Global attractor for reaction-diffusion equations

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$$
\begin{cases}
u_t - \Delta u + f(u) = g & \text{in } \Omega \times (0, +\infty) \\
u = 0 & \text{on } \Gamma \times (0, +\infty) \\
u(0) = u_0 & \text{in } \Omega
\end{cases}
$$

- $\Omega \subset \mathbb{R}^n$ bdd with smooth bdry Γ (e.g. of class $\mathcal{C}^{1,1}$)
- $f \in C^1(\mathbb{R})$
- $\exists c_1, c_2, c_3 \geq 0$ s.t.

$$c_1 |y|^6 - c_2 \leq f(y)y \leq c_3(|y|^6 + 1) \quad \forall y \in \mathbb{R}$$

- $f'(y) \geq \gamma \quad \forall y \in \mathbb{R}$, for some $\gamma \in \mathbb{R}$
- $A = -\Delta : D(A) = H^2(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega)$

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Theorem

Let $T > 0$ *and* $g \in H^{-1}(\Omega)$.
For any $u_0 \in L^2(\Omega)$, $\exists!\, u \in C([0,T]; L^2(\Omega))$ *(weak solution) s.t.*

$$\begin{aligned}
&u \in L^2(0,T; H_0^1(\Omega)) \cap L^6(\Omega \times (0,T)) \\
&u_t \in L^2(0,T; H^{-1}(\Omega)) + L^{6/5}(\Omega \times (0,T)) \subset L^{6/5}(0,T; H^{-1}(\Omega)) \\
&u_t + Au + f(u) = g \quad \text{in } H^{-1}(\Omega), \ \text{a.e. in } (0,T) \\
&u(0) = u_0
\end{aligned}$$

Moreover, the map $u_0 \mapsto u(t)$ *is Lipschitz continuous from* $L^2(\Omega)$ *to itself for any fixed* $t \geq 0$*, that is*

$$\|u_1(t) - u_2(t)\|_{L^2} \leq C(T, f, \Omega)\|u_{01} - u_{02}\|_{L^2}, \quad \forall\, t \geq 0$$

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Remark

On account of the previous theorem, we can now define the semigroup

$$S(t)u_0 = u(t) \qquad \forall t \geq 0$$

and we have that $(L^2(\Omega), S(t))$ *is a dynamical system*

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Theorem

The following dissipative estimate holds

$$(1) \quad \|u(t)\|_{L^2}^2 \leq \|u_0\|_{L^2}^2 e^{-\lambda_1 t} + \frac{2c_2}{\lambda_1} + \frac{1}{\lambda_1}\|g\|_{H^{-1}}^2 \quad \forall t \geq 0$$

where λ_1 is a constant depending on Ω.

Moreover, we also have

$$(2) \quad \int_t^{t+1} \left(\|u(\tau)\|_{H^1}^2 + \|u(\tau)\|_{L^6}^6 \right) d\tau \leq C(f)(1+\|u_0\|_{L^2}^2+\|g\|_{H^{-1}}^2)$$

for all $t \geq 0$ and some $C(f) > 0$ depending also on Ω.

Proof. Take $v = u(t) \in L^2(\Omega)$ as test function. Then we have

$$\langle u_t(t), u(t)\rangle + \langle A(u)(t), u(t)\rangle + (f(u(t)), u(t)) = \langle g, u(t)\rangle$$

from which

$$\frac{d}{dt}\|u(t)\|_{L^2}^2 + 2\|\nabla u(t)\|_{L^2}^2 + 2(f(u(t)), u(t)) = 2\langle g, u(t)\rangle$$

Recall that, if Ω is bdd, it holds $|{}_{H^{-1}}\langle F, v\rangle_{H_0^1}| \leq \|F\|_{H^{-1}}\|\nabla v\|_{L^2}$.

Thus we deduce (using also the bounds on $f(y)y$)

$$\frac{d}{dt}\|u(t)\|_{L^2}^2 + 2\|\nabla u(t)\|_{L^2}^2 + 2c_1\|u(t)\|_{L^6}^6 \leq 2c_2 + 2\|g\|_{H^{-1}}\|\nabla u(t)\|_{L^2}$$

and an application of the Young's inequality gives

$$(*) \quad \frac{d}{dt}\|u(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 + 2c_1\|u(t)\|_{L^6}^6 \leq 2c_2 + \|g\|_{H^{-1}}^2$$

