Appunti del Corso di

Equazioni di Evoluzione

Equazioni paraboliche
ed iperboliche astratte

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Chapter 1

Preliminaries

1.1 Fractional Sobolev spaces and trace theorems

Let Ω be a regular, bounded domain in $\mathbb{R}^N$ and $\Gamma = \partial\Omega$. Let $p \in [1, \infty)$. As in [1, p. 312], it is possible to define a family of spaces intermediate between $L^p(\Omega)$ and $W^{1,p}(\Omega)$. Let $s \in (0, 1)$, we define

$$W^{s,p}(\Omega) = \left\{ u \in L^p(\Omega) \, : \, \frac{|u(x) - u(y)|}{|x - y|^{s+N/p}} \in L^p(\Omega \times \Omega) \right\} ,$$

with the natural norm associated. Set $H^s(\Omega) = W^{s,2}(\Omega)$. If s is real, not integer and $s > 1$, instead, we set $s = m + \sigma$, with $m = [s]$ and we can define the space $W^{s,p}(\Omega)$ as follows:

$$W^{s,p}(\Omega) = \{ u \in W^{m,p}(\Omega) \, : \, D^\alpha u \in W^{\sigma,p}(\Omega) \quad \forall \alpha \, : \, |\alpha| = m \} .$$

Then, the *trace* operator $u \mapsto u_{|\Gamma}$ defined on the regular functions u (cf. [1, p. 314]) has a unique prolongation to the linear and continuous operator

$$\gamma \, : \, W^{1,p}(\Omega) \to W^{1-1/p,p}(\Gamma)$$

and there exists a linear continuous operator

$$\mathcal{R} \, : \, W^{1-1/p,p}(\Gamma) \to W^{1,p}(\Omega)$$

such that $\gamma(\mathcal{R}(v)) = v$ for all $v \in W^{1-1/p,p}(\Gamma)$.

1.2 The Hilbert triplet

Let V and H be two real Hilbert spaces satisfying

$$V \quad \text{is a vectorial subspace of } H \text{ dense in } H\,, \tag{1.1}$$

$$\text{the immersion of } V \text{ in } H \text{ is continuous,} \tag{1.2}$$

$$V \quad \text{is separable,} \tag{1.3}$$

which implies that also H and V' are separable because $\overline{V} = H$, and V' is separable because it is a dual space of a reflexive and separable Banach space ([1, p. 73]). Denote respectively by $\|\cdot\|$, $|\cdot|$, and $\|\cdot\|_*$ the norms in the three spaces V, H, and V'. By $(\!(\cdot,\cdot)\!)$, $(\cdot,\cdot)$, and $(\!(\cdot,\cdot)\!)_*$ the corresponding scalar products, and by $\langle\cdot,\cdot\rangle$ the duality between V' and V. Condition (1.2) is equivalent to the existence of a positive constant c_* such that $|v| \le c_*\|v\|$ for all $v \in V$. Fixed $u \in H$, consider the functional

$$Iu\,:\, v \to (u,v), \quad v \in V\,.$$

Then, Iu is linear and continuous on V. Indeed, applying the Schwarz inequality, we have

$$|(u,v)| \le |u||v| \le c_*|u|\|v\| \quad \forall v \in V\,.$$

Hence, $Iu \in V'$ and $\|Iu\|_* \le c_*|u|$. Hence the application from H to V' defined as $I\,:\, u \to Iu$ is linear and continuous. It is also injective due to (1.1). Indeed, $Iu = 0$ implies $(u,v) = 0$ for all $v \in V$, hence $u \in V^\perp$ in H, i.e. $u = 0$ for (1.1) (cf. [1, p. 133]). Hence, we can identify the element $u \in H$ with the element $Iu \in V'$. Hence, we have the inclusion $H \subseteq V'$ and the inequality

$$\|u\|_* \le c_*|u| \quad \forall u \in H$$

and we usually say that (V, H, V') is an *Hilbert triplet.* The identification can be resumed in the formula

$$\langle u,v\rangle = (u,v) \quad \forall u \in H,\ \forall v \in V\,.$$

It is possible, moreover, to prove that the immersion of V into V' is dense too.

1.3 Vector valued function spaces

Cf. [4], [5, p. 649], [11, p. 332]), [13]. For the proofs cf. also [6].

Let W be a generic Hilbert space. Then, we define the Banach space $C^0([0,T];W)$ as the space of continuous functions from $[0,T]$ with values in W and endow it with the norm

$$\|v\|_{C^0([0,T];W)} = \max_{t\in[0,T]} \|v(t)\|_W\,.$$

In case $W = L^2(\Omega)$, with Ω a sufficiently regular bounded domain in $\mathbb{R}^N$, $N = 1, 2, 3$, we have that $u \in C^0([0,T]; L^2(\Omega))$ means: if $t_n \to t_0$ as $n \nearrow \infty$, then $u(t_n) \to u(t_0)$ in $L^2(\Omega)$. Let us note that, e.g., taking $\Omega = (0,1)$, $T = 1$, $u(t) = \chi_{(t,1)}$, then $u \in C^0([0,T]; L^2(\Omega))$ but $u \notin C^0(\Omega \times (0,T))$.

Analogously, we can define the space $C^1([0,T]; W)$. Moreover, we want now to define the space $L^p(0,T;W)$. In order to do that, we need first to introduce the space $\mathcal{D}(0,T;W)(= C_c^\infty(0,T;W))$ as the space of C^∞ functions from $(0,T)$ with values in W, which are 0 near 0 and T. Then, we say that $u : (0,T) \to W$ is strongly measurable if there exists a sequence $u_n : (0,T) \to W$ of simple functions (i.e. finite combinations with coefficients in W of characteristic functions of Lebesgue-measurable subsets of $(0,T)$) such that $u_n(t) \to u(t)$ as $n \nearrow \infty$ strongly in W and for a.e. $t \in (0,T)$. Then - due to the Pettis theorem - we have that in a separable space the strong measurability is equivalent to the scalar measurability, i.e. for all $w' \in W'$ the function $t \mapsto \langle w', u(t)\rangle$ is measurable on $(0,T)$. This is not true for general (not separable) spaces. Indeed, it is possible to prove that $L^\infty(\Omega \times (0,T)) \neq L^\infty((0,T); L^\infty(\Omega))$.

