Universal Attractor for some Singular Phase Transition Systems

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joint work with Giulio Schimperna *(Pavia)*

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Plan of the talk

- ♢ Give some examples of models describing phase-transitions processes
- ♢ Focus the attention on the Penrose-Fife model ("singular model"): discute its derivation and its "thermodynamical consistency"
- ♢ Study well-posedness of one particular system of PDE's coming from a model of Penrose-Fife type
- ♢ Show existence of the universal attractor for such system and list some open problems

Phase transitions

Definition. By the term phase transitions we mean the transformations of one substance from one phase to another.

Examples: ferromagnetic, liquid-vapor, phase separation (cf. binary alloys).

Variables

- θ: the relative temperature
- χ: the order parameter (it may be the density or concentration of some substances, such as one of the components in an alloy).

Example 1. The Stefan problem

It describes solid-liquid phase transitions (cf. J. Stefan in 1889)

- χ is the proportion of (e.g.) the liquid phase
- $\ell > 0$ is the latent heat
- $e = \ell\chi + \theta$ is the internal energy
- $\mathbf{q}$ is the heat flux. Ex.: the Fourier law $\mathbf{q} = -\kappa\nabla\theta$
- $\theta = 0$ the fusion-solidification temperature.

It consists in finding the couple (θ, χ) such that

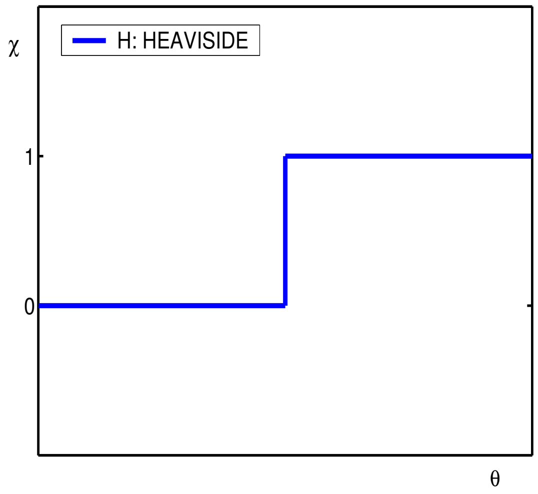
$$\begin{aligned} &\partial_t(\ell\chi + \theta) + \operatorname{div}\mathbf{q} = \mathit{known} \\ &\chi \in H(\theta). \end{aligned} \tag{Ste}$$

WEAK FORMULATION OF THE STEFAN PROBLEM

Constitutive relation between θ and χ:

where $\theta < 0$ it is $\chi = 0$	solid
where $\theta > 0$ it is $\chi = 1$	liquid
where $\theta = 0$ it is $0 \leq \chi \leq 1$	mushy region.

Hence, the admissible couples are the ones belonging to the graph



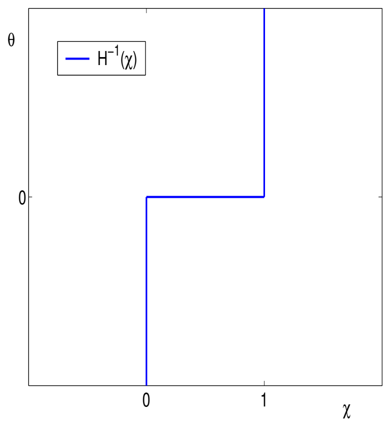
$\chi \in H(\theta)$:

EXAMPLE 2. PHASE RELAXATION

It describes supercooling and superheating (A. VISINTIN in 1985)

$$\partial_t(\theta + \ell\chi) - \Delta\theta = known$$

$$\theta \in H^{-1}(\chi) + \mu\partial_t\chi.$$



Note. For $\mu = 0$ we recover the STEFAN model.

