Weak formulation of a degenerating PDE system for phase transitions and damage

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joint work with Riccarda Rossi (University of Brescia)

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Part 1. Presentation of the problem and deduction of the PDE system via modelling

Part 2. Our most recent results (work in progress with Riccarda Rossi): weak solvability of the 3D degenerating PDE system

The scope

The analysis of the initial boundary-value problem for the following PDE system:

$$c(\vartheta)\vartheta_t + \chi_t\vartheta - \rho\vartheta\,\mathrm{div}\,\mathbf{u}_t - \mathrm{div}(k(\vartheta)\nabla\vartheta) = g$$

$$\mathbf{u}_{tt} - \mathrm{div}(a(\chi)\varepsilon(\mathbf{u}_t) + b(\chi)\varepsilon(\mathbf{u}) - \rho\vartheta\mathbf{1}) = \mathbf{f}$$

$$\chi_t + \mu\partial I_{(-\infty,0]}(\chi_t) - \Delta_p\chi + W'(\chi) \ni -b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} + \vartheta$$

which describes a thermoviscoelastic system in a reference domain $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$ during a time interval $[0, T]$

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- ϑ is the absolute temperature of the system
- $\mathbf{u}$ the vector of *small displacements*
- χ is the order parameter, standing for the local proportion of one of the two phases in *phase transitions* ($\chi = 0$: solid phase and $\chi = 1$: liquid phase, and $0 < \chi < 1$ in the so-called *mushy regions*)
- χ is the damage parameter, assessing the soundness of the material in *damage* (for the completely *damaged* $\chi = 0$ and the *undamaged* state $\chi = 1$, respectively, while $0 < \chi < 1$: *partial damage*)
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$\Longrightarrow$ We shall approximate the system with a non-degenerating one, where we replace the momentum equation with

$$\mathbf{u}_{tt} - \mathrm{div}((a(\chi) + \delta)\varepsilon(\mathbf{u}_t) + b(\chi)\varepsilon(\mathbf{u}) - \rho\vartheta\mathbf{1}) = \mathbf{f} \qquad \text{for } \delta > 0$$

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Note: in both these results we assumed χ_0 separated from the thresholds 0 and 1 and we prove (exploiting a sufficient coercivity condition on W at the thresholds 0 and 1) that the solution χ during the evolution continues to stay separated from 0 and 1 $\Longrightarrow$ prevent degeneracy (the operators are uniformly elliptic)

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The goal (joint work in progress with R. Rossi): to establish a **global existence result in 3D** using a suitable notion of solution and without enforcing the separation property, i.e. **allowing for degeneracy**

The model

Free energy and Dissipation, cf. [Frémond]

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The free-energy $\mathcal{F}$:

$$\mathcal{F} = \int_\Omega \left(f(\vartheta) + b(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} + \frac{1}{p}|\nabla\chi|^p + W(\chi) + \rho\vartheta \mathrm{tr}(\varepsilon(\mathbf{u})) - \vartheta\chi \right) \mathrm{d}x$$

- f is a concave function, $\rho \in \mathbb{R}$ a thermal expansion coefficient
- $b \in C^2(\mathbb{R}; [0, +\infty))$, e.g., $b(\chi) = 1 - \chi$ in phase transitions, $b(\chi) = \chi$ in damage
- $p > d$: we need the embedding of $W^{1,p}(\Omega)$ into $C^0(\overline{\Omega})$
- $W = \widehat{\beta} + \widehat{\gamma}$, $\widehat{\gamma} \in C^2(\mathbb{R})$, $\widehat{\beta}$ proper, convex, l.s.c., $\overline{\mathrm{dom}(\widehat{\beta})} = [0, 1]$

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The pseudo-potential $\mathcal{P}$:

$$\mathcal{P} = \frac{k(\vartheta)}{2}|\nabla\vartheta|^2 + \frac{1}{2}|\chi_t|^2 + a(\chi)\frac{|\varepsilon(\mathbf{u}_t)|^2}{2} + \mu I_{(-\infty,0]}(\chi_t)$$

- k the heat conductivity: coupled conditions with the specific heat $\mathrm{c}(\vartheta) = f(\vartheta) - \vartheta f'(\vartheta)$
- $a \in C^1(\mathbb{R}; [0, +\infty))$, e.g., $a(\chi) = \chi$
- $\mu = 0$: reversible case, $\mu = 1$: irreversible case