Hence, from now on we suppose W to be separable and we introduce the following Banach space

$$L^p(0,T;W) = \{u : (0,T) \to W \text{ measurable} : t \mapsto \|u(t)\|_W \text{ belongs to } L^p(0,T)\},$$

endowed with the following norm (in case $p \in [1, +\infty)$)

$$\|u\|_{L^p(0,T;W)} = \left(\int_0^T \|u(t)\|_W^p \, dt\right)^{1/p},$$

and in case $p = \infty$, we take

$$\|u\|_{L^\infty(0,T;W)} = \sup(\text{ess})_{t\in(0,T)} \|u(t)\|_W ,$$

Then, we have the following result

Theorem 1.3.1. *Let W be a separable Banach space. Then, for every $p \in [1,\infty]$ $L^p(0,T;W)$ is a Banach space and if W is Hilbert, then also $L^2(0,T;W)$ is an Hilbert space with the following scalar product $(u,v)_{L^2(0,T;W)} = \int_0^T (u(t), v(t))_W \, dt$.*

Notice that, for every $p \in [1,\infty)$, $L^p(0,T; L^p(\Omega)) = L^p(\Omega \times (0,T))$.

Now, we can introduce a generalized notion of derivative

Theorem 1.3.2. *Let W be an Hilbert separable space and u, $w \in L^1(0,T;W)$. Then the following three conditions are equivalent*

$$\text{there esists } c \in W \text{ such that } u(t) = c + \int_0^t w(s)\,ds, \quad \text{a.e. in } [0,T]\,, \tag{1.4}$$

$$\int_0^T (w(t), v(t))\,dt = -\int_0^T (u(t), v'(t))\,dt, \quad \forall v \in \mathcal{D}(0,T;W)\,, \tag{1.5}$$

$$\int_0^T (w(t), z)\varphi(t)\,dt = -\int_0^T (u(t), z)\varphi'(t)\,dt, \quad \forall z \in W,\ \forall \varphi \in \mathcal{D}(0,T)\,. \tag{1.6}$$

Definition 1.3.3. *Let W be an Hilbert separable space and u, $w \in L^1(0,T;W)$. We say that w is the* derivative *of u in $L^1(0,T;W)$ if w satisfies one of the equivalent conditions of Thm. 1.3.2. Its uniqueness follows from the density of $\mathcal{D}(0,T;W)$ in $L^1(0,T;W)$.*

Note that in case $u \in C^1([0,T];W)$ this notion of derivative coincides with the *classical* one.

Then, we can define the space $H^1(0,T;W)$ as the space of functions $u \in L^2(0,T;W)$ such that $u' \in L^2(0,T;W)$. Proceeding by induction on k, we can, analogously, define the spaces $H^{k+1}(0,T;W) = \{v \in H^k(0,T;W) : v' \in H^k(0,T;W)\}$, and we have that the space $H^k(0,T;W)$ is an Hilbert space with respect to the following norm and scalar product

$$\|v\|^2_{H^k(0,T;W)} = \sum_{i=0}^k \int_0^T \|v^{(i)}(t)\|^2_W\,dt\,,$$

$$(u,v)_{H^k(0,T;W)} = \sum_{i=0}^k \int_0^T (u^{(i)}(t), v^{(i)}(t))\,dt\,.$$

Let V be an Hilbert separable space and define the following space

$$W := \{u \in L^2(0,T;V) \,:\, u' \in L^2(0,T;V')\}\,, \tag{1.7}$$

endowed with the norm

$$\|v\|^2_W := \int_0^T \left(\|v(t)\|^2 + \|v'(t)\|^2_*\right)\,dt\,. \tag{1.8}$$

This space is an Hilbert space linked to the usual Sobolev spaces by the formula

$$W = L^2(0,T;V) \cap H^1(0,T;V')$$

and the norm (1.8) is equivalent to the following one

$$\|v\|^2_W := \|v\|^2_{L^2(0,T;V)} + \|v'\|^2_{L^2(0,T;V')}\,.$$

Theorem 1.3.4. *The following continuous inclusion holds true*

$$W \hookrightarrow C^0([0,T];H)\,. \tag{1.9}$$

Moreover, if w, $v \in W$, then the following formulae hold true for every couple of points τ, $t \in [0,T]$:

$$\int_\tau^t \left(\langle w'(s), v(s)\rangle + \langle v'(s), w(s)\rangle\right) ds = (w(t),v(t)) - (w(\tau),v(\tau)), \tag{1.10}$$

$$\int_\tau^t \langle v'(s), v(s)\rangle = \frac{1}{2}|v(t)|^2 - \frac{1}{2}|v(\tau)|^2\,. \tag{1.11}$$

In the study of abstract Cauchy problems it is useful to have an immersion and a formula analogous to (1.10–1.11) linked to the operator A, which intervenes in the problem we want to deal with.

Theorem 1.3.5. *Let $A : V \to V'$ be a linear and continuous operator, let $D(A;H) = A^{-1}(H)$ and endow it with the graph norm*

$$\|v\|^2_{D(A;H)} = \|v\|^2 + |Av|^2 \quad \forall v \in D(A;H).$$

Suppose, moreover, that A is symmetric (i.e. $\langle Au, v\rangle = \langle Av, u\rangle$ for all $u, v \in V$) and it satisfies the following weak coercivity *assumption:*

$$\langle Av, v\rangle + \lambda|v|^2 \geq L\|v\|^2 \quad \forall v \in V\,.$$

Then, the following continuous inclusion holds true:

$$L^2(0,T;D(A;H)) \cap H^1(0,T;H) \hookrightarrow C^0([0,T];V)\,. \tag{1.12}$$

Moreover, if w, $v \in L^2(0,T;D(A;H)) \cap H^1(0,T;H)$, then the following formulae hold true for every couple of points τ, $t \in [0,T]$:

$$\int_\tau^t \left((Aw(s), v'(s)) + (Av(s), w'(s))\right) ds = \langle Aw(t), v(t)\rangle - \langle Aw(\tau), v(\tau)\rangle\,, \tag{1.13}$$

$$\int_\tau^t (Av(s), v'(s)) = \frac{1}{2}\langle Av(t), v(t)\rangle - \frac{1}{2}\langle Av(\tau), v(\tau)\rangle\,. \tag{1.14}$$

1.4 Gronwall lemma

Lemma 1.4.1. *Let a, $b \geq 0$, $\varphi \in L^1(0,T)$ such that*

$$\varphi(t) \leq a + b\int_0^t \varphi(s)\,ds \quad \text{a.e. in } (0,T)\,.$$

Then, the following inequality holds true:

$$\varphi(t) \le a e^{bt} \quad \text{a.e. in } (0,T)\,.$$

Hence, there exists a positive constant c depending only on b and T such that

$$\varphi(t) \le c\, a \quad \text{a.e. in } (0,T)\,.$$

1.5 Generalized Poincarè inequality

Lemma 1.5.1. *Let V and X be two Hilbert spaces and $L : V \to X$ be a linear and compact operator. Let $\|\cdot\|_0$ be a norm in V associated to a scalar product in V. Then for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for all $v \in V$ $\|Lv\|_X \le \varepsilon \|v\|_V + C_\varepsilon \|v\|_0$.*