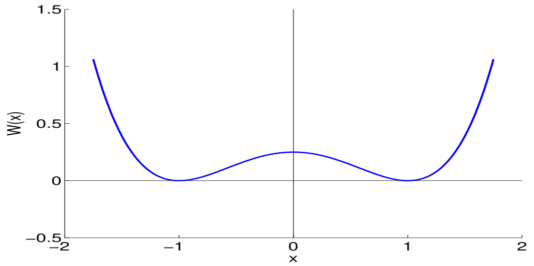
EXAMPLE 3. "CLASSICAL" PHASE FIELD

It includes interfacial energy effects (G. CAGINALP in 1986, 1989)

$$\partial_t(\theta + \ell\chi) - \Delta\theta = known \tag{Cag1}$$

$$\theta = \mathcal{W}'(\chi) - \varepsilon\Delta\chi + \mu\partial_t\chi \tag{Cag2}$$

$\mathcal{W}$ is the classical DOUBLE WELL potential



Note. It is "thermodynamically consistent" only for θ "near" to the crtical temperature. For ε, $\mu = 0$ and $\mathcal{W}' = H^{-1}$, we recover again the STEFAN model.

EXAMPLE 4. THE PENROSE-FIFE MODEL
("SINGULAR MODEL")

It is a "thermodynamically consistent" model (O. PENROSE and P.C. FIFE in 1990), not only around the critical temperature ϑ_c, but for "EVERY VALUE OF THE ABSOLUTE TEMPERATURE": $\vartheta > 0$.

$$\partial_t(\vartheta + \lambda(\chi)) + \operatorname{div} \mathbf{q} = known \qquad \text{(PF1)}$$

$$\frac{\lambda'(\chi)}{\vartheta_c} - \frac{\lambda'(\chi)}{\vartheta} = \beta(\chi) + \sigma'(\chi) - \varepsilon \Delta \chi + \mu \partial_t \chi. \qquad \text{(PF2)}$$

- ◇ $\lambda'(\chi)$ is the *latent heat* ($\lambda'(\chi) = \ell$ in Ex. 1, 2, 3),
- ◇ β is a maximal monotone graph, $\sigma'(\chi)$ is a smooth function of χ ($\beta = H^{-1}$ in Ex. 1, 2 and $\beta + \sigma' = \mathcal{W}'$ in Ex. 3).

Relations with the previous models

- <u>With the Stefan model.</u> If $\lambda' = \ell$, $\sigma'(\chi) = \ell\chi/\vartheta_c$, $\varepsilon = \mu = 0$, and $\beta = H^{-1}$, we recover the Stefan model (cf. [P. Colli, J. Sprekels, '96]), indeed we have:
$$\left[\ell\Big(\frac{1}{\vartheta_c} - \frac{1}{\vartheta}\Big) \in H^{-1}(\chi)\right] \quad \times \quad (\vartheta_c\vartheta)\ell^{-1} \implies (\vartheta - \vartheta_c) \in H^{-1}(\chi).$$
Taking now $\theta = \vartheta - \vartheta_c$, we obtain the Stefan condition (Ste).

- <u>With the Caginalp model.</u> Assuming $\vartheta \sim \vartheta_c$, then it makes sense to introduce the *relative temperature* $\theta = (\vartheta - \vartheta_c)$ and linearize the kinetic equations (PF1–PF2) with respect to θ, obtaining the so-called Caginalp system (Cag1–Cag2).

Focus the attention on the Penrose-Fife model

We are now interested in

- ◇ The derivation of the Penrose-Fife models.
- ◇ Find the systems of PDE's from well-known thermodynamical relations.
- ◇ Show that these models are "Thermodynamically consistent".

Since we are interested in the evolutionary aspects, we set:

$\vartheta : Q \to \mathbb{R}^+$, $\chi : Q \to \mathbb{R}$, where $Q := \Omega \times (0,T)$ and $\Omega \subset \mathbb{R}^N$ $(N \leq 3)$ is bounded connected with smooth $\partial\Omega =: \Gamma$, $\Sigma := \Gamma \times (0,T)$.

CONSERVED/NON-CONSERVED DYNAMICS

Regarding the evolution of χ, we have first to distinguish between:

- CONSERVED CASE: χ is the density of concentration of one substance: as one component in an alloy $\Longrightarrow$ phase separation.
  - Then, it follows from the "Principle of conservation of matter" that the dynamical process cannot change the total amount of this substance in the system (provided that there is no mass flux over the boundary).
- NON-CONSERVED CASE: χ is the local proportion of one of the two phases, e.g.: the fraction of liquid $\Longrightarrow$ solid-liquid phase transitions.