The modelling

The momentum equation

$$\mathbf{u}_{tt} - \operatorname{div} \sigma = \mathbf{f} \quad \left(\sigma = \sigma^{nd} + \sigma^{d} = \frac{\partial \mathcal{F}}{\partial \varepsilon(\mathbf{u})} + \frac{\partial \mathcal{P}}{\partial \varepsilon(\mathbf{u}_t)}\right) \quad \text{becomes}$$

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$$B - \operatorname{div}\mathbf{H} = 0 \quad \left(B = \frac{\partial \mathcal{F}}{\partial \chi} + \frac{\partial \mathcal{P}}{\partial \chi_t}, \mathbf{H} = \frac{\partial \mathcal{F}}{\partial \nabla \chi}\right) \quad \text{becomes}$$

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The internal energy balance

$$e_t + \operatorname{div}\mathbf{q} = g + \sigma : \varepsilon(\mathbf{u}_t) + B\chi_t + \mathbf{H}\cdot\nabla\chi_t \quad \left(e = \mathcal{F} - \vartheta\frac{\partial\mathcal{F}}{\partial\vartheta}, \quad \mathbf{q} = \frac{\partial\mathcal{P}}{\partial\nabla\vartheta}\right)$$

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$$\boxed{c(\vartheta)\vartheta_t + \chi_t\vartheta - \rho\vartheta\operatorname{div}\mathbf{u}_t - \operatorname{div}(k(\vartheta)\nabla\vartheta)) = g + |\chi_t|^2 + a(\chi)|\varepsilon(\mathbf{u}_t)|^2}$$

The analysis

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3) the *low regularity of the temperature variable*: difficulties in dealing with the coupling between ϑ and $\mathbf{u}$ equations in case $\rho \neq 0$

4) the *doubly nonlinear* character of the phase equation:
   - the nonsmooth graph $\partial\widehat{\beta}$,
   - the nonlinear p-Laplacian operator $-\Delta_p\chi$ (however regularizing)
   - the non-smooth constraint $\partial I_{(-\infty,0]}(\chi_t)$ in the irreversible case $\mu = 1$

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- We replace the momentum equation with a non-degenerating one

$$\mathbf{u}_{tt} - \operatorname{div}((a(\chi) + \delta)\varepsilon(\mathbf{u}_t) + b(\chi)\varepsilon(\mathbf{u}) - \rho\vartheta\mathbf{1}) = \mathbf{f}, \quad \delta > 0 \qquad (1)$$

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- Our first result states the existence of solutions to the non-degenerating system in the *reversible* case, i.e. with $\mu = 0$ in

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- In the *irreversible* case ($\mu = 1$) a major difficulty stems from the simultaneous presence in (2) of $\partial I_{(-\infty,0]}(\chi_t)$, $W'(\chi)$, and $-\Delta_p\chi$. We follow the approach of [Heinemann, Kraus, 2010] and consider a suitable weak formulation of (2) consisting of a *one-sided* variational inequality and of an energy inequality

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- For the analysis of the degenerate limit $\delta \searrow 0$ we have carefully adapted techniques from [Bouchitté, Mielke, Roubíček, 2009] and [Mielke, Roubíček, Zeman, 2011] to the case of a *rate-dependent* equation for χ, also coupled with the temperature equation

Energy vs Enthalpy

In order to deal with the low regularity of ϑ, rewrite the internal energy equation

$$\mathrm{c}(\vartheta)\vartheta_t + \chi_t\vartheta - \rho\vartheta \operatorname{div} \mathbf{u}_t - \operatorname{div}(k(\vartheta)\nabla\vartheta)) = g$$

as the enthalpy equation

$$w_t + \chi_t\Theta(w) - \rho\Theta(w) \operatorname{div} \mathbf{u}_t - \operatorname{div}(K(w)\nabla w)) = g \quad \text{where}$$

$$w = h(\vartheta) := \int_0^{\vartheta} \mathrm{c}(s)\,\mathrm{d}s, \quad \Theta(w) := \begin{cases} h^{-1}(w) & \text{if } w \geq 0, \\ 0 & \text{if } w < 0, \end{cases} \quad K(w) := \frac{k(\Theta(w))}{\mathrm{c}(\Theta(w))}$$

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We assume that

- $\mathrm{c} \in C^0([0,+\infty);[0,+\infty))$
- $\exists\,\sigma_1 \geq \sigma > \frac{2d}{d+2}$: $c_0(1+\vartheta)^{\sigma-1} \leq \mathrm{c}(\vartheta) \leq c_1(1+\vartheta)^{\sigma_1-1} \implies h$ is strictly increasing