Using Poincaré's inequality, we get

$$\frac{d}{dt}\|u(t)\|^2_{L^2} + c_P\|u(t)\|^2_{L^2} \leq 2c_2 + \|g\|^2_{H^{-1}} \quad \forall t \geq 0$$

and standard Gronwall's lemma yields estimate (1):

$$\|u(t)\|^2_{L^2} \leq \|u_0\|^2_{L^2} e^{-c_P t} + (2c_2 + \|g\|^2_{H^{-1}}) \int_0^t e^{-c_P(t-s)} ds$$

$$\leq \|u_0\|^2_{L^2} e^{-c_P t} + \frac{1}{c_P}(2c_2 + \|g\|^2_{H^{-1}}) \quad \forall t \geq 0$$

Then, integrating $(*)$ on $(t, t+1)$, we also recover estimate (2):

$$\int_t^{t+1} \left(\|\nabla u(\tau)\|^2_{L^2} + 2c_1\|u(\tau)\|^6_{L^6}\right) d\tau \leq \|u(t)\|^2_{L^2} + 2c_2 + \|g\|^2_{H^{-1}}$$

$$\leq \|u_0\|^2_{L^2} e^{-c_P t} + (1/c_P + 1)(2c_2 + \|g\|^2_{H^{-1}})$$

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Corollary

$(L^2(\Omega), S(t))$ has a bounded (in $L^2(\Omega)$) absorbing set.

Proof. Denote by $B(\rho) \subset L^2(\Omega)$ a generic ball with radius $\rho > 0$. If $u_0 \in B(\rho)$ then, on account of (1), we have

$$\|u(t)\|_{L^2}^2 \leq \rho e^{-\lambda_1 t} + \frac{1}{\lambda_1}(2c_2 + \|g\|_{H^{-1}}^2) \quad \forall t \geq 0$$

Choosing any $t_{B(\rho)} \geq 0$ such that

$$\rho e^{-\lambda_1 t_{B(\rho)}} \leq 1/\lambda_1(2c_2 + \|g\|_{H^{-1}}^2)$$

then we easily get

$$\|u(t)\|_{L^2}^2 \leq 2/\lambda_1(2c_2 + \|g\|_{H^{-1}}^2) := \rho_0 \quad \forall t \geq t_{B(\rho)}$$

We conclude that $B(\rho_0)$ is a bounded absorbing set.

Preliminary result: Uniform Gronwall's lemma

Lemma

Let η be an absolutely continuous nonnegative function on $[t_0, \infty)$ and $\phi, \psi \in L^1_{loc}([t_0, +\infty))$ two nonnegative functions (a.e.) such that

- $$\int_t^{t+r} \phi(s)\, ds \le a_1, \quad \int_t^{t+r} \psi(s)\, ds \le a_2, \quad \int_t^{t+r} \eta(s)\, ds \le a_3$$
  $\forall\, t \ge t_0$, *for some positive constants r and a_j*
- $$\frac{d}{dt}\eta(t) \le \phi(t)\eta(t) + \psi(t) \quad a.e.\ in\ [t_0, \infty)$$

Then

$$\eta(t+r) \le \left(\frac{a_3}{r} + a_2\right) e^{a_1} \quad \forall\, t \ge t_0$$

Theorem

$(L^2(\Omega), S(t))$ has an absorbing set B_1 that is bounded in $H_0^1(\Omega)$

Proof. We argue formally by taking $v = u_t(t)$ as test function (to be rigorous we should build a Faedo-Galerkin scheme). This gives

$$(u_t(t), u_t(t)) + \langle A(u)(t), u_t(t)\rangle + (f(u(t)), u_t(t)) = \langle g, u_t(t)\rangle$$

from which ($u = 0$ on $\Gamma \times (0, T) \Rightarrow u_t = 0$ on $\Gamma \times (0, T)$)

$$\|u_t(t)\|_{L^2}^2 + (\nabla u(t), \nabla u_t(t)) + (f(u(t)), u_t(t)) = \langle g, u_t(t)\rangle$$

Setting $F(y) = \int_0^y f(z)dz$, then we get

$$(a) \quad \frac{d}{dt}\Big(\|\nabla u(t)\|_{L^2}^2 + 2(F(u(t)), 1) - 2\langle g, u(t)\rangle\Big) + 2\|u_t(t)\|_{L^2}^2 = 0$$

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On account of assumptions on f, it the can be shown that there exist $c_i' \geq 0$ such that (see notes on reaction-diffusion eqs)

$$(b) \quad c_1'|y|^6 - c_2' \leq F(y) \leq c_3'(|y|^6 + 1) \quad \forall y \in \mathbb{R}.$$

Let us set

$$(*)\, E(u(t)) = \|\nabla u(t)\|_{L^2}^2 + 2(F(u(t)), 1) - 2\langle g, u(t)\rangle + 2c_2'|\Omega| + 2\|g\|_{H^{-1}}^2$$