THE PENROSE-FIFE SYSTEMS

Generalizations ([O. PENROSE, P.C. FIFE, '90]) to the case of a NON-CONSTANT ϑ of the ALLEN-CAHN or GINZBURG-LANDAU ('77) and the CAHN-HILLIARD ('58) equations, respectively:

$$\partial_t e + \operatorname{div}\left[\kappa(\vartheta)\nabla\left(\frac{\delta\mathcal{S}}{\delta e}\right)\right] = 0 \qquad \text{(energy)}$$

$$\partial_t \chi = M(\vartheta,\chi)\frac{\delta\mathcal{S}}{\delta\chi} \qquad \text{(non-conserved)}$$

or

$$\partial_t \chi + \operatorname{div}\left[M(\vartheta,\chi)\nabla\left(\frac{\delta\mathcal{S}}{\delta\chi}\right)\right] = 0. \qquad \text{(conserved)}$$

$\mathcal{S} := \int_\Omega S\,dx$ is the *entropy functional*,
e is the *internal energy* and we have here 0 heat source in (energy),
κ is the *conductivity*, and M the *mobility* of the system.

Boundary conditions: this is why "conserved"

If there is no mass-flux $\left[\mathbf{j} := \nabla\left(\delta\mathcal{S}/\delta\chi\right)\right]$ through the boundary:

$$[\,\partial_t\chi = -\operatorname{div}\mathbf{j} \quad \text{in } Q] + [\,\mathbf{j}\cdot\nu = 0 \quad \text{on } \Sigma\,]$$

$$\Downarrow$$

$$\frac{d}{dt}\int_\Omega \chi(x,t)\,dx = \int_\Omega \partial_t\chi(x,t)\,dx = 0 \quad \text{for every } t \in [0,T]$$

$$\Downarrow$$

$$\int_\Omega \chi(x,t)\,dx = \int_\Omega \chi(x,0)\,dx \quad \text{for every } t \in [0,T]...$$

the spatial mean value of χ is preserving during the evolution.

Thermodynamical consistency

- The value of the *entropy* functional $\mathcal{S}$ cannot decrease along solution paths (in agreement with the "Second law of Thermodynamics").

- Let us do the computations in the non-conserved case, with Neumann-homogeneous boundary conditions on $\delta\mathcal{S}/\delta e$, and with constant κ and M:

$$\frac{d\mathcal{S}}{dt} = \int_\Omega \frac{\delta\mathcal{S}}{\delta e}\frac{\partial e}{\partial t} + \frac{\delta\mathcal{S}}{\delta\chi}\frac{\partial\chi}{\partial t} = \int_\Omega \left\{\kappa\left|\nabla\left(\frac{\delta\mathcal{S}}{\delta e}\right)\right|^2 + M\left(\frac{\delta\mathcal{S}}{\delta\chi}\right)^2\right\} \geq 0.$$

Indeed from (energy) and (non-conserved), we have that

$$\frac{\partial e}{\partial t} = -\kappa\,\mathrm{div}\,\nabla\left(\frac{\delta S}{\delta e}\right) \quad \text{and} \quad \frac{\partial\chi}{\partial t} = M\frac{\delta S}{\delta\chi}.$$

THE FREE-ENERGY FUNCTIONAL

The thermodynamical relations linking $\mathcal{S}$ with the *free-energy functional* $\mathcal{F} := \int_\Omega F\,dx$, and with e are:

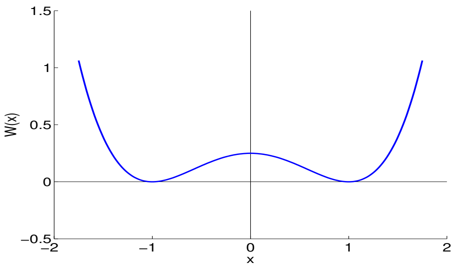
$$e = F + \vartheta S, \quad \text{with} \quad S = -\frac{\partial F}{\partial \vartheta}.$$

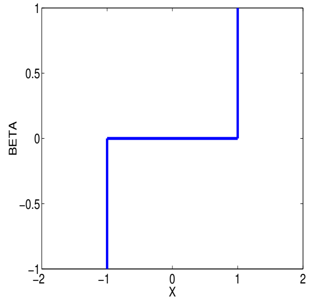
Take F as the GINZBURG-LANDAU free-energy:

$$F(\vartheta,\chi) = -\vartheta \ln(\vartheta) + \frac{\vartheta}{2}|\nabla\chi|^2 + \vartheta\Big[\widehat{\beta}(\chi) + \sigma(\chi) + \lambda(\chi)\Big(\frac{1}{\vartheta} - \frac{1}{\vartheta_c}\Big)\Big]$$

