcal{H}}^2 \leq c\|Z_0\|_{\mathcal{H}}^2$$

which leads to the last assertion. □

8.5 Proof of Theorem 8.8

As we already pointed out, it will be enough to show that the attractor $\mathcal{A}$ of the semigroup $U(t)$ acting on $\mathcal{H}$ has finite fractal dimension.

Using the same notation of Theorem 8.7, given any unitary vector $Z = (\Theta, X, Y)$ belonging to the domain of $L + N'(\zeta)$, using (8.11) and (8.12), where $\delta_0 > 0$ is defined in (8.16), we have

$$\begin{aligned}((L+N'(\zeta))Z, Z)_{\mathcal{H}} \ &= \ \left((L+N'(\zeta))\left(\frac{\Theta}{\sqrt{\kappa}}, X, \sqrt{\kappa}\frac{Y}{\sqrt{\kappa}}\right), \left(\kappa\frac{\Theta}{\sqrt{\kappa}}, X, \kappa\frac{Y}{\sqrt{\kappa}}\right)\right)_{\mathcal{H}} \\ &\leq \ -\delta_0\left(\|\Theta\|_V^2 + \|X\|_W^2 + \|Y\|_{\mathcal{M}}^2\right) + c\left(\|\Theta\|_H^2 + \|X\|_V^2\right).\end{aligned}$$

Therefore we conclude that $L + N(\zeta) \leq M$, where M is the diagonal operator acting on $H \oplus V \oplus \mathcal{M}$ defined by

$$M = \begin{pmatrix} c\mathbb{I} - \delta_0(\mathbb{I} - \Delta) & 0 & 0 \\ 0 & c\mathbb{I} - \delta_0(\mathbb{I} - \Delta) & 0 \\ 0 & 0 & -\delta_0\mathbb{I} \end{pmatrix}.$$

Directly from the definition of Tr_m, it is apparent that

$$\mathrm{Tr}_m(L+N'(\zeta)) \leq \mathrm{Tr}_m(M).$$

Finally, since M is diagonal, we have

$$\mathrm{Tr}_m(M) = \sup_Q \sum_{i=1..m} (MZ_i, Z_i)_{\mathcal{H}}$$

where the supremum is taken over the projections Q of the form $Q_1 \oplus Q_2 \oplus Q_3$. This amounts to consider vectors Z_i where only one of the three components is non-zero (and in fact of norm one in its space). Let then $m_1+m_2+m_3 = m$ be the number of vectors Z_i of the form $(\Theta, 0, 0)$, $(0, X, 0)$ and $(0, 0, Y)$, respectively. Applying Lemma 8.6, we obtain that there exists a positive constant λ such that

$$\mathrm{Tr}_m(M) \leq -m_1(\delta_0 \lambda m_1^{2/3} - c) - m_2(\delta_0 \lambda m_2^{2/3} - c) - \delta_0 m_3$$

which gives at once

$$q_m \leq -m_1(\delta_0 \lambda m_1^{2/3} - c) - m_2(\delta_0 \lambda m_2^{2/3} - c) - \delta_0 m_3.$$

Since as m goes to infinity at least one of the m_j's goes to infinity, it is clear that there exists $\tilde{m}$ big enough such that $q_{\tilde{m}} < 0$. Thus, the proof follows from Theorem 8.5 and Theorem 8.7. $\square$

Remark 8.9. It is possible to extend the above results to a particular non-autonomous case (see, e.g., [57]). Indeed, the uniform attractor $\mathcal{A}$ of the family $\{U_f(t,\tau), f \in \mathrm{H}(g)\}$ has finite fractal dimension, when the time dependent function g is *quasiperiodic*, namely, a function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ of the form (see, e.g., [2])

$$g(x,t) = G(x, \kappa t) = G(x, \kappa_1 t, \cdots, \kappa_m t)$$

where $G(\cdot, \varpi) \in C^1(\mathbb{T}^m, H)$ is a 2π-periodic function of ϖ on the m-dimensional torus $\mathbb{T}^m$ and $\kappa = (\kappa_1, \cdots, \kappa_m)$ are rationally independent numbers. It is immediate to check that g is translation compact in $L^2_{\mathrm{loc}}(\mathbb{R}; H)$, and $f \in \mathrm{H}(g)$ if and only if

$$f(x,t) = G(x, \kappa t + \varpi_0)$$

for some $\varpi_0 \in \mathbb{T}^m$.