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Assume moreover
[If $\rho = 0$:] the function $k : [0,+\infty) \to [0,+\infty)$ is continuous, and

$$\exists c_2,\, c_3 > 0 \ \ \forall \vartheta \in [0,+\infty) \;:\quad c_2\mathrm{c}(\vartheta) \leq k(\vartheta) \leq c_3(\mathrm{c}(\vartheta)+1)$$

[If $\rho \neq 0$:] $\exists c_\rho > 0\, \exists q > \frac{d+2}{2d} \;:\; K(w) = c_\rho\left(|w|^{2q}+1\right) \quad \forall w \in [0,+\infty)$

The non-degenerate case

The approximating non-degenerate Problem [P$_\delta$]

Given $\delta > 0$, $\mu \in \{0, 1\}$, find (measurable) functions

$$w \in L^r(0,T;W^{1,r}(\Omega)) \cap L^\infty(0,T;L^1(\Omega)) \cap \mathrm{BV}([0,T];W^{1,r'}(\Omega)^*)$$
$$\mathbf{u} \in H^1(0,T;H^2(\Omega;\mathbb{R}^d)) \cap W^{1,\infty}(0,T;H^1_0(\Omega)) \cap H^2(0,T;L^2(\Omega;\mathbb{R}^d))$$
$$\chi \in L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega))$$

for every $1 \le r < \frac{d+2}{d+1}$, fulfilling the initial conditions

$$\mathbf{u}(0,x) = \mathbf{u}_0(x), \quad \mathbf{u}_t(0,x) = \mathbf{v}_0(x) \qquad \text{for a.e. } x \in \Omega$$
$$\chi(0,x) = \chi_0(x) \qquad \text{for a.e. } x \in \Omega$$

the equations (for every $\varphi \in \mathrm{C}^0([0,T];W^{1,r'}(\Omega)) \cap W^{1,r'}(0,T;L^{r'}(\Omega))$ and $t \in (0,T]$)

$$\int_\Omega \varphi(t)\, w(t)(\mathrm{d}x) - \int_0^t \int_\Omega w\varphi_t \,\mathrm{d}x + \int_0^t \int_\Omega \chi_t \Theta(w)\varphi \,\mathrm{d}x$$
$$- \rho \int_0^t \int_\Omega \mathrm{div}\mathbf{u}_t \Theta(w)\varphi \,\mathrm{d}x + \int_0^t \int_\Omega K(w)\nabla w \nabla\varphi \,\mathrm{d}x = \int_0^t \int_\Omega g\varphi + \int_\Omega w_0\varphi(0)\,\mathrm{d}x$$

$\mathbf{u}_{tt} - \mathrm{div}\,((a(\chi) + \delta)\varepsilon(\mathbf{u}_t) + b(\chi)\varepsilon(\mathbf{u})) - \rho\nabla\Theta(w) = \mathbf{f}$ in $H^{-1}(\Omega;\mathbb{R}^d)$ a.e. in $(0,T)$

and the subdifferential inclusion (in $W^{1,p}(\Omega)^*$ and a.e. in $(0,T)$)

$$\chi_t + \mu\partial I_{(-\infty,0]}(\chi_t) - \Delta_p\chi + \beta(\chi) + \gamma(\chi) \ni -b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} + \Theta(w)$$

Theorem 1 [The reversible case $\mu = 0$]

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Let $\mu = 0$ and $\rho = 0$, assume the previous Hypotheses and the conditions:

$$\mathbf{f} \in L^2(0,T;L^2(\Omega)), \quad g \in L^1(0,T;L^1(\Omega)) \cap L^2(0,T;H^1(\Omega)')$$

$$\vartheta_0 \in L^{\sigma_1}(\Omega) \quad \text{whence} \quad w_0 := h(\vartheta_0) \in L^1(\Omega)$$

$$\mathbf{u}_0 \in H^2_0(\Omega), \quad \mathbf{v}_0 \in H^1_0(\Omega) \quad \chi_0 \in \mathrm{dom}(\Delta_p), \quad \widehat{\beta}(\chi_0) \in L^1(\Omega)$$