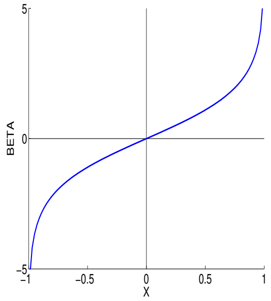
- ◇ $\lambda(\chi)$ is a first or second order degree polinomial function of χ,
- ◇ $\sigma(\chi)$ is a smooth function of χ and $\widehat{\beta}(\chi)$ is a proper, l.s.c., and convex function, e.g., $\sigma(\chi) + \widehat{\beta}(\chi)$ can be:
  - – the classical "double-well": $\mathcal{W} = (\chi^2 - 1)^2$ or
  - – a constraint: $\sigma(\chi) + I_{[-1,1]}(\chi)$ with a smooth σ.

- Double-well case: $\mathcal{W} = (\widehat{\beta} + \sigma)(\mathcal{X}) = (\mathcal{X}^2 - 1)^2$:





- Subdifferential case: $\beta := \partial\widehat{\beta} = \partial I_{[-1,1]} \sim H^{-1}$:



- Logarithmic case: $\beta := \partial\widehat{\beta} = \log(1 + \mathcal{X}) - \log(1 - \mathcal{X})$:

Entropy functional and final systems

$$\begin{cases} S(\vartheta,\chi) = -\dfrac{\partial F(\vartheta,\chi)}{\partial\vartheta} = 1 + \ln\vartheta - \dfrac{1}{2}|\nabla\chi|^2 - \widehat{\beta}(\chi) - \sigma(\chi) + \dfrac{\lambda(\chi)}{\vartheta_c} \\ e = F + \vartheta S = \vartheta + \lambda(\chi). \end{cases}$$

Hence, the energy balance equation (with 0 heat source) becomes:

$$\partial_t(\vartheta + \lambda(\chi)) + \operatorname{div}\left[\kappa(\vartheta)\nabla\left(\frac{1}{\vartheta}\right)\right] = 0. \tag{E}$$

And the two phase equations (with mobility $M = 1$) become:

$$\partial_t\chi - \Delta\chi + \beta(\chi) + \sigma'(\chi) + \lambda'(\chi)\left(\frac{1}{\vartheta} - \frac{1}{\vartheta_c}\right) \ni 0 \tag{NC}$$

$$\partial_t\chi - \Delta\left[-\Delta\chi + \beta(\chi) + \sigma'(\chi) + \lambda'(\chi)\left(\frac{1}{\vartheta} - \frac{1}{\vartheta_c}\right)\right] \ni 0 \tag{C}$$

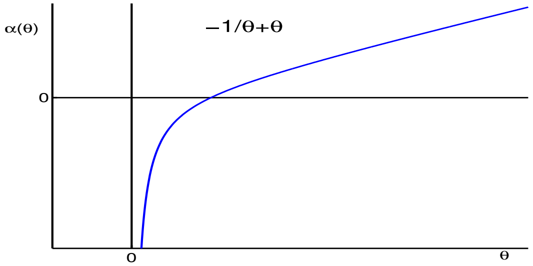
Some problems concerning
Penrose-Fife models

- Well-posedness (cf. [E.R., PhD Thesis, '03] for some references): existence and uniqueness for <u>variational formulations</u> of Cauchy problems associated to:
  - different choices of heat flux laws [different choices of $\mathbf{q}$],
  - different boundary conditions for ϑ [Robin (Third type), Neumann, or Dirichlet].
- Long time behaviour of solutions, in particular, the study of existence of the <u>universal attractor</u> (cf. [E.R., G. Schimperna, submitted, '03]).

CHOICE OF THE HEAT FLUX LAW

◇ <u>For $\kappa(\vartheta) = 1$.</u> Then $\mathbf{q} = -\nabla(-1/\vartheta)$. It furnish a GOOD COUPLING with the Penrose-Fife phase equation.