Appendix A

Compactness results and Gronwall lemmas

For the reader's convenience, we report here below some technical tools, which are used in this thesis.

Since we deal with nonlinear equations, we need some results concerning weak convergences and compactness theorems.

The following lemma gives sufficient conditions to identify a weak limit (see Lemma 1.3, in [46] for the proof).

Lemma A.1. *Let Ω be an open bounded set in $\mathbb{R}^3$ and I be a bounded interval in $\mathbb{R}$. If F is a bounded subset of $L^p(\Omega \times I)$, where $p \in \mathbb{N}$ lies in the open interval $(1, \infty)$, and there exist a sequence $\{g_i, i \in \mathbb{N}\}$ of elements of F such that*

$$g_i \longrightarrow g$$

a.e. in $\Omega \times I$, then

$$g_i \longrightarrow g$$

weakly in $L^p(\Omega \times I)$.

We recall two useful compactness lemmas (for the proof we refer to Section 6.4.7 in [54] and Corollary 4, Section 8 in [55], respectively).

Lemma A.2. *Assume X, B and Y are Banach spaces such that $X \hookrightarrow B \hookrightarrow Y$ with compact embedding $X \hookrightarrow B$. Then for every $\varepsilon > 0$ there exists a constant $c(\varepsilon)$ such that*

$$\|u\|_B \leq \varepsilon \|u\|_X + c(\varepsilon)\|u\|_Y$$

for every $u \in X$.

Lemma A.3. *Assume X, B and Y are Banach spaces such that $X \hookrightarrow B \hookrightarrow Y$ with compact embedding $X \hookrightarrow B$ and let I be compact in $\mathbb{R}$.*

1. *If F is a bounded subset of $L^r(I; X)$, with $1 \leq r < +\infty$ and $\partial F = \{\partial_t f : f \in F\}$ is bounded in $L^1(I; Y)$, then F is relatively compact in $L^r(I; B)$.*

2. *If F is a bounded subset of $L^\infty(I;X)$ and ∂F is bounded in $L^r(I;Y)$, with $r>1$, then F is relatively compact in $C(I;B)$.*

The following compactness lemma for weighted-valued spaces is used in Chapter 7 (see Lemma 5.5. in [53] for the proof).

Lemma A.4. *Let $\alpha \in C(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$ be a non-negative and non-increasing function, and assume X, B and Y are Hilbert spaces such that $X \hookrightarrow B \hookrightarrow Y$, with compact embedding $X \hookrightarrow B$.*

Let $F \subset L^2_\alpha(\mathbb{R}^+;B)$ satisfy the following hypothesis:

1. *F is bounded in $L^2_\alpha(\mathbb{R}^+;X) \cap H^1_\alpha(\mathbb{R}^+;Y)$;*
2. $\sup_{\eta\in F} \|\eta(s)\|^2_B \le h(s) \quad$ a.e. *for some $h \in L^1_\alpha(\mathbb{R}^+)$.*

Then F is relatively compact in $L^2_\alpha(\mathbb{R}^+;B)$.

We conclude with two Gronwall-type Lemmas, which subsume Lemma A.5 in [8] and some results in [52].

Lemma A.5. *Let Φ, m_1, m_2, m_3 be four non-negative, summable function on the interval $[\tau,T] \subset \mathbb{R}$. Then the differential inequality*

$$\frac{d}{dt}\Phi^2(t) \le m_1(t)\Phi^2(t) + m_2(t)\Phi(t) + m_3(t) \qquad \text{a.e.} t \in [\tau,T]$$

implies

$$\Phi^2(t) \le 2e^{\mu_1}\Phi^2(\tau) + e^{2\mu_1}\|m_2\|^2_{L^1(\tau,t)} + 2e^{\mu_1}\|m_3\|_{L^1(\tau,t)}$$

for all $t \in [\tau,T]$, where $\mu_1 = \|m_1\|_{L^1(\tau,T)}$.

Lemma A.6. *Let Φ be an absolutely continuous, non negative function on the interval $[\tau,+\infty)$; let $m_0 \ge 0$ and m_1, m_2 two locally summable functions where $m_1 \in \mathcal{T}_1$ and m_2 is non negative. Then the differential inequality*

$$\frac{d}{dt}\Phi^2(t) + \varepsilon\Phi^2(t) \le m_0 + \|m_1(t)\|_H\Phi(t) + m_2(t) \qquad a.e.\ t \in [\tau,+\infty)$$

which holds for some $\varepsilon > 0$ implies

$$\Phi^2(t) \le 2\Phi^2(\tau)e^{-\varepsilon(t-\tau)} + \frac{2m_0}{\varepsilon} + \frac{e^\varepsilon}{(1-e^{-\varepsilon/2})^2}\|m_1\|^2_{\mathcal{T}_1} + \int_\tau^t m_2(s)e^{-\varepsilon(t-s)}ds$$

for all $t \in [\tau,+\infty)$.