Proof. Suppose, by contradiction, that there exists $\varepsilon > 0$ such that for all $c > 0$ there exists $v \in V$ such that $\|Lv\|_X > \varepsilon\|v\|_V + c\|v\|_0$. Then, there exist $\varepsilon > 0$ and a sequence $\{v_n\}$ in V such that $\|Lv_n\|_X > \varepsilon\|v_n\|_V + n\|v_n\|_0$ for all $n \in \mathbb{N}$. Then, $v_n \neq 0$ because $Lv_n > 0$, hence, we can take the sequence $u_n = \dfrac{v_n}{\|v_n\|_V}$ and we have $\|u_n\|_V = 1$ and $\|Lu_n\|_X > \varepsilon + n\|u_n\|_0$. But, since, L is a compact operator and $u_n \to u$ weakly in V. Then, $Lu_n \to Lu$ in X as $n \nearrow \infty$. Hence, for a subsequence n_k of n it holds true: $n_k\|u_{n_k}\| \le \text{const}$, which means that $u_{n_k} \to 0$ as $k \nearrow \infty$. But we have

$$\|u\|_0^2 = (u,u)_0 = \lim_{k \nearrow \infty} (u, u_{n_k})_0 \le \liminf \|u\|_0 \|u_{n_k}\|_0 = 0\,.$$

Hence, $u = 0$, and, since $u_{n_k} \to u(=0)$ weakly in V, we have also $Lu_{n_k} \to Lu(=0)$ in X, which is in contradiction with $\|Lu_n\|_X > \varepsilon$ for all $n \in \mathbb{N}$. This concludes the proof.

Lemma 1.5.2. *Let V, H, W, Z be four Hilbert spaces and $A : V \to W$, $B : V \to Z$ be linear and continuous operators such that $V \subseteq H$ with compact immersion, $\mathrm{Ker}(A) \cap \mathrm{Ker}(B) = \{0\}$, and there exists a positive constant M such that $\|v\|_V \le M\left(\|Av\|_W + \|v\|_H\right)$ for all $v \in V$. Then, there exists a positive constant C such that $\|v\|_H \le C\left(\|Av\|_W + \|Bv\|_Z\right)$ for all $v \in V$ and the norm $\|Av\|_W + \|Bv\|_Z$ is equivalent to the norm $\|v\|_V$.*

Proof. We can use Lemma 1.5.1 with $\|v\|_0 = \|Av\|_W + \|Bv\|_Z$, $X = H$ and L the immersion of V in H. Then, we have

$$\begin{aligned}\|v\|_H &\le \varepsilon\|v\|_V + c_\varepsilon\|v\|_0 = \varepsilon\|v\|_V + c_\varepsilon\left(\|Av\|_W + \|Bv\|_Z\right)\\ &\le \varepsilon M\left(\|Av\|_W + \|v\|_H\right) + c_\varepsilon\left(\|Av\|_W + \|Bv\|_Z\right)\\ &\le \varepsilon M\|v\|_H + (\varepsilon M + c_\varepsilon)\left(\|Av\|_W + \|Bv\|_Z\right)\,.\end{aligned}$$

Then, we can take $\varepsilon = 1/2M$, obtaining the desired estimate.

Applications. Let Ω be a bounded, Lipschitz, connected subset of $\mathbb{R}^N$, then we can apply the previous lemma to the following cases:

1) $V = H^1(\Omega)$, $H = L^2(\Omega)$, $W = L^2(\Omega)^N$, $Z = \mathbb{R}$, $A = \nabla$, $Bv = \int_{\partial\Omega} v\,ds$. Then, we get the *standard* Poincarè inequality

$$\|v\|_{L^2(\Omega)} \leq c_p \left(\|\nabla v\|_{L^2(\Omega)^N} + \left| \int_{\partial\Omega} v\,ds \right| \right) \quad \forall v \in H^1(\Omega)\,, \tag{1.15}$$

and, in particular,

$$\|v\|_{L^2(\Omega)} \leq c_p \|\nabla v\|_{L^2(\Omega)^N} \quad \forall v \in H^1_0(\Omega)\,. \tag{1.16}$$

2) $V = H^1(\Omega)$, $H = L^2(\Omega)$, $W = L^2(\Omega)^N$, $Z = \mathbb{R}$, $A = \nabla$, $Bv = \int_{\Gamma_0} v\,ds$, where Γ_0 is an open connected subset of $\partial\Omega$ with positive measure. Then, we get the following Poincarè inequality

$$\|v\|_{L^2(\Omega)} \leq c_p \left(\|\nabla v\|_{L^2(\Omega)^N} + \left| \int_{\Gamma_0} v\,ds \right| \right) \quad \forall v \in H^1(\Omega)\,,$$

and, in particular,

$$\|v\|_{L^2(\Omega)} \leq c_p \|\nabla v\|_{L^2(\Omega)^N} \quad \forall v \in H^1_{\Gamma_0}(\Omega)\,,$$

where $H^1_{\Gamma_0}(\Omega) = \{v \in H^1(\Omega) \,:\, v_{|\Gamma_0} = 0\}$.

3) Ω sufficiently regular, $V = H^2(\Omega)$, $H = H^1(\Omega)$, $n = \#\{$multindexes α such that $|\alpha| = 2\}$ $W = L^2(\Omega)^n$, $Z = L^2(\Omega)$, $A \,:\, v \mapsto \{D^\alpha v\,,\, |\alpha| = 2\}$, $Bv = v$. Then, we get the following inequality

$$\|v\|_{H^1(\Omega)} \leq C \left(\sum_{|\alpha|=2} \|D^\alpha v\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)} \right) \quad \forall v \in H^2(\Omega)\,,$$

hence, the norm $\sum_{|\alpha|=2} \|D^\alpha v\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}$ is equivalent to the norm $\|v\|_{H^2(\Omega)}$.

Chapter 2

Abstract linear parabolic equations

We resume here some of the results concerning some abstract linear parabolic equations. The theory that we cannot completely explain here is contained in [5, 11] and a more detailed and sophisticated theory is explained in [8, 9]. Regarding the regularity results for elliptic equations cf., e.g., [3], [7, Thms. 2.2.2.3, 3.2.1.2], and [5, Cap. 6].

