Formulazione duale per un modello di transizione di fase: buona positura e comportamento per tempi lunghi

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joint work with E. Bonetti (Pavia) and M. Frémond (Paris)

Plan of the Talk

We discuss here a new approach to phase transitions with thermal memory based on a new formulation of the internal energy balance by means of the entropy leading to a nonlinear and possibly singular PDE system. We proceed as follows:

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- we state the long-time behaviour results holding true for particular choices of the nonlinearities involved

  E. Bonetti, E.R., Commun. Pure Appl. Anal., to appear

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where

- σ and λ are smooth functions accounting for the non-convex part of the internal energy and the latent heat, respectively
- $\widehat{\beta} : \mathbb{R} \to [0, \infty]$ is a general proper, convex, and lower-semicontinuous function
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to the following general free energy functional:

$$\Psi(\vartheta, \chi, \nabla\chi) = -\widehat{\alpha}^*(\vartheta) - \lambda(\chi)\vartheta + \sigma(\chi) + \widehat{\beta}(\chi) + \frac{\nu}{2}|\nabla\chi|^2$$

- $\widehat{\alpha}^* : \mathbb{R} \to \mathbb{R}$ is the convex conjugate of $\widehat{\alpha}$

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to the standard one in case $\widehat{\alpha}^*(\vartheta) = c_v\vartheta(\log\vartheta - 1)$:

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- Since, c_v in the applications may also not be constant, we can allow every form for $c_v = c_v(\vartheta)$ such that $\widehat{\alpha}(\vartheta)$ is convex - e.g., if $c_V(\vartheta) = \vartheta^\gamma$, for $\vartheta \in (0, \bar{\vartheta})$ with $\gamma \geq 0$ - since $c_v(\vartheta) = -\vartheta\left(\partial^2\Psi/\partial\vartheta^2\right)$, then we have $\widehat{\alpha}^*(\vartheta) = \vartheta^{\gamma+1}/\left[\gamma(\gamma+1)\right]$

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  $$\widehat{\alpha}(u) = u^{\frac{\gamma+1}{\gamma}}/(\gamma+1)$$

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Indeed, we can compute the conjugate function

$$p^*(\chi_t, \nabla\vartheta) = \text{sup\_}_{\mathbf{Q}}\{-\nabla\vartheta\cdot\mathbf{Q} - p(\chi_t, -\mathbf{Q})\},$$ from which

it follows $\nabla\vartheta = -\alpha'(u)\mathbf{Q}$ and $-\mathbf{Q} = \nabla u$ because $\vartheta = \alpha(u)$.

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Hence, we recover the following form for the pseudo-potential of dissipation

$$\Phi(\chi_t, \nabla\vartheta) = p^*(\chi_t, \nabla\vartheta) = \frac{1}{2}|\chi_t|^2 + \frac{1}{2\alpha'(\alpha^{-1}(\vartheta))}|\nabla\vartheta|^2.$$

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From which, if $\alpha'(\alpha^{-1}(\vartheta)) = \vartheta$, like, e.g., in case $\alpha(u) = \exp(u)$, we recover the standard form of Φ:

$$\Phi(\chi_t, \nabla\vartheta) = \frac{1}{2}|\chi_t|^2 + \frac{|\nabla\vartheta|^2}{2\vartheta}.$$

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THE PRINCIPLE OF VIRTUAL POWER for microscopic motion - for any subdomain $D \subset \Omega$ and any virtual microscopic velocity v - reads

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In absence of external actions we get

$\boxed{B - \operatorname{div} \mathbf{H} = 0}$ in Ω with $\mathbf{H} \cdot \mathbf{n} = 0$ on $\partial\Omega$.

The phase inclusion

The equilibrium equation

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$$\chi_t - \nu\Delta\chi + \beta(\chi) + \sigma'(\chi) - \alpha(s - \lambda(\chi))\lambda'(\chi) \ni 0 \quad \text{in } \Omega$$

$$\text{and } \partial_{\mathbf{n}}\chi = 0 \text{ on } \partial\Omega$$

where $\alpha = \widehat{\alpha}'$ and $\beta = \partial\widehat{\beta}$.

Possible choices of the potentials $\widehat{\beta}$

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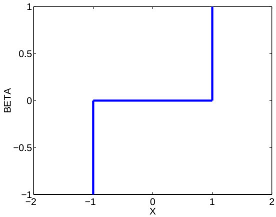
Subdifferential case: $\beta := \partial\widehat{\beta} = \partial I_{[-1,1]}$:



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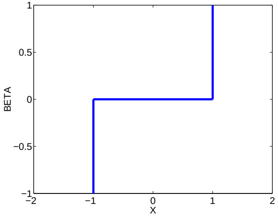
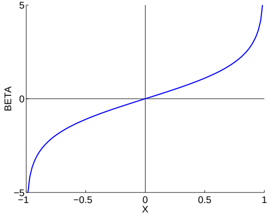


Logarithmic case: $\beta := \partial\widehat{\beta} = \log(1+\chi) - \log(1-\chi)$:



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Then, if we take - as before - the following form for the power of internal actions:

$$P_{\text{int}}(D, \chi_t) = -\int_D (B\chi_t + \mathbf{H} \cdot \nabla\chi_t) \, d\Omega,$$

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we get exactly that there exists $\mathbf{q}$ such that

$$E_t + \mathrm{div}\,\mathbf{q} = \frac{\partial E}{\partial \chi}\chi_t + \frac{\partial p}{\partial \chi_t}\chi_t + \frac{\partial E}{\partial(\nabla\chi)}\nabla\chi_t \quad \text{in } \Omega.$$

The energy balance

Hence, the first principle of thermodynamics reads

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With

$$E(s, \chi, \nabla\chi) = \widehat{\alpha}(s - \lambda(\chi)) + \sigma(\chi) + \widehat{\beta}(\chi) + \frac{\nu}{2}|\nabla\chi|^2,$$

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and, denoting by $u = s - \lambda(\chi)$, it gives:

$$\alpha(u)\,(s_t + \operatorname{div} \mathbf{Q}) = \alpha'(u)|\nabla u|^2 + \chi_t^2 \text{ in } \Omega$$

where

- we recall that $\alpha(s - \lambda(\chi)) = \widehat{\alpha}'(s - \lambda(\chi)) = \dfrac{\partial E}{\partial s}$,
- and we have chosen $\mathbf{q}$ such that $\mathbf{q}/\alpha(u) = \mathbf{Q} = -\nabla u$.

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- $\alpha(u) = \alpha(s-\lambda(\chi)) = \widehat{\alpha}'(s-\lambda(\chi)) = \dfrac{\partial E}{\partial s}(=\vartheta) > 0$ (ϑ is the absolute temperature) – because we have assumed $\widehat{\alpha}$ to be increasing,
- $\alpha' > 0$ – because we have assumed $\widehat{\alpha}$ to be convex.

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- $\alpha' > 0$ – because we have assumed $\widehat{\alpha}$ to be convex.

Divide by $\alpha(u) > 0$ the internal energy balance, getting

$$s_t + \operatorname{div}\mathbf{Q} \geq 0,$$

that is just the pointwise Clausius-Duhem inequality .

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where $u = s - \lambda(\chi)$, dividing by $\alpha(u)$, and using the small perturbations assumption (cf. [Germain]) - which allow us to neglect the higher order dissipative contributions on the right hand side -

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where $u = s - \lambda(\chi)$, dividing by $\alpha(u)$, and using the small perturbations assumption (cf. [Germain]) - which allow us to neglect the higher order dissipative contributions on the right hand side - we obtain the following equation for u

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- we let $\alpha = \partial\widehat{\alpha}$ be a general MAXIMAL MONOTONE GRAPH (maybe also multivalued),
- we include in the internal energy balance memory effects, i.e. the term $-\operatorname{div}\int_{-\infty}^{t} k(t-\tau)\nabla\alpha(u(\tau))\,d\tau$ on the left hand side of (EB).

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Put it on the right hand side. Then, we aim to find suitably regular (u, χ) solving in a proper sense:

$$\begin{aligned}
&(u + \lambda(\chi))_t - \Delta(u + k * \alpha(u)) \ni r \quad \text{in } \Omega \\
&\partial_{\mathbf{n}}(u + k * \alpha(u)) \ni h \quad \text{on } \partial\Omega \\
&\chi_t - \nu\Delta\chi + \beta(\chi) + \sigma'(\chi) - \lambda'(\chi)\alpha(u) \ni 0 \quad \text{in } \Omega \\
&\partial_{\mathbf{n}}\chi = 0 \quad \text{on } \partial\Omega \\
&u(0) = u_0, \quad \chi(0) = \chi_0 \quad \text{in } \Omega.
\end{aligned}$$

We must suppose from now on λ' constant ($= 1$ for simplicity).

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- An existence (of weak solutions) result under general assumptions on the nonlinearity α for a graph β with domain the whole $\mathbb{R}$ and with at most a polynomial growth at ∞
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- An existence-long-time behaviour (of solutions) result in case $\alpha = \exp$ and for a general β

Hypotheses 1

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- $\Omega \subset \mathbb{R}^3$ bdd connected domain with sufficiently smooth boundary $\Gamma := \partial\Omega$
- $t \in [0, \infty]$, $Q_t := \Omega \times (0, t)$, $\Sigma_t := \Gamma \times (0, t)$,
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Suppose moreover that