Remark A.7. Observe that if the function $m_2 \in \mathcal{T}^1(\mathbb{R})$, then

$$\int_\tau^t m_2(y)\exp[-\varepsilon(t-y)]dy \le c\|m_2\|_{\mathcal{T}^1(\mathbb{R})}$$

where c is a proper constant which depends only on ε. Indeed

$$\begin{aligned}
&\int_\tau^t m_2(y)e^{-\varepsilon(t-y)}dy = \sum_{r=0}^{[t-\tau]-1}\int_r^{r+1} m_2(t-s)e^{-\varepsilon s}ds + \int_{[t-\tau]}^{t-\tau} m_2(t-s)e^{-\varepsilon s}ds \\
&\leq \sum_{r=0}^{[t-\tau]-1} e^{-\varepsilon r}\int_r^{r+1} m_2(t-s)ds + e^{-\varepsilon([t-\tau])}\int_{[t-\tau]}^{[t-\tau]+1} m_2(t-s)ds \\
&\leq \sum_{r=0}^{[t-\tau]-1} e^{-\varepsilon r}\int_{t-r-1}^{t-r} m(y)dy + e^{-\varepsilon([t-\tau])}\int_{t-[t-\tau]-1}^{t-[t-\tau]} m(y)dy \\
&\leq \sup_{r\in\mathbb{R}}\int_r^{r+1} m_2(y)dy\sum_{r=0}^{[t-\tau]} e^{-\varepsilon r} \leq \sup_{r\in\mathbb{R}}\int_r^{r+1} m_2(y)dy\sum_{r=0}^{\infty} e^{-\varepsilon r} \\
&\leq \sup_{r\in\mathbb{R}}\int_r^{r+1} m_2(y)dy\int_0^\infty e^{-\varepsilon(x-1)}dx = \sup_{r\in\mathbb{R}}\int_r^{r+1} m_2(y)dy\left[-\frac{e^{-\varepsilon(x-1)}}{\varepsilon}\right]_0^\infty \leq \frac{e^\varepsilon}{\varepsilon}\|m_2\|_{\mathcal{T}^1(\mathbb{R})}.
\end{aligned}$$

The thesis of Lemma A.6 could then be restated in the form

$$\Phi^2(t) \leq 2\Phi^2(\tau)e^{-\varepsilon(t-\tau)} + \frac{2m_0}{\varepsilon} + \frac{e^\varepsilon}{(1-e^{-\varepsilon/2})^2}\|m_1\|^2_{\mathcal{T}_1} + \frac{e^\varepsilon}{\varepsilon}\|m_2\|_{\mathcal{T}^1(\mathbb{R})}.$$

Notice also that the following inequality

$$\int_\tau^t m(y)e^{-\varepsilon(t-y)}dy \leq \frac{e^\varepsilon}{\varepsilon}\sup_{r\geq\tau}\int_r^{r+1} m(y)dy$$

holds for every non-negative locally summable function m on $(\tau,+\infty)$.

Bibliography

[1] S. Aizicovici, V. Barbu: *Existence and asymptotic results for a system of integro-partial differential equations*, NoDEA Nonlinear Differential Equations Appl. **3** (1996), 1–18.

[2] L. Amerio, G. Prouse: *Almost-periodic functions and functional equations*, Van Nostrand, New York, 1971.

[3] A.V. Babin, M.I. Vishik: *Attracteurs maximaux dans les équations aux dérivées partielles*, Pitman, Boston, Mass.-London, 1985.

[4] A.V. Babin, M.I. Vishik: *Attractors of evolution equations*, North-Holland, Amsterdam, 1992.

[5] E. Bonetti: *Some asymptotic analysis for hyperbolic relaxed Stefan problems with memory*, Asymptot. Anal. **20** (1999), 241–261.

[6] G. Bonfanti, F. Luterotti: *Global solution to a phase-field model with memory and quadratic nonlinearity*, Adv. Math. Sci. Appl. **9** (1999), 523–538.