◇ <u>For $\kappa(\vartheta) \sim 1 + \vartheta^2$.</u> Then $\mathbf{q} = -\nabla\alpha(\vartheta)$ $(\alpha \sim -1/\vartheta + \vartheta)$. It furnish MORE COERCIVENESS:



◇ <u>Law $\mathbf{q} = -\nabla[-1/\vartheta + k * \vartheta]$</u>, where the convolution is: $[k * \vartheta](t) = \int_0^t k(t-s)\vartheta(s)ds$.

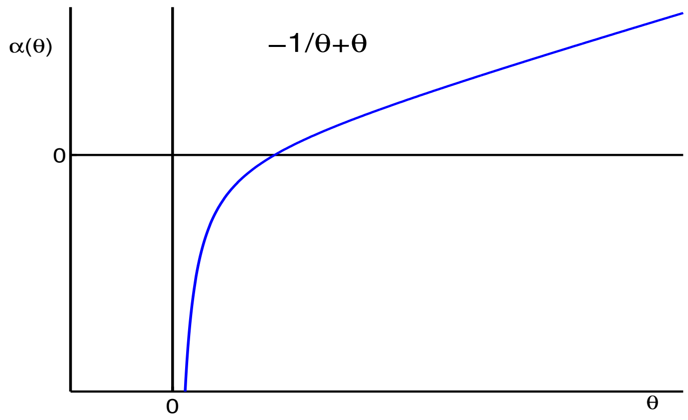
◇ <u>For $\kappa(\vartheta) = \vartheta^2$.</u> Then $\mathbf{q} = -\nabla\vartheta$ (Fourier law). It gives a DIFFICULT COUPLING with the Penrose-Fife phase equation.

Our Choice of Heat Flux

The Energy balance equation

$$\partial_t\big(\vartheta + \lambda(\chi)\big) - \Delta\alpha(\vartheta) = g \quad \text{in } Q := \Omega \times (0,T)$$

- g heat source
- λ' smooth latent heat (here $\lambda'(\chi) \equiv 1$)
- $\alpha(\vartheta) \sim -1/\vartheta + \vartheta$:

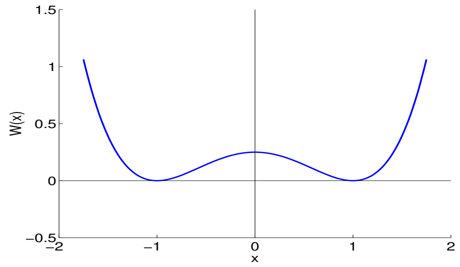


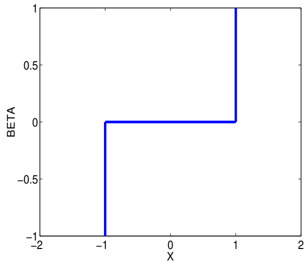
Kinetic equation for the phase field
The non-conserved case

$$\partial_t \chi - \Delta \chi + \beta(\chi) + \sigma'(\chi) \ni -\lambda'(\chi)/\vartheta \quad \text{in } Q$$

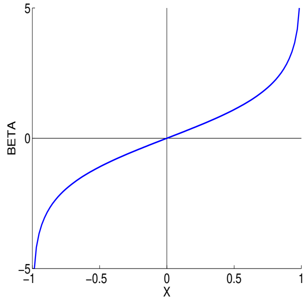
- σ' is a smooth function
- $\beta = \partial \widehat{\beta}$ is a maximal monotone graph.

- The double-well case: $W = \widehat{\beta} + \sigma = (\chi^2 - 1)^2$:





- The subdifferential case: $\beta = \partial I_{[-1,1]}$:



- The logarithmic case: $\beta = \log(1 + \chi) - \log(1 - \chi)$:

Our Initial and boundary conditions
Third type (Robin) conditions on $\alpha(\vartheta)$

$$-\partial_{\mathbf{n}}\alpha(\vartheta) = n_0\big(\alpha(\vartheta) - h\big), \quad \partial_{\mathbf{n}}\chi = 0,$$

n_0 : positive heat transmission coefficient,

h : function of ϑ_Γ,

$\vartheta(0) = \vartheta_0, \quad \chi(0) = \chi_0$ initial conditions.