$\beta = \partial\widehat{\beta}$, $\alpha = \partial\widehat{\alpha}$, with $\widehat{\beta}$, $\widehat{\alpha} : \mathbb{R} \to (-\infty, +\infty]$ are proper, convex, and lower semicontinuous

$$\sigma \in C^2(D(\beta)), \quad \sigma'' \in L^\infty(D(\beta)), \quad \nu \geq 0$$

$$k \in W^{2,1}(0, t), \quad k(0) \geq 0, \quad k \equiv 0 \text{ if } k(0) = 0,$$

$$r \in L^2(Q_t) \cap L^1(0, T; L^\infty(\Omega)), \quad h \in L^\infty(\Sigma_t),$$

$$\langle R(t), v \rangle = \int_\Omega r(\cdot, t) v + \int_\Gamma h(\cdot, v) v_{|_\Gamma} \quad \forall v \in V$$

$$u_0 \in H,\ \widehat{\alpha}(u_0) \in L^1(\Omega),\ \chi_0 \in H,\ \nu\chi_0 \in V,\ \widehat{\beta}(\chi_0) \in L^1(\Omega).$$

Existence result for a general α

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Thm 1.Let T be a positive final time, HYPOTHESIS 1 be satisfied with $t = T$, and suppose moreover that $\nu > 0$, $k(0) > 0$, and there exists $p < 5$ such that

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then there exists at least a couple (u, χ) with the regularity properties

$$u \in H^1(0, T; V') \cap L^2(0, T; V), \ \chi \in H^1(0, T; H) \cap L^\infty(0, T; V),$$
$$\alpha_{V',V}(u) \in L^2(0, T; V'),$$
$$1 * \alpha_{V',V}(u) \in L^2(0, T; V) \cap C^0(0, T; H)$$

solving, a.e. in $(0, T)$, the PDE system:

$$\partial_t(u + \chi) + Au + A(k * \alpha_{V',V}(u)) \ni R, \quad \text{in } V', \qquad (1)$$
$$\partial_t\chi + \nu A\chi + \beta(\chi) + \sigma'(\chi) - \alpha_{V',V}(u) \ni 0 \quad \text{in } V', \qquad (2)$$
$$u(0) = u_0, \quad \chi(0) = \chi_0 \quad \text{a.e. in } \Omega.$$

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- $\alpha(u) = -1/u$: we recover, e.g., the Penrose-Fife system

The case α Lipschitz continuous

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$$u \in C^0([0,T]; H) \cap L^2(0,T; V), \quad \xi \in L^2(Q_T),$$
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THE PROOF IS A SUITABLE ADAPTATION OF THE ONE OF [BONETTI, COLLI, FRÉMOND, 2003] HOLDING TRUE IN CASE $\beta = \partial I_{[0,1]}$, $\sigma' = \vartheta_c$

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Thm 3. Let HYPOTHESIS 1 hold and suppose that

(i) $k \in W^{1,1}(0,\infty)$ is of strongly positive type, i.e. $\exists\, \eta > 0$ such that

$$\widetilde{k}(t) := k(t) - \eta \exp(-t) \quad \text{is of positive type;}$$

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Then, the ω-limit:

$$\omega(u_0, \chi_0, \nu) := \{(u_\infty, \chi_\infty) \in H \times H, \nu\chi_\infty \in V : \exists\, t_n \to +\infty, \\ (u(t_n), \chi(t_n)) \to (u_\infty, \chi_\infty) \text{ in } V' \times (V' \cap \nu H)\}$$

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is a compact, connected subset ($\neq \emptyset$) of $V' \times (V' \cap \nu H)$ and $\forall (u_\infty,\chi_\infty) \in \omega(u_0,\chi_0,\nu)$, $\exists\, \xi_\infty \in \beta(\chi_\infty)$ such that:

$$u_\infty = \frac{1}{|\Omega|}\left(-\int_\Omega \chi_\infty + c_0 + m\right),$$

$$\nu A\chi_\infty + \xi_\infty + \sigma'(\chi_\infty) = \alpha\left(\frac{1}{|\Omega|}\left(-\int_\Omega \chi_\infty + c_0 + m\right)\right),$$

where $c_0 = \int_\Omega u_0 + \int_\Omega \chi_0$, $m = \int_0^\infty \left(\int_\Omega r(s) + \int_\Gamma h(s)\right) ds$.

The case $\alpha = \exp$

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Then, there exists at least a quadruple $(u, \vartheta, \chi, \xi)$ such that $\vartheta = \alpha(u) = \exp(u)$, $\xi \in \beta(\chi)$ a.e.,

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Let $(u,\chi) : (0,\infty) \to H \times V$ be a solution on $(0,+\infty)$ associated to (u_0,χ_0). Then, the ω-limit set of a single trajectory (u,χ) defined in $(0,+\infty)$:

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THE CASE ν, $k = 0$ has been studied in [Bonetti, in "Dissipative phase transitions" (ed. P. Colli, N. Kenmochi, J. Sprekels) (2006)]

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In general, we cannot conclude that the whole trajectory $\{(u(t), \chi(t))\ t \geq 0\}$ tends to (u_∞, χ_∞) weakly in $H \times V$ and strongly in $V' \times H$ as $t \to +\infty$. This is mainly due to the presence of the anti-monotone term $\sigma'(\chi_\infty)$.

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- To get uniqueness in case of a general α (not Lipschitz-continuous). Problem: the doubly-nonlinear character of the system.