Let Ω be an open bounded subset of $\mathbb{R}^n$ with *sufficiently regular* boundary Γ. Denote by ν the outward normal derivative to the boundary Γ and by Δ and ∇ the spatial Laplacian and gradient, respectively. Then, we consider the following Cauchy problem for the heat equation:

$$\partial_t u - \Delta u = f \quad \text{in } \Omega \times (0,T)\,, \tag{2.1}$$
$$\nabla u \cdot \nu + \alpha u = g \quad \text{on } \Gamma \times (0,T)\,, \tag{2.2}$$
$$u(\cdot,0) = u_0 \quad \text{in } \Omega\,, \tag{2.3}$$

where, in a *classical* formulation, the unknown u and the source f are regular functions defined in $\overline{\Omega} \times [0,T]$, the data α, g are defined and sufficiently regular on Γ and $\Gamma \times [0,T]$, respectively, such that $\alpha(x) \in [0,+\infty)$ for all $x \in \Gamma$. Finally, u_0 is a sufficiently regular initial datum in $\overline{\Omega}$. Let us note that the boundary condition (2.2) is called Robin boundary condition in case $\alpha > 0$. Instead, in case $\alpha = 0$, it corresponds to the Neumann boundary condition.

2.1 Variational formulation.

Let now v be a regular function in $\overline{\Omega}$, multiply equation (2.1) by v, integrating by parts in Ω and using the boundary condition (2.2), we *formally* get

$$\int_\Omega \partial_t u\, v\, dx + \int_\Omega \nabla u \cdot \nabla v\, dx + \int_\Gamma \alpha(\sigma)\, u\, v\, d\sigma = \int_\Omega f(x,t)\, v\, dx + \int_\Gamma g(\sigma,t)\, v\, d\sigma\,,$$

$d\sigma$ denoting the superficial element. Now, it is natural to use the spaces $V = H^1(\Omega)$ and $H = L^2(\Omega)$, the bilinear form:

$$a(w,v) = \int_\Omega \nabla w \cdot \nabla v\, dx + \int_\Gamma \alpha(\sigma)\, w\, v\, d\sigma, \quad w,\, v \in V\,, \tag{2.4}$$

and the element $h(t) \in V'$ such that

$$\langle h(t), v\rangle = \int_\Omega f(t)\, v\, dx + \int_\Gamma g(\sigma,t)\, v\, d\sigma, \quad w,\, v \in V\,. \tag{2.5}$$

Naturally, in the integrals on Γ we have used the notation v, even if rigorously, we should use the trace of $v \in V$ on Γ and indicate it with the symbol $v_{|\Gamma}$. Notice that, due to the trace theorems, this trace belongs to the space $H^{1/2}(\Gamma) \hookrightarrow L^2(\Gamma)$. Then, our problem can be re-written as

$$\frac{d}{dt}\langle u(t), v\rangle + \langle Au(t), v\rangle = \langle h(t), v\rangle \quad \forall v \in V\,, \tag{2.6}$$

where A denotes the linear operator associated to the bilinear form a, i.e.

$$\langle Aw, v\rangle = a(w,v), \quad w,\, v \in V\,. \tag{2.7}$$

If we are able to interpret $u'(t)$ as an element of V', then, we can rewrite (2.6) as

$$\langle u'(t), v\rangle + \langle Au(t), v\rangle = \langle h(t), v\rangle \quad \forall v \in V\,. \tag{2.8}$$

This is a consequence of the following equalities, holding true for all $v \in V$ and for every $\varphi \in \mathcal{D}(0,T)$,

$$\begin{aligned}\int_0^T \frac{d}{dt}\langle u(t), v\rangle\, \varphi(t)\, dt &= -\int_0^T \langle u(t), v\rangle\, \varphi'(t)\, dt = -\int_0^T \langle u(t)\varphi'(t), v\rangle\, dt \\ &= \langle -\int_0^T u(t)\varphi'(t)\, dt, v\rangle = \int_0^T \langle u'(t), v\rangle\, \varphi(t)\, dt\,.\end{aligned}$$

2.2 Abstract equation

We generalize here our setting introducing the *abstract* Cauchy problem in the framework of an abstract Hilbert triplet (V, H, V'), which, in the particular case given above, would be $(H^1(\Omega), L^2(\Omega), (H^1(\Omega))')$. Denote as in the previous Chapter, respectively by $\|\cdot\|$, $|\cdot|$, and $\|\cdot\|_*$ the norms in the three spaces V, H, and V'. By $(\!(\cdot,\cdot)\!)$, $(\cdot,\cdot)$, and $(\!(\cdot,\cdot)\!)_*$ the corresponding scalar products, and by $\langle\cdot,\cdot\rangle$ the duality between V' and V. Hence, we search for $u \in W$, where $W = \{u \in L^2(0,T;V), \quad u' \in L^2(0,T;V')\}$ is defined in (1.7), solving the abstract Cauchy problem:

$$u'(t) + Au(t) = h(t) \quad \text{in } V' \text{ and a.e. in } (0,T)\,, \tag{2.9}$$

$$u(\cdot,0) = u_0\,. \tag{2.10}$$

For the properties of the space W the reader can refer to Chapter 1. Let us only note that $W \hookrightarrow C^0([0,T];H)$, hence, the initial condition (2.10) turns out to be meaningful and the natural assumptions on the data are

$$h \in L^2(0,T;V')\,, \tag{2.11}$$

$$u_0 \in H\,. \tag{2.12}$$

Note that assumption (2.11) is satisfied for the particular problem (2.1–2.3), e.g., in case $f \in L^2(\Omega \times (0,T))$, $g \in L^2(\Gamma \times (0,T))$. The assumption we make on A is that it satisfies the following *weak coercivity* property: there exist λ, $L > 0$ such that

$$\langle Av, v\rangle + \lambda|v|^2 \geq L\|v\|^2 \quad \forall v \in V\,. \tag{2.13}$$

Note that assumption (2.13) is satisfied for the particular problem (2.1–2.3), e.g., in case $\alpha \in L^\infty(\Gamma)$, $\alpha \geq 0$.