Goals of the talk

- Give the existence result (known)
- Prove the uniqueness and continuous dependence results
- Choose a suitable phase space and prove continuity of the associated semigroup
- Show the existence of the universal attractor
- Tell something about the *conserved* case and of λ quadratic
- Make a comparison with other works
- Cite some related open problems

Variational formulation

$$(\vartheta + \chi)_t + J\big(\alpha(\vartheta)\big) = f \quad \text{in } V', \quad \text{a.e. in } (0,T) \qquad (\mathrm{E})$$

$$\chi_t + B\chi + \beta(\chi) + \sigma'(\chi) \ni -1/\vartheta \quad \text{a.e. in } Q \qquad (\mathrm{N})$$

$+$ I.C.

with $V := H^1(\Omega) \hookrightarrow H := L^2(\Omega) \hookrightarrow V'$

$$\langle Ju, v\rangle = \int_\Omega \nabla u \nabla v + \int_{\partial\Omega} uv \quad \forall u, v \in V$$

$$\langle Bu, v\rangle = \int_\Omega \nabla u \nabla v \quad \forall u, v \in V$$

$$\langle f, v\rangle = \int_\Omega gv + \int_{\partial\Omega} hv \quad \forall v \in V.$$

THE EXISTENCE RESULT

[P. Colli and Ph. Laurençot, Physica D, '98]

If $\vartheta_0 > 0$, $\widehat{\alpha}(\vartheta_0) \in L^1(\Omega)$, $\chi_0 \in V$, $\widehat{\beta}(\chi_0) \in L^1(\Omega)$

Then we have the energy estimate:

$$(E) \times \alpha(\vartheta) + (N) \times \chi_t$$

which (for any $T > 0$) leads to existence of solutions.

Note:

- $\widehat{\alpha}(\vartheta_0) = \int_0^{\vartheta_0} \alpha(s)ds \geq \vartheta_0^2 + \log^- \vartheta_0 - 1/2$, hence
- $\widehat{\alpha}(\vartheta_0) \in L^1(\Omega) \Leftrightarrow \vartheta_0 \in H, \log^- \vartheta_0 \in L^1(\Omega)$.

CONTINUOUS DEPENDENCE

Let (ϑ_i, χ_i) $(i = 1, 2)$ be two solutions of $(E)_i - (N)_i$ corresponding to the data $(\vartheta_{0i}, \chi_{0i})$. Then

$$\left[(E)_1 - (E)_2\right] \times J^{-1}\left[(\vartheta + \chi)_1 - (\vartheta + \chi)_2\right]$$
$$\left[(N)_1 - (N)_2\right] \times (\chi_1 - \chi_2)$$

$$\Downarrow$$

$$\|(\vartheta + \chi)_1 - (\vartheta + \chi)_2(t)\|_{V'} + \|(\chi_1 - \chi_2)(t)\|_H$$
$$\leq const\left[\|(\vartheta_{01} + \chi_{01}) - (\vartheta_{02} + \chi_{02})\|_{V'} + \|\chi_{01} - \chi_{02}\|_H\right]$$

(Lipschitz continuity)

CHOICE OF THE PHASE SPACE

Uniqueness/continuous dependence :

$$\chi_0 \in H,\ \vartheta_0 > 0,\ (\vartheta_0 + \chi_0) \in V' \qquad \text{(weaker)} \qquad \text{(Uni)}$$

Existence : $\chi_0 \in V,\ \widehat{\beta}(\chi_0) \in L^1(\Omega),$

$$\vartheta_0 > 0,\ \vartheta_0 \in H,\ \log^- \vartheta_0 \in L^1(\Omega) \qquad \text{(stronger)} \quad \text{(Exi)}$$

<u>POSSIBLE CHOICES</u>

- Prove existence under (Uni): IMPOSSIBLE!
- Take a large Phase Space (Uni) with (Exi) as additional constraint $\Longrightarrow$ *weak attractor* (N. Kenmochi et al.)
- <u>Our choice</u>: take a small Phase Space (Exi) refining the continuous dep. result $\Longrightarrow$ strong attractor.

The Phase Space and the Semigroup

$\mathcal{X} := \{(\vartheta, \chi) \ : \ \text{(Exi) holds}\}$ with the distance:

$$d_{\mathcal{X}}\big((\vartheta_1, \chi_1), (\vartheta_2, \chi_2)\big) := \|\vartheta_1 - \vartheta_2\|_H + \int_\Omega \big|\log^- \vartheta_1 - \log^- \vartheta_2\big|$$

$$+ \|\chi_1 - \chi_2\|_V + \int_\Omega \big|\widehat{\beta}(\chi_1) - \widehat{\beta}(\chi_2)\big|$$

is a complete metric space and

$S(t) : \mathcal{X} \to \mathcal{X}$ s.t.

