On a doubly nonlinear Volterra equation

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In the last years, a number of mathematical models for phase transition which are consistent with the basic principles of thermodinamics have been introduced. In particular, some of them are based on an entropy balance and an equation like

$$\partial_t \beta(u) + \operatorname{div} \mathbf{Q} = f \tag{1}$$

plays an important role. In (1), u is the absolute temperature, $\mathbf{Q} \in \mathbb{R}^n$ is the entropy flux density, and f is an entropy source. Moreover, β is a real monotone function defined in some interval, and the choice $\beta = \ln$, the logarithm, looks particularly appropriate. On the other hand, it is well known that memory effects should be taken into account. Hence, it is interesting to couple equation (1) with a constitutive law of the form

$$\mathbf{Q} = -\big(\alpha(\nabla u) + k * \alpha(\nabla u)\big) \tag{2}$$

where $\alpha : \mathbb{R}^n \to \mathbb{R}^n$ and $k : (0,T) \to \mathbb{R}$ are given, and to study the corresponding initial-boundary value problems in $\Omega \times (0,T)$, where Ω is a bounded domain in $\mathbb{R}^n$.

On the other hand, any reasonable choice of α leads to a monotone graph in $\mathbb{R}^n \times \mathbb{R}^n$ containing the origin. Therefore, the following is fulfilled

$$\int_\Omega \beta(v)(-\operatorname{div} \alpha(\nabla v)) = \int_\Omega \beta'(v) \nabla v \cdot \alpha(\nabla v) - \int_{\partial\Omega} \beta(v)\, \alpha(\nabla v) \cdot \mathbf{n} \geq 0 \tag{3}$$

for every smooth v satisfying $\beta(v)\alpha(\nabla v) \cdot \mathbf{n} \leq 0$ on $\partial\Omega$, where $\mathbf{n}$ is the unit normal vector on the boundary. In particular, (3) holds if v satisfies suitable boundary conditions.

In a joint work with Ulisse Stefanelli (Pavia), an existence result for a more general problem has been obtained via implicit time discretization, and the present talk deals with the outline of such a result. Namely, we consider the abstract Cauchy problem

$$(B(u))' + A(u) + k * A(u) \ni f \quad \text{in } (0,T) \qquad \text{and} \qquad B(u)|_{t=0} \ni v^0 \tag{4}$$

where f and v^0 are (quite general) prescribed data. More precisely, we are given a reflexive Banach space V and a Hilbert space H such that $V \subset H \subset V^*$ with compact embeddings (after identifying H with H^*, its dual space). Moreover the operators $A : V \to V^*$ and $B : H \to H$ are maximal monotone, possibly multivalued, and B is allowed to be both degenerate and singular. Indeed, our assumptions on the structure are (essentially) just the following: $k \in BV(0,T)$; A is everywhere defined and (suitably) bounded and coercive on V; $B = \partial\psi$, where $\psi : H \to (-\infty, +\infty]$ is proper, convex, and lower semicontinuous, with $V \cap D(\psi) \neq \emptyset$; A and B satisfy a compatibility condition in the direction of (3).