Remark 2.2.1. Notice that this weak coercivity property is satisfied, e.g., in case $A = B + C$, where B is a V-elliptic operator, i.e. it satisfies (2.13) with $\lambda = 0$ and C is a linear and continuous operator from V in H. The most easy case is the case in which $V = H$ and they have finite dimension. Then, we are solving a Cauchy problem for a system of ODEs, and, fixed a base for V, the operator A can be written in terms of a $m \times m$- matrix. The V-ellipticity - in this case - is equivalent to the fact that the matrix is positive definite, while the problem is solvable for every matrix A and, indeed, (2.13) is satisfied for every matrix A (for proper choices of λ and L, depending on A, obviously). Finally, in the general case of an Hilbert triplet our problem, under assumption (2.13), can be reduced to the V-elliptic case with the change of unknown $u(t) = e^{\lambda t} w(t)$. Then the initial condition (2.10) becomes $w(0) = u_0$ and (2.8) can be rewritten as

$$w'(t) + (A + \lambda I)w(t) = e^{-\lambda t} h(t) \quad \text{for a.e. } t \in (0, T),$$

where I denotes the immersion of V in V'. Hence, it is clear that (2.13) is equivalent to the V-ellipticity of $A + \lambda I$ with ellipticity constant L. Hence, if we can solve the problem in the V-elliptic case, we can do it also in the weakly coercive case. Unfortunately the two sets of assumptions can not be interchanged in order to study the long time behavior of solutions as $t \nearrow \infty$. For this reason and also in view of possible generalizations of equation (2.9), we will assume hypothesis (2.13) in our analysis.

Now, we are in the position of stating our well-posedness result for the abstract Cauchy problem (2.9–2.10).

Theorem 2.2.2. *Let $A : V \to V'$ be a linear and continuous operator satisfying the weak coercivity assumption* (2.13). *Then, for every h and u_0 verifying hypotheses (2.11–2.12), there exists a unique $u \in W$ (cf.* (1.7)*) solving the Cauchy problem (2.9–2.10). Moreover, u satisfies the following estimate*

$$\|u\|_W \leq c \left(|u_0| + \|h\|_{L^2(0,T;V')} \right) , \tag{2.14}$$

where c depends only on L, λ, T and on the norm of the operator A.

Proof. Let us start proving *uniqueness* of solutions. Let us test (2.9) by $u(t)$ and integrate the resulting equation over $(0, t)$ with $t \in [0, T]$, getting

$$\int_0^t \langle u'(s), u(s) \rangle \, ds + \int_0^t \langle Au(s), u(s) \rangle \, ds = \int_0^t \langle f(s), u(s) \rangle \, ds \,. \tag{2.15}$$

Using the fact that $u \in W$, we can apply (1.11) and rewrite the first integral on the left-hand side as

$$\int_0^t \langle u'(s), u(s) \rangle \, ds = \frac{1}{2} |u(t)|^2 - \frac{1}{2} |u_0|^2 \,. \tag{2.16}$$

Then, we can estimate the second integral on the left-hand side in (2.15) using (2.13) in the following way

$$\int_0^t \langle Au(s), u(s) \rangle \, ds \geq L \int_0^t \|u(s)\|^2 \, ds - \lambda \int_0^t |u(s)|^2 \, ds \,, \tag{2.17}$$

while we can estimate the integral on the right-hand side as follows:

$$\int_0^t \langle f(s), u(s)\rangle \, ds \le \int_0^t \|f(s)\|_* \|u(s)\| \, ds \le \frac{L}{2}\int_0^t \|u(s)\|^2 \, ds + \frac{1}{2L}\int_0^t \|f(s)\|_*^2 \, ds \, . \tag{2.18}$$

Combining (2.15–2.18), we get

$$\begin{aligned} &\frac{1}{2}|u(t)|^2 - \frac{1}{2}|u_0|^2 + L\int_0^t \|u(s)\|^2 \, ds - \lambda \int_0^t |u(s)|^2 \, ds \\ &\le \frac{L}{2}\int_0^t \|u(s)\|^2 \, ds + \frac{1}{2L}\int_0^t \|f(s)\|_*^2 \, ds \, , \end{aligned}$$

from which we deduce

$$|u(t)|^2 + L\int_0^t \|u(s)\|^2 \, ds \le |u_0|^2 + 2\lambda \int_0^t |u(s)|^2 \, ds + \frac{1}{L}\int_0^t \|f(s)\|_*^2 \, ds \, .$$

Hence, we can apply the Gronwall lemma 1.4.1 with the choices

$$\varphi(t) = |u(t)|^2 + L\int_0^t \|u(s)\|^2 \, ds, \quad a = |u_0|^2 + \frac{1}{L}\int_0^t \|f(s)\|_*^2 \, ds, \quad b = 2\lambda \, .$$

We get the estimate, holding true for every $t \in [0, T]$

$$|u(t)|^2 + L\int_0^t \|u(s)\|^2 \, ds \le c\left(|u_0|^2 + \frac{1}{L}\int_0^t \|f(s)\|_*^2 \, ds\right) \, ,$$

where c depends only on L, λ, and T. This implies:

$$\|u\|^2_{C^0([0,T];H)} + \|u\|^2_{L^2(0,T;V)} \le c\left(|u_0|^2 + \|f\|^2_{L^2(0,T;V')}\right) \, , \tag{2.19}$$

for a new constant c depending again only on L, λ, and T. In case of null data, this implies that $u \equiv 0$, and we get uniqueness. Moreover, estimate (2.14) follows from the following inequalities

$$\|Au\|_{L^2(0,T;V')} \le M\|u\|_{L^2(0,T;V)}, \quad \|u'\|_{L^2(0,T;V')} \le \|Au\|_{L^2(0,T;V')} + \|f\|_{L^2(0,T;V')} \, ,$$

where M denotes the norm of the operator A.

Let us proceed now proving *existence*. The procedure will consist in

1. finding a solution for a suitable approximating problem
2. proving a-priori estimates independent of the approximating parameter
3. using compactness results in order to pass to the limit in the approximating problem and finding a solution to (2.9–2.10).

The approximation. We use the *Faedo-Galerkin* scheme, which consists in discretizing the operator A by means of a discretization of the space V. Using the fact that V is a separable space, we fix a non-decreasing subsequence $\{V_n\}$ of subspaces of V of finite dimension whose union $V_\infty := \bigcup_n V_n$ is densely embedded in V. Then, fixed n, we consider the following approximated problem. Find $u_n \in H^1(0,T;V_n)$ such that

$$\langle u_n'(t), v\rangle + \langle Au_n(t), v\rangle = \langle h(t), v\rangle \quad \forall v \in V_n, \text{ and a.e. in } (0,T)\,, \tag{2.20}$$

$$\langle u_n(0), v\rangle = \langle u_0, v\rangle \quad \forall v \in V_n\,. \tag{2.21}$$