[7] G. Bonfanti, F. Luterotti: *Regularity and convergence results for a phase-field model with memory*, Math. Meth. Appl. Sci. **21** (1998), 1085–1105.

[8] H. Brézis: *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North-Holland, Amsterdam, 1973.

[9] H. Brézis: *Analyse fonctionnelle, théorie et applications*, Masson, Paris, 1983.

[10] D. Brochet, D. Hilhorst, A. Novick– Cohen: *Maximal attractor and inertial sets for a conserved phase-field model*, Adv. Differential Equations **1** (1996), 547–578.

[11] M. Brokate, J. Sprekels: *Hysteresis and phase transitions*, Springer-Verlag, New York, 1996.

[12] G. Caginalp: *The dynamics of a conserved phase-field system: Stefan-like, Hele-Shaw, and Cahn-Hilliard models as asymptotic limits*, IMA J. Appl. Math **44** (1990), 77–94.

[13] V.V. Chepyzhov, M.I. Vishik: *Nonautonomous evolution equations and their attractors*, Russian J. Math. Phys. **1** (1993), 165–190,

[14] V.V. Chepyzhov, M.I. Vishik: *Attractors of nonautonomous dynamical systems and their dimensions*, J. Math. Pures Appl. (9) **73** (1994), 279–333.

[15] V.V. Chepyzhov, M.I. Vishik: *Non-autonomous evolutionary equations with translation compact symbols and their attractor*, C. R. Acad. Sci. Paris Sér. I Math. **321** (1995), 153–158.

[16] L. Chiusano, P. Colli: *Positive kernel and zero time relaxation in a phase-filed model with memory*, Commun. Appl. Anal. **2** (1998), 531–550.

[17] B.D. Coleman, M.E. Gurtin: *Equipresence and constitutive equations for rigid heat conductors*, Z. Angew. Math. Phys **18** (1967), 199–208.

[18] P. Colli, G. Gilardi, M. Grasselli: *Convergence of phase fieldto phase relaxation models with memory*, Ann. Univ. Ferrara Ser. VII (N.S.) Suppl **41** (1996), 1–14.

[19] P. Colli, G. Gilardi, M. Grasselli: *Asymptotic analysis of a phase field model with memory for vanishing time relaxation*, Hiroshima Math. J. **29** (1999), 117–143.

[20] P. Colli, G. Gilardi, M. Grasselli: *Asymptotic justification of the phase-field model with memory*, Comm. Appl. Nonlinear Anal. **6** (1999), 1–27.

[21] P. Colli, G. Gilardi, M. Grasselli: *Errata: Asymptotic justification of the phase-field model with memory*, Comm. Appl. Nonlinear Anal. **7** (2000), 101–102.

[22] P. Colli, G. Gilardi, M. Grasselli: *Global smooth solutions to the standard phase-field model with memory*, Adv. Differential Equations **3** (1997) 453–486.

[23] P. Colli, G. Gilardi, M. Grasselli: *Well posedness of the weak formulation for the phase-field model with memory*, Adv. Differential Equations **3**, (1997) 487–508.

[24] P. Colli, G. Gilardi, M. Grasselli, G. Schimperna: *The conserved phase-field model with memory*, Adv. Math. Sci. Appl. (to appear).

[25] P. Colli, G. Gilardi, M. Grasselli, G. Schimperna: *Global existence for the conserved phase field model with memory and quadratic nonlinearity*, Portugal. Math. (to appear).

[26] P. Colli, G. Gilardi, Ph. Laurençot, A. Novick–Cohen: *Uniqueness and long-time behavior for the conserved phase-field system with memory*, Discrete Contin. Dynam. Systems **5** (1999), 375–390.

[27] P. Colli, P. Laurençot: *Existence and stabilization of solutions to the phase-field model with memory*, J. Integral Equations Appl. **10** (1998), 169–194.

[28] P. Colli, P. Laurençot: *Uniqueness of weak solutions to the phase-field model with memory*, J. Math. Sci. Univ. Tokyo **5** (1998), 459–476.

[29] P. Constantine, C. Foias, R. Temam: *Attractors representing turbulent flows*, Mem. Amer. Math. Soc. **53**, 1985.

[30] C.M. Dafermos: *Asymptotic stability in viscoelasticity*, Arch. Rational Mech. Anal. **37**, (1970), 297–308.

[31] R. Dautray, J.L. Lions: *Analyse mathématique et calcul numerique pour les sciences et les techniques*, Masson, Paris, 1987.