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Then,

1. Problem $[\mathrm{P}_\delta]$ admits a solution $(w, \mathbf{u}, \chi)$, such that there exists

$$\xi \in L^2(0,T;L^2(\Omega)),\ \xi(x,t) \in \beta(\chi(x,t)) \text{ for a.e. } (x,t) \in \Omega \times (0,T) :$$

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Let $\mu = 0$ and $\rho = 0$, assume the previous Hypotheses and the conditions:

$$\mathbf{f} \in L^2(0,T;L^2(\Omega)), \quad g \in L^1(0,T;L^1(\Omega)) \cap L^2(0,T;H^1(\Omega)')$$

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Theorem 2 [The irreversible case $\mu = 1$]

Let $\mu = 1$, $\rho = 0$, and take the previous assumptions with $\widehat{\beta} = I_{[0,+\infty)}$. Then,

[1.] Problem $[P_\delta]$ admits a *weak solution* $(w, \mathbf{u}, \chi)$, which, beside fulfilling the enthalpy and momentum equations, satisfies $\chi_t(x,t) \leq 0$ for almost all $t \in (0,T)$, and $(\forall \varphi \in L^p(0,T; W^{1,p}_-(\Omega)) \cap L^\infty(Q))$ the *one-sided* inequality

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[3.] In case $\rho \neq 0$ an analogous statement to the reversible case holds true

The isothermal case: uniqueness

Let $\rho \in \mathbb{R}$. In addition to the previous hypotheses, assume that

the function *a* is constant

Then, the isothermal reversible system admits a unique solution $(\mathbf{u}, \chi)$ which continuously depends on the data

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Uniqueness of solutions for the irreversible system, even in the isothermal case, is still an open problem. This is mainly due to the triply nonlinear character of the χ equation.

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  - the Boccardo-Gallouët-type estimates combined with the Gagliardo-Nirenberg inequality applied to the enthalpy equation in order to obtain an $L^r(0, T; W^{1,r}(\Omega))$-estimate on the enthalpy w

The degenerating case

Hypotheses

Consider the irreversible case with the $s-$Laplacian (the previous results still hold true in this case), $\rho = 0$, and $a(\chi) = \chi$, $b(\chi) = \chi + \delta$:

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$$+ \int_0^t \int_\Omega K(w) \nabla w \nabla \varphi \,\mathrm{d}x = \int_0^t \int_\Omega g\varphi + \int_\Omega w_0 \varphi(0)\,\mathrm{d}x\,,$$

$$\mathbf{u}_{tt} - \mathrm{div}\left((\chi + \delta)\varepsilon(\mathbf{u}_t) + (\chi + \delta)\varepsilon(\mathbf{u})\right) = \mathbf{f} \text{ in } H^{-1}(\Omega; \mathbb{R}^d) \text{ a.e. in } (0, T)$$

and the subdifferential inclusion (in $W^{1,p}(\Omega)^*$ and a.e. in $(0, T)$)

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where

$$A_s : H^s(\Omega) \to H^s(\Omega)^* \quad \text{with } s > \frac{d}{2}, \quad \langle A_s\chi, w\rangle_{H^s(\Omega)} := a_s(\chi, w) \text{ and}$$

$$a_s(z_1,z_2) := \int_\Omega \int_\Omega \frac{(\nabla z_1(x) - \nabla z_1(y)) \cdot (\nabla z_2(x) - \nabla z_2(y))}{|x-y|^{d+2(s-1)}}\,\mathrm{d}x\,\mathrm{d}y$$

The energy estimate

Rewrite the momentum equation

$$\partial_t^2 \mathbf{u}_\delta - \mathrm{div}((\chi + \delta)\varepsilon(\partial_t \mathbf{u}_\delta)) - \mathrm{div}((\chi + \delta)\varepsilon(\mathbf{u}_\delta)) = \mathbf{f}$$

using the new variables (*quasi-stresses*) $\boldsymbol{\mu}_\delta := \sqrt{\chi_\delta + \delta}\,\varepsilon(\partial_t \mathbf{u}_\delta)$, and $\boldsymbol{\eta}_\delta := \sqrt{\chi_\delta + \delta}\,\varepsilon(\mathbf{u}_\delta)$:

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The total energy inequality for $(w_\delta, \mathbf{u}_\delta, \chi_\delta)$ is

$$\begin{aligned}
&\int_\Omega w_\delta(t)(\mathrm{d}x) + \frac{1}{2}\int_\Omega |\partial_t \mathbf{u}_\delta(t)|^2\,\mathrm{d}x + \int_s^t\int_\Omega |\partial_t\chi_\delta|^2\,\mathrm{d}x + \frac{1}{2}\int_s^t |\boldsymbol{\mu}_\delta(r)|^2 \\
&\quad + \frac{|\boldsymbol{\eta}_\delta(t)|^2}{2} + \frac{1}{2}a_s(\chi_\delta(t),\chi_\delta(t)) + \int_\Omega W(\chi_\delta(t))\,\mathrm{d}x \\
&\le \int_\Omega w_\delta(s)(\mathrm{d}x) + \frac{1}{2}\int_\Omega |\partial_t \mathbf{u}_\delta(s)|^2\,\mathrm{d}x + \frac{|\boldsymbol{\eta}_\delta(s)|^2}{2} + \frac{1}{2}a_s(\chi_\delta(s),\chi_\delta(s)) \\
&\quad + \int_\Omega W(\chi_\delta(s))\,\mathrm{d}x + \int_s^t\int_\Omega \mathbf{f}\cdot\partial_t\mathbf{u}_\delta\,\mathrm{d}x + \int_s^t\int_\Omega g\,\mathrm{d}x
\end{aligned}$$

Passage to the limit for $\delta \searrow 0$

Theorem 3 [The degenerate case]

Under the previous assumptions, there exist

$\mathbf{u} \in W^{1,\infty}(0,T;L^2(\Omega)) \cap H^2(0,T;H^{-1}(\Omega))$, $\boldsymbol{\mu} \in L^2(0,T;L^2(\Omega))$, $\boldsymbol{\eta} \in L^\infty(0,T;L^2(\Omega))$,

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for all $\varphi \in L^2(0,T;W^{s,2}_+(\Omega)) \cap L^\infty(Q)$ with $\mathrm{supp}(\varphi) \subset \{\chi > 0\}$,

together with the *total energy inequality* (for almost all $t \in (0,T]$)

$$\int_\Omega w(t)(\mathrm{d}x) + \int_0^t \int_\Omega |\chi_t|^2\,\mathrm{d}x + \frac{1}{2}\int_0^t |\boldsymbol{\mu}(r)|^2 + \int_\Omega W(\chi(t))\,\mathrm{d}x + \mathcal{J}(t)$$

$$= \int_\Omega w_0\,\mathrm{d}x + \frac{1}{2}\int_\Omega |\mathbf{v}_0|^2\,\mathrm{d}x + \frac{1}{2} b(\chi_0)|\varepsilon(\mathbf{u}_0)|^2 + \frac{1}{2} a_s(\chi_0,\chi_0) + \int_\Omega W(\chi_0)\,\mathrm{d}x$$

$$+ \int_0^t \int_\Omega \mathbf{f} \cdot \mathbf{u}_t\,\mathrm{d}x\mathrm{d}r + \int_0^t \int_\Omega g\,\mathrm{d}x \quad \text{with}$$

$$\int_0^t \mathcal{J}(r)\,\mathrm{d}r \geq \frac{1}{2}\int_0^t \left(\int_\Omega |\mathbf{u}_t(r)|^2\,\mathrm{d}x + |\boldsymbol{\eta}(r)|^2 + a_s(\chi(r),\chi(r)) \right)$$

A comparison between the solution notions

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Subtracting from the *degenerate energy inequality* the weak enthalpy equation tested by 1, we recover (a.e. in $(0, T]$):

$$\int_0^t \int_\Omega |\chi_t|^2 \,\mathrm{d}x\,\mathrm{d}r + \|\chi(t)\|^2_{H^s(\Omega)} + \int_\Omega W(\chi(t))\,\mathrm{d}x$$

$$\le \|\chi_0\|^2_{H^s(\Omega)} + \int_\Omega W(\chi_0)\,\mathrm{d}x + \int_0^t \int_\Omega \chi_t \left(-\frac{|\varepsilon(\mathbf{u})|^2}{2} + \Theta(w)\right)\mathrm{d}x\,\mathrm{d}r$$

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4. the fact that we neglige the thermal expansion, i.e. we take $\rho = 0$, is due to the low regularity estimates we have on $\operatorname{div}\mathbf{u}_t$ for $\delta = 0$, which does not allow to pass to the limit in $\rho\operatorname{div}(\mathbf{u}_t)\Theta(w)$ when $\delta \searrow 0$

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These are the reasons why we have restricted the analysis of **the degenerate limit** to the **irreversible system**, with the **nonlocal *s*-Laplacian operator**.