If m denotes the dimension of V_n, chosen a basis $(v^1,\ldots,v^m)$ of V_n, we can represent the value $u_n(t)$ of the solution u_n (which we are searching for) in the following form:

$$u_n(t) = \sum_{j=1}^{m} y_j(t) v^j\,.$$

Moreover, denote by $\vec{f}$ and $\vec{y}$ the vectors of components, respectively, $f_j(t) = \langle h(t), v^j\rangle$, $y_{0j} = \langle u_0, v^j\rangle$, B and D the $m\times m$-matrix of elements defined, respectively, by $B_{ij} = (v^j, v^i)$, $D_{ij} = a(v^j, v^i)$. Then, we can write the approximated problem (2.20–2.21) as: find the vector $\vec{y}$ - whose general value is the column vector $\vec{y}(t) \in \mathbb{R}^m$: $(y_1(t),\ldots,y_m(t))$ - satisfying

$$B\vec{y'}(t) + D\vec{y}(t) = \vec{f}(t) \quad \text{a.e. in } (0,T)\,, \tag{2.22}$$

$$\vec{y}(0) = \vec{y}_0\,. \tag{2.23}$$

Since the vectors v^i are independent, it turns out that the matrix B is invertible. Moreover, the components $f_i \in L^2(0,T)$ for all $i = 1,\ldots,m$; indeed, for all $i = 1,\ldots,m$, the following inequalities hold true:

$$|f_i(t)| \le \|h(t)\|_* \|v^i\| \le \|h(t)\|_* \max_i \|v^i\|\,.$$

The Cauchy problem (2.22–2.23) is a Cauchy problem for a linear system of m ODEs in m unknowns with right-hand side in $L^2(0,T)$. Hence it admits a unique solution $\vec{y}$, whose components y_i belong to $H^1(0,T)$. Finally, the approximated problem (2.20–2.21) has a unique solution $u_n \in H^1(0,T;V_n)$.

A-priori estimates. Repeating the procedure already used in order to prove uniqueness of solutions, let us take $v = u_n(t)$ in (2.20), obtaining the following estimate

$$\|u_n\|^2_{C^0([0,T];H)} + \|u_n\|^2_{L^2(0,T;V)} \le C\left(|u_0|^2 + \|h\|^2_{L^2(0,T;V')}\right)\,, \tag{2.24}$$

where C is independent of n. In order to obtain this estimate, here we have used the fact that, since $u_n(0)$ is the orthogonal projection in H of u_0 on V_n (which is a closed subspace of H, because it has finite dimension), then, we have $|u_n(0)| \leq |u_0|$. Hence, the positive constant C in (2.24) is the same constant as in (2.19), and, in particular, it turns out to be independent of n. Moreover, let us notice that, by definition of orthogonal projection, we have

$$|u_n(0) - u_0| = \min_{w \in V_n} |u_0 - w| \leq |u_0 - v| \quad \forall v \in V_n \,.$$

This estimate implies that

$$u_n(0) \to u_0 \quad \text{in } H \quad \text{as } n \nearrow \infty \,. \tag{2.25}$$

Indeed, since V_∞ is densely embedded in V, it is dense also in H. Hence, we can choose a sequence of elements $\{v_k\}$ of V_∞ which tends to u_0 in H. Fixed k, choose n_k such that $v_k \in V_{n_k}$. Then, the subsequence $d_{n_k} = |u_{n_k} - u_0|$ tends to zero as $k \nearrow \infty$. Since the sequence $d_n = |u_n - u_0|$ is non increasing ($V_n \subseteq V_{n+1}$), then the whole sequence d_n tends to zero as $n \nearrow \infty$.

Passage to the limit. By means of compactness arguments, we deduce that the following convergences hold true (at least for a subsequence of $n \nearrow \infty$)

$$u_n \to u \quad \text{weakly in } L^2(0,T;V) \,, \tag{2.26}$$

$$u_n \to u \quad \text{weakly star in } L^\infty(0,T;H) \,. \tag{2.27}$$

Let us notice that, whence we will have identified the limit function u as solution of the problem, since we just know that it is unique, we can conclude that the previous convergences hold true for all the sequence $n \nearrow \infty$ and not only up to a subsequence. Hence, for simplicity of notation we will use the symbol $\{u_n\}$ to denote the extracted subsequence. Moreover, since A is linear, we also get

$$Au_n \to Au \quad \text{weakly in } L^2(0,T;V') \,. \tag{2.28}$$

Now, we need to show that $h - Au$ satisfies the definition of u'. Note that we cannot use $v = u_n'(t)$ in (2.20) as test function in order to pass to the limit in u_n', because we do not have (as it was instead true in case of the uniqueness proof) that $u_n' = h - Au_n$. This equality holds true only restricted to the space V_n. Fix an arbitrary function $\varphi \in \mathcal{D}(0,T)$, fix m and $z \in V_m$. If $n \geq m$, we can choose in (2.20) $v(t) = \varphi(t)z$, because $V_m \subseteq V_n$. Integrate the resulting equation over $(0,T)$ and use the integration by parts in time formula, getting

$$-\int_0^T \langle u_n(t), z\rangle \varphi'(t)\,dt = \int_0^T \langle h(t) - Au_n(t), z\rangle \varphi(t)\,dt \quad \forall z \in V_m \quad \forall m \leq n \,.$$

We can now pass to the limit as $n \nearrow \infty$ in this equality getting

$$-\int_0^T \langle u(t), z\rangle \varphi'(t)\, dt = \int_0^T \langle h(t) - Au(t), z\rangle \varphi(t)\, dt \quad \forall z \in V_m\,. \tag{2.29}$$

Since m was arbitrary the same equality holds true for all $z \in V_\infty$, which is dense in V and so it holds true also for all $z \in V$. Indeed, fixed $z \in V$, we can choose $z_k \in V_\infty$ such that z_k converges to z in V, and, written (2.29) for z_k, since

$$\varphi z_k \to \varphi z \quad \text{and } \varphi' z_k \to \varphi' z \quad \text{in } L^2(0,T;V)\,,$$

passing to the limit in (2.29) as $k \nearrow \infty$, we conclude that (2.29) holds true for all $z \in V$.