$(\vartheta_0, \chi_0) \mapsto (\vartheta(t), \chi(t))$ (the solution of the problem)

is strongly continuous for all $t \in [0, T]$

(no longer Lipschitz continuous).

PRELIMINARIES

- Def. $\mathcal{B}_0$ is an *absorbing set* for a semigroup $S(t)$ on a metric space (X, d_X) iff
  - $\mathcal{B}_0$ is bdd
  - $\forall B \subset X$ bdd $\exists\, T_B \geq 0$ s.t. $S(t)B \subset \mathcal{B}_0 \quad \forall t \geq T_B$.
- Theorem. Let $S(t)$ be a strongly continuous semigroup on a c.m.s. (X, d_X). Moreover, if
  - $S(t)$ admits an absorbing set $\mathcal{B}_0$;
  - $\forall B \subset X$ bdd $\exists\, t_B > 0$ s.t. $\bigcup_{t \geq t_B} S(t)B$ is compact in X,

  then $S(t)$ admits a *universal attractor* $\mathcal{A}$ that is

$$\mathcal{A} = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} S(t)\mathcal{B}_0}.$$

DISSIPATIVITY

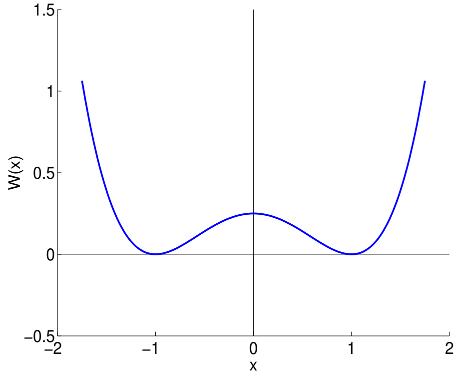
[E.R., G. Schimperna, submitted, (2003)]

If we assume that there esist k_1, $k_2 > 0$ such that

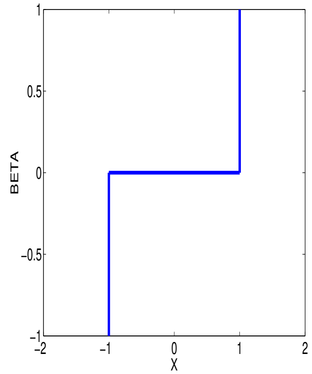
$$s \geq k_1 r^3 - k_2 \quad \forall r \in D(\beta), \quad \forall s \in \beta(r), \tag{Beta}$$

then the semigroup $S(t)$ possesses an absorbing set $\mathcal{B}_0$ bdd w.r.t. $d_{\mathcal{X}}$.

Note that (Beta) holds in both the two cases:



and



How to prove dissipativity?

<u>If</u> there exists a nonnegative functional $\Phi(\vartheta, \chi) \geq d_{\mathcal{X}}\big((\vartheta, \chi), (1, 0)\big)$ such that for some c_1, $c_2 > 0$ there holds

$$\Phi(\vartheta, \chi) \leq \Phi(\vartheta_0, \chi_0) e^{-c_1 t} + c_2 \quad \forall t \geq 0 \tag{Phi}$$

$$\Downarrow$$

the system is dissipative...

THE SOLUTION IS...

Take the function Φ as

$$\Phi(\vartheta, \chi) = -\int_\Omega \log\vartheta + \|\vartheta\|_H^2 + \|\chi\|_V^2 + \int_\Omega \widehat{\beta}(\chi) + c.$$

Then, it is trivial to prove that it is nonnegative and that

$$\Phi(\vartheta, \chi) \geq d_{\mathcal{X}}\big((\vartheta, \chi), (1, 0)\big).$$

Moreover (Phi) comes from the following tests:

$$[(E) \times (\vartheta - 1/\vartheta) + (N) \times (\chi_t + \chi)].$$

Hence the system turns out to be dissipative.