Using Young's inequality $(|\langle g, u(t)\rangle| \leq \|g\|_{H^{-1}}^2 + \frac{1}{4}\|\nabla u(t)\|_{L^2}^2)$ and (b), we obtain

$$(c) \qquad E(u(t)) \geq \frac{1}{2}\|\nabla u(t)\|_{L^2}^2 + 2c_1'\|u(t)\|_{L^6}^6 \geq 0$$

$$(d) \quad E(u(t)) \leq \frac{3}{2}\|\nabla u(t)\|_{L^2}^2 + 2c_3'\|u(t)\|_{L^6}^6 + 4\|g\|_{H^{-1}}^2 + 2(c_2' + c_3')|\Omega|$$

Remark

Let $\mathcal{B}$ be any bounded set in $L^2(\Omega)$. Thanks to the previous corollary we know that after a time $t_{\mathcal{B}} \geq 0$ all the trajectories starting from $u_0 \in \mathcal{B}$ enter in $B_0 = B(\rho_0)$, $B(\rho_0) \subset L^2(\Omega)$ being an absorbing set.
Hence we can always write, $\forall\, t \geq t_{\mathcal{B}}$, $\forall\, \tau \geq 0$,

$$S(t)u_0 = S(t - t_{\mathcal{B}} + t_{\mathcal{B}})u_0 = S(t - t_{\mathcal{B}})S(t_{\mathcal{B}})u_0 = S(\tau)u_1$$

where $u_1 \in B_0$.
Hence, for the sake of simplicity, in the following part of the proof we will take $t \geq 0$ and $u_0 \in B_0$.

Going back to (d), taking $u_0 \in B_0$ and applying inequality (2) of the previous theorem, we deduce

$$\sup_{t \geq 0} \int_t^{t+1} E(u(\tau))d\tau \leq C(\rho_0)$$

where $C(\rho_0)$ is a positive constant depending also on f, g, Ω. Moreover, recalling (a) and definition $(*)$ one has

$$\frac{d}{dt}(E(u(t))) \leq 0$$

Then, an application of the uniform Gronwall's lemma (with $\eta = E(u)$, $\phi = 0$, $\psi = 0$, $r = 1$, $t_0 = 0$) gives

$$E(u(t)) \leq C(\rho_0) \quad \forall\, t \geq 1$$

Recalling again definition $(*)$ and Poincaré's inequality we conclude

$$\|u(t)\|_{H^1} \leq \widetilde{C}(\rho_0) \quad \forall\, t \geq 1$$

Finally, on account of the previous remark, for any bounded set $\mathcal{B} \subset L^2(\Omega)$ we have

$$(e) \quad \|S(t)\mathcal{B}\|_{H^1} \leq \widetilde{C}(\rho_0) \quad \forall\, t \geq t_{\mathcal{B}} + 1$$

Hence

$$B_1 = \{x \in H_0^1(\Omega) \,:\, \|x\|_{H^1} \leq \widetilde{C}(\rho_0)\}$$

is an absorbing set in $L^2(\Omega)$ that is bounded in $H_0^1(\Omega)$.

Remark

As by-product we deduce

$$\int_{t^*}^{+\infty} \|u_t(\tau)\|_{L^2}^2 d\tau \leq C(\rho_0), \ \ \textit{where } t^* = t_{\mathcal{B}} + 1$$

Proof. Recall definition $(*)$ and (a). Then we have

$$\frac{1}{2}\frac{d}{dt}(E(u(t))) + \|u_t(t)\|_{L^2}^2 = 0$$

Integrating on (t^*, t) and using (c) and (d) we get

$$\int_{t^*}^{t} \|u_t(\tau)\|_{L^2}^2 \leq \frac{E(t^*)}{2} \leq C(f,\Omega)(\|\nabla u(t^*)\|_{L^2}^2 + \|u(t^*)\|_{L^6}^6 + \|g\|_{H^{-1}}^2 + 1)$$

Passing to the limit for $t \to +\infty$ and on account of (e) we get the estimate.

Corollary

$(L^2(\Omega), S(t))$ is a compact dynamical system which has a (connected) global attractor $\mathcal{A}$ that is bounded in $H_0^1(\Omega)$. Moreover, for any bounded absorbing set B_0 it holds

$$\mathcal{A} = \omega(B_0)$$

Proof. In the previous theorem we proved the existence of a bounded absorbing set $B_1 \subset L^2(\Omega)$ that is closed and bounded in $H_0^1(\Omega)$. Hence, B_1 is compact in $L^2(\Omega)$.

Using corollary 3.27 (see notes on attractors) we get the existence of the global attractor $\mathcal{A} \subseteq B_1 \subset H_0^1(\Omega)$.

On account of Theorem 3.17, $\mathcal{A}$ is also connected (as $L^2(\Omega)$).

Finally, an application of theorem 3.26 gives $\mathcal{A} = \omega(B_0)$, for any bounded absorbing set B_0.