We need now to obtain the initial condition (2.21). If we knew that $u_n \to u$ weakly in W then it would converge also to u weakly in $C^0([0,T];H)$ as $n \nearrow \infty$. Then we have $u_n(0) \to u(0)$ weakly in H, and, due to (2.25), we get also $u_n(0) \to u_0$ in H, which, due to the uniqueness of the limit, implies $u(0) = u_0$. Hence, let us prove that $u_n \to u$ weakly in W as $n \nearrow \infty$. Fix $m \in \mathbb{N}$ such that $m \le n$ and choose in (2.20) $v(t) = \varphi(t)z$, where $\varphi \in C^\infty([0,T])$, $\varphi(T) = 0$. Integrate the resulting equation over $(0,T)$ and get

$$\int_0^T \langle u_n'(t), z\rangle \varphi(t)\, dt + \int_0^T \langle Au_n(t), z\rangle \varphi(t)\, dt = \int_0^T \langle h(t), z\rangle \varphi(t)\, dt\,. \tag{2.30}$$

Integrating by parts in time the first term, we obtain

$$\int_0^T \langle u_n'(t), z\rangle \varphi(t)\, dt = -\int_0^T \langle u_n(t), z\rangle \varphi'(t)\, dt - (u_0, z\varphi(0))\,. \tag{2.31}$$

Hence, passing to the limit as $n \nearrow \infty$ in (2.30) and using (2.31), one deduces

$$-\int_0^T \langle u(t), z\rangle \varphi'(t)\, dt + \int_0^T \langle Au(t), z\rangle \varphi(t)\, dt = \int_0^T \langle h(t), z\rangle \varphi(t)\, dt + (u_0, z\varphi(0)), \tag{2.32}$$

$$\forall z \in V_m,\ \varphi \in C^\infty([0,T])\ :\ \varphi(T) = 0\,.$$

Using again the density of V_∞ in V, as before, we deduce that (2.32) holds true for all $z \in V$. Now, we use the fact that $u \in W$ and so we can integrate the first term in (2.32) getting

$$-\int_0^T \langle u(t), z\rangle \varphi'(t)\, dt = \int_0^T \langle u'(t), z\rangle \varphi(t)\, dt + (u(0), z\varphi(0)), \tag{2.33}$$

$$\forall z \in V,\ \varphi \in C^\infty([0,T])\ :\ \varphi(T) = 0\,. \tag{2.34}$$

Since we know that u is a solution to (2.9), using (2.33) in (2.32), we obtain that two terms cancel out and we recover

$$(u(0), z\varphi(0)) = (u_0, z\varphi(0)), \quad \forall z \in V,\ \varphi \in C^\infty([0,T]) \,:\, \varphi(T) = 0.$$

Choosing, e.g., $\varphi(t) = T - t$ and using the fact that V is densely embedded in H, we finally get the desired initial condition $u(0) = u_0$ in H (cf. (2.10)).

Remark 2.2.3. Note that there are different techniques one could employ in order to solve the problem. For example one could apply a discretization in time method, which corresponds to replace the time derivative with the incremental differences and to study the corresponding elliptic problem. Otherwise in case of nonlinear operators in the equation, one can regularize it with more regular functions.

Let us proceed here with the following regularity result

Theorem 2.2.4. *Let $A : V \to V'$ be a linear and continuous operator satisfying the weak coercivity assumption* (2.13). *Suppose moreover that A is symmetric, i.e. $\langle Au, v\rangle = Av, u$ for all u, $v \in V$. Let $h \in L^2(0,T;H)$ and $u_0 \in V$. Then, the solution u to the Cauchy problem (2.9–2.10) is such that u', $Au \in L^2(0,T;H)$, $u \in C^0([0,T];V)$ and it satisfies the following estimate*

$$\|u'\|_{L^2(0,T;H)} + \|Au\|_{L^2(0,T;H)} \le c\left(\|u_0\| + \|h\|_{L^2(0,T;H)}\right), \tag{2.35}$$

where c depends only on L, λ, T and on the norm of the operator A.

Proof. Let us consider the same approximated problem as for Theorem 2.2.2, but take as initial condition the following one

$$u_n(0) = u_{0n}\,,$$

where u_{0n} is the projection of u_0 on V_n with respect to the scalar product of V. We will need in the proof that the initial datum u_{0n} of u_n satisfies $u_{0n} \to u_0$ in V and $\|u_{0n}\| \le c\|u_0\|$. We choose now in (2.20) $v = u_n'(t)$ - which turns out to be an admissible choice because $u_n \in H^1(0,T;V_n)$ - and integrate the resulting equation over $(0,t)$, with $t \in (0,T)$, getting

$$\int_0^t |u_n'(s)|^2\, ds + \int_0^t \langle Au_n(s), u_n'(s)\rangle\, ds = \int_0^t (h(s), u_n'(s))\, ds\,. \tag{2.36}$$

Then, we can estimate the second term on the left-hand side and the the term on the right-hand side, respectively, from below and above, getting

$$\begin{aligned}\int_0^t \langle Au_n(s), u_n'(s)\rangle\, ds &= \frac{1}{2}\langle Au_n(t), u_n(t)\rangle - \frac{1}{2}\langle Au_{0n}, u_{0n}\rangle \\ &\geq \frac{L}{2}\|u_n(t)\|^2 - \frac{\lambda}{2}|u_n(t)|^2 - \frac{M}{2}\|u_{0n}\|^2 \\ &\geq \frac{L}{2}\|u_n(t)\|^2 - \frac{\lambda}{2}|u_n(t)|^2 - c\|u_0\|^2 , \\ \int_0^t (h(s), u_n'(s))\, ds &\leq \frac{1}{2}\int_0^T |h(s)|^2\, ds + \frac{1}{2}\int_0^T |u_n'(s)|^2\, ds .\end{aligned}$$

Here we have used the symmetry of A together with (2.13) (cf. also Theorem 1.3.5) and we have denoted by M the norm of the operator A, $\|A\|_{\mathcal{L}(V;V')}$, while c depends also on M. Hence, we get

$$\frac{1}{2}\int_0^t |u_n'(s)|^2\, ds + \frac{L}{2}\|u_n(t)\|^2 \leq \frac{\lambda}{2}|u_n(t)|^2 + c\|u_0\|^2 + \frac{1}{2}\int_0^T |h(s)|^2\, ds . \tag{2.37}$$

As it results in the proof of Theorem 2.2.2, the sequence $\{u_n\}$ is bounded independently of n in $C^0([0,T];H)$. Hence the right-hand side in bounded independently of n and so it follows that $u_n' \to u'$ in $L^2(0,T;H)$ (up to a subsequence of $n \nearrow \infty$). The limit u' is the same as the one in $L^2(0,T;V')$ and so we deduce the following estimate

$$\|u'\|_{L^2(0,T;H)} \leq \liminf_n \|u_n'\|_{L^2(0,T;H)} \leq c\left(\|u_0\| + \|h\|_{L^2(0,T;H)}\right) ,$$

with a new constant c depending only on L, λ, T and on the norm of the operator A. By comparison in (2.9), we deduce that also Au is bounded (with the same estimate of u') in $L^2(0,T;H)$.