EXISTENCE OF THE ATTRACTOR

[E.R., G. Schimperna, submitted, (2003)]

- If (Beta) holds, then the absorbing set $\mathcal{B}_0$ turns out to be bdd in the space

$$\mathcal{V} := \{(\vartheta, \chi) \in \mathcal{X} \, : \, \vartheta \in V, \, 1/\vartheta \in V, \chi \in H^2(\Omega)\}.$$

- Moreover, since $\mathcal{V}$ is compact in $\mathcal{X}$, then $S(t)$ posseses a unique compact universal attractor $\mathcal{A}$.
- Note that our choice of $\mathcal{X}$ gives information also on the asymptotic behaviour of $1/\vartheta$, not only of ϑ.

About the conserved case

[E.R., G. Schimperna, submitted, (2003)]

The *conserved* system is the one coupling the *energy balance* equation (E) with the phase-equations:

$$\partial_t \chi - \Delta w = 0 \tag{C1}$$

$$w \in -\Delta \chi + \beta(\chi) + \sigma'(\chi) + 1/\vartheta \tag{C2}$$

indeed "(C1)+$\partial_{\mathbf{n}} w = 0$"$\Longrightarrow \chi_\Omega(t) = (\chi_0)_\Omega \ \forall t \in [0,T]$.

- In this case the Phase Space is

$$\mathcal{X}_{\eta_1,\eta_2} := \{(\vartheta,\chi) \in \mathcal{X} \,:\, \eta_1 \le \chi_\Omega \le \eta_2\}$$

  with $\eta_1, \eta_2 \in int(D(\beta))$.

- We find again the existence of the Universal Attractor; the bound for the attractor will depend on η_1 and η_2.

ABOUT THE CASE OF A QUADRATIC LATENT HEAT λ

[E.R. and G. Schimperna, submitted (2003)]

The problem here consists in finding a dissipative estimate.
If there exists a nonnegative functional $\Phi(\vartheta, \chi) \geq d_{\mathcal{X}}\big((\vartheta, \chi), (1, 0)\big)$ such that for some c_1, $c_2 > 0$ there holds

$$\Phi(\vartheta, \chi) \leq \Phi(\vartheta_0, \chi_0) e^{-c_1 t} + c_2 \quad \forall t \geq 0 \tag{Phi}$$

$$\Downarrow$$

the system is dissipative...

... In case of

- λ linear: $[(E) \times (\vartheta - 1/\vartheta) + (N) \times (\chi_t + \chi)] \Longrightarrow$ (Phi);
- λ quadratic: $[(E) \times \vartheta] \Longrightarrow |\int_\Omega \lambda'(\chi)\chi_t\vartheta| \leq ???$
  - We need a previous bound for χ_t and χ in order to estimate it. This can be obtained with the trick:

$$\begin{aligned} \underline{\text{First do:}} \quad & [(E) \times (\vartheta^{1/4} - 1/\vartheta) + (N) \times (\chi_t + \chi)] \\ \underline{\text{and then:}} \quad & [(E) \times \vartheta] \end{aligned}$$

...

A COMPARISON WITH...

[W. Shen and S. Zheng, Nonlinear Anal., '02] (conserved case), [W. Shen and S. Zheng, Appl. Math. Lett., '02] (non-conserved case):

- 1-D space setting, $\alpha(\vartheta) \sim -1/\vartheta$, quadratic λ, double-well case
- With a *strong constraint* imposed a priori on the initial data
- They proved existence of an attractor depending on the *constraint*

...AND WITH...

[A. Ito, N. Kenmochi, Adv. Math. Sci. App., '00]:

- 1-D space setting, $\alpha(\vartheta) \sim -1/\vartheta$, with $(E) + \varepsilon\vartheta$, logarithmic β
- In a bigger phase space (cf. (Uni)) in which they prove Lip. continuity for $S(t) \mapsto$ *inertial set*
- Their absorbing set is able to absorb only set bdd w.r.t. a stronger metric (similar to our $d_{\mathcal{X}}$) $\Longrightarrow$ *weak attractor*
- They do not get information about $1/\vartheta$

Related problems

- The regularity and the dimension of the attractor
- Existence of the attractor for (conserved and non-conserved) Penrose-Fife model with $\alpha(\vartheta) \sim -1/\vartheta$ in a 3-D space setting.
  Problem: lack of parabolicity in ϑ
- The case of a quadratic λ in the conserved case (even well-posedness is difficult)
- The case of a parabolic-hyperbolic system (with an hyperbolic equation for the phase variable) [E.R. and G. Schimperna, in preparation, '03]