[32] C. Giorgi, M. Grasselli, V. Pata: *Uniform attractors for a phase-field model with memory and quadratic nonlinearity*, Indiana Univ. Math. J. **48** (1999), 1395–1445.

[33] C. Giorgi, M. Grasselli, V. Pata: *Well-posedness and longtime behavior of the phase-field model with memory in a history space setting*, Quart. Appl. Math. (to appear).

[34] C. Giorgi, A. Marzocchi, V. Pata: *Asymptotic behavior of a semilinear problem in heat conduction with memory*, NoDEA Nonlinear Differential Equations Appl. **5** (1998), 333–354.

[35] C. Giorgi, A. Marzocchi, V. Pata: *Uniform attractors for a non-autonomous semilinear heat equation with memory*, Quart. Appl. Math. (to appear).

[36] M. Grasselli: *Asymptotic analysis of a phase-field model with memory*, "Free Boundary Problems: Theory and Applications" (I. Athanasopoulos, G. Makrakis, and J.F. Rodrigues, Eds.), 273–286, Chapman & Hall/CRC Res. Notes Math. **409**, Boca Raton 1999.

[37] M. Grasselli, V. Pata: *Longtime behavior of a homogenized model in viscoelastodynamics*, Discrete Contin. Dynam. Systems **4** (1998), 339–358.

[38] M. Grasselli, V. Pata: *Upper semicontinuous attractors for a hyperbolic phase-field model with memory*, submitted.

[39] M.E. Gurtin, A.C. Pipkin: *A general theory of heat conduction with finite wave speeds*, Arch. Rational Mech. Anal. **31** (1968), 113–126.

[40] J. Hale: *Asymptotic behavior of dissipative systems*, Math. Surv. Monogr. **25**, Amer. Math. Soc., Providence, 1988.

[41] A. Haraux: *Linear semigroups in Banach spaces*, Pitman Res. Notes Math. Ser. 152, 1986.

[42] A. Haraux: *Systèmes dynamiques dissipatifs et applications*, Masson, Paris, 1991.

[43] J. Jäckle: *Heat conduction and relaxation in liquids of high viscosity*, Physica A **162** (1990), 337-404.

[44] D.D. Joseph, L. Preziosi: *Heat waves*, Rev. Modern Phys. **61** (1989), 41–73.

[45] D.D. Joseph, L. Preziosi: *Addendum to the paper "Heat waves" [Rev. Mod. Phys. 61 (1989), 41–73]*, Rev. Modern Phys. **62** (1990), 375–391.

[46] J.L. Lions: *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Gauthier-Villars, Paris, 1969.

[47] J.L. Lions, E. Magenes: *Non-homogeneous boundary value problem and applications II*, Springer-Verlag, New York-Heidelberg, 1972.

[48] Z. Liu, S. Zheng: *Semigroups associated with dissipative systems* Chapman & Hall/CRC Research Notes in Mathematics **398**, Boca Raton, 1999.

[49] R. Mañé: *On the dimensions of the compact invariant sets of certains nonlinear maps*, Springer, Berlin-New York, 1981.

[50] A. Novick–Cohen: *The Cahn-Hilliard equation: mathematical and modeling perspectives*, Adv. Math. Sci. Appl. **8** (1998), 965–985.

[51] A. Novick–Cohen: *Conserved phase-field equations with memory*, Curvature flows and related topics (1994), 179–197, GAKUTO Internat. Ser. Math. Sci. Appl., 5, Gakkōtosho, Tokyo, 1995.

[52] V. Pata, G. Prouse, M.I. Vishik: *Traveling waves of dissipativee non-autonomous hyperbolic equations in a strip*, Adv. Differential Equations **3** (1998), 249–270.

[53] V. Pata, A. Zucchi: *Attractors for a damped hyperbolic equation with linear memory*, submitted.

[54] M. Renardy, R.C. Rogers: *An introduction to partial differential equations*, Springer-Verlag, New York, 1993.

[55] J. Simon: *Compact sets in* $L^p(0,T;B)$, Ann. Mat. Pura Appl. (4) **146** (1987), 65–96.

[56] R. Temam: *Infinite-dimensional dynamical systems in mechanics and physics*, Springer-Verlag, New York, 1997.

[57] M.I. Vishik: *Asymptotic behaviour of solutions of evolutionary equations*, Cambridge University Press, Cambridge, 1992.

[58] A. Visintin: *Models of phase transition*, Birkhauser Boston Inc., Boston, 1996.