A generation problem for a differential parabolic equation degenerating at ∞

A. Favaron (Milan)

Consider the inverse problem consisting in recovering the time and space dependent memory kernel k in the following integrodifferential parabolic equation

$$m(x)D_t u(t,x) = \Delta_x u(t,x) - cu(t,x) + [k(\cdot,\rho(x)) * \Delta_x u(\cdot,x)](t) \\ +[D_\eta k(\cdot,\rho(x)) * \nabla_x u(\cdot,\rho(x))](t) + f(t,x)\,, \quad (t,x) \in [0,T] \times \Omega\,, \qquad (1)$$

where Ω is an unbounded open subset of $\mathbf{R}^n$ having (possibly) *many* branches at ∞, ρ is a given function from Ω to $\mathbf{R}$, c is a positive constant and m is a nonnegative function possibly vanishing at some points $x \in \Omega$. Dealing with such a problem we are faced with the problem of determining which interpolation space between $W^{2,p}(\Omega)$ and $L^p(\Omega)$ the gradient $\nabla_x u$ belongs to. Indeed, the relation $\nabla_x u \in W^{1,p}(\Omega)$ holding in the non-degenerate case, does not in the degenerate one, since the semigroup $\{e^{t\Delta_x}\}_{t\geq 0}$ associated with equation (1) does not verify the usual estimates $\|\Delta_x^j e^{t\Delta_x}\|_{\mathcal{L}(L^p(\Omega);L^p(\Omega))} \leq C_j t^{-j}$, $t > 0$, $j \in \mathbf{N}\cup\{0\}$, but it does the estimates $\|\Delta_x^j e^{t\Delta_x}\|_{\mathcal{L}(L^p(\Omega);L^p(\Omega))} \leq C_j t^{(1/p)-j-1}$. Such a trouble led us to analyze the spectral equation $[\Delta_x - \lambda m(x)]u(x) = g(x)$, $g \in L^p(\Omega)$, $\lambda \in \mathbf{C}$, in order to find suitable weighted estimates involving u, ∇u and (possibly) the second space derivatives of u. To derive such estimates we are forced to require $|\nabla m(x)| \leq Cm(x)$, $x \in \Omega$, which implies that the zeros of m may occur only when $|x|$ goes to $+\infty$. Since the classical a priori estimates of Agmon, Douglis and Nirenberg work only for unbounded domains like $\mathbf{R}^n$ and $\mathbf{R}^n_+$, to solve our problem we have to use some more recent a priori estimates due to Cavaliere, Transirico and Troisi (Le Matematiche **51** (1996), 87–104) involving the function space VMO and Morrey spaces under the additional assumption that the unbounded domain Ω should verify an internal cone property, too.