Remark 2.2.5. First of all, let us notice that to have $h \in L^2(0,T;H)$ it is necessary that $g = 0$ in (2.5). Let us note that for the regularity result the weak coercivity assumption (2.13) can be reduced to the V-ellipticity of a, i.e. to the case in which $\lambda = 0$ in (2.13). Indeed, we could re-write equation (2.9) as

$$u' + Au + \lambda u = h + \lambda u .$$

Then, using the existence Theorem 2.2.2, we deduce that the right-hand side $h + \lambda u$ is in H, if $h \in H$, and the operator $A + \lambda I$ is weakly coercive in case A is V-elliptic. So, using Theorem 2.2.4, we can estimate the norm of u' and Au in $L^2(0,T;H)$ in terms of the norms of h and λu in $L^2(0,T;H)$. Observe that the norm of λu in $L^2(0,T;H)$ is already estimated in terms of the data thanks to Theorem 2.2.2. Moreover, let us note that from its proof it is easy to see that we could prove Theorem 2.2.4 under less restrictive assumptions on A and on h. E.g., we could treat the case in which A is substituted by the sum $A+B$ where A is the symmetric operator verifying the requested assumptions and B is a linear continuous operator from V with values in H. Moreover we can suppose that $h = h_1 + h_2$, where $h_1 \in L^2(0,T;H)$ and $h_2 \in H^1(0,T;V')$. Moreover, regarding the initial datum, we can observe that by choosing as test function in (2.20) $v = \zeta u_n'(t)$ with $\zeta \in C^\infty([0,T])$ such that $\zeta(0) = 0$, we do not need anymore the datum u_0 to be in V and we obtain analogous regularity results in the time interval $[\delta, T]$ for all $\delta \in (0,T)$. This phenomenon can be called *regularizing effect.*

Application 1. Let Ω be an open bounded subset of $\mathbb{R}^n$ with *sufficiently regular* boundary Γ. Denote by ν the outward normal derivative to the boundary Γ and by Δ and ∇ the spatial Laplacian and gradient, respectively. We would like to show here to which types of problems the previous abstract results can be applied. Let us consider, e.g., the following second order bilinear form defined on $H^1(\Omega) \times H^1(\Omega)$

$$a(u,v) = \int_\Omega \Big[(M(x)\nabla u)\cdot\nabla v + (\vec{b}(x)\nabla u)v + (\vec{c}(x)u)\cdot\nabla v + d(x)uv \Big]\, dx\,. \tag{2.38}$$

Suppose that all the coefficients are in $L^\infty(\Omega)$ and M satisfies

$$(M(x)\xi)\xi \geq L|\xi|^2 \quad \forall \xi \in \mathbb{R}^n, \quad \text{and a.e. in } \Omega\,.$$

Under these assumptions $a(u,v)$ is a bilinear, continuous, weakly coercive form on $H^1(\Omega) \times H^1(\Omega)$.

Suppose that $V = H_0^1(\Omega)$, then, consider the following *Cauchy-Dirichlet* problem:

$$\begin{cases} \partial_t u - \operatorname{div}(M\nabla u + \vec{c}u) + \vec{b}\cdot\nabla u + du = f & \text{in } \Omega\times(0,T)\,, \\ u(0) = u_0, \quad u = 0 & \text{on } \Gamma\times(0,T)\,. \end{cases} \tag{2.39}$$

In order to apply Theorem 2.2.2 we need $g \in L^2(0,T;H^{-1}(\Omega))$, $u_0 \in L^2(\Omega)$ and we have existence and uniqueness of solutions. We can apply the regularity result Theorem 2.2.4 in case $f \in L^2(0,T;L^2(\Omega))$ and $u_0 \in H_0^1(\Omega)$, a symmetric (i.e. $\vec{c} = \vec{b} = \vec{0}$, M symmetric). Moreover, in case Ω is of class $C^{1,1}$ and M is Lipschitz continuous, we can apply also the regularity results for elliptic equations, obtaining $u \in L^2(0,T;H^2(\Omega))$ and

$$\|u\|_{L^2(0,T;H^2(\Omega))} \leq C \left(\|f\|_{L^2(\Omega\times(0,T))} + \|u_0\|_{H^1(\Omega)} \right)\,,$$

where C is a positive constant depending on Ω, the coefficients and on T. Note that we cannot apply the regularity result in case, e.g., $u_0 \in H^1(\Omega)$ and it is not 0 at the boundary of Ω. It is a compatibility condition between the initial datum and the boundary datum (which is 0 in this case).

Let $V = H^1(\Omega)$ and consider now the *Cauchy-Neumann* problem:

$$\begin{cases} \partial_t u - \operatorname{div}(M\nabla u + \vec{c}u) + \vec{b}\cdot\nabla u + du = f & \text{in } \Omega \times (0,T)\,, \\ u(0) = u_0 \quad \text{in } \Omega, \quad (M\nabla u + \vec{c}u)\cdot\nu = g & \text{on } \Gamma \times (0,T)\,. \end{cases} \tag{2.40}$$

If $f \in L^2(0,T;L^2(\Omega) = L^2(\Omega \times (0,T))$, $g \in L^2(0,T;L^2(\Gamma)) = L^2(\Gamma \times (0,T))$, $u_0 \in H$, we can apply Theorem 2.2.2 and we get existence and uniqueness of solutions. To apply the regularity Theorem 2.2.4, we need $f \in L^2(0,T;L^2(\Omega))$, $g = 0$, and $u_0 \in H^1(\Omega)$, a symmetric (i.e. $\vec{c} = \vec{b} = \vec{0}$, M symmetric). Moreover, if Ω is of class $C^{1,1}$ and M is Lipschitz continuous, we get $u \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega))$.

Let $V = \{v \in H^1(\Omega) : v_{|\Gamma_0} = 0\}$, where (Γ_0, Γ_1) is a partition of Γ. Consider now the *Cauchy-mixed* problem:

$$\begin{cases} \partial_t u - \operatorname{div}(M\nabla u + \vec{c}u) + \vec{b}\cdot\nabla u + du = f & \text{in } \Omega \times (0,T)\,, \\ u(0) = u_0 \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma_0 \times (0,T),\ (M\nabla u + \vec{c}u)\cdot\nu = g & \text{on } \Gamma_1 \times (0,T)\,. \end{cases}$$

If $f \in L^2(0,T;L^2(\Omega)) = L^2(\Omega \times (0,T))$, $g \in L^2(0,T;L^2(\Gamma_1)) = L^2(\Gamma_1 \times (0,T))$, $u_0 \in L^2(\Omega)$, we can apply Theorem 2.2.2 and we get existence and uniqueness of solutions. To apply the regularity Theorem 2.2.4, we need $f \in L^2(0,T;L^2(\Omega))$, $g = 0$, and $u_0 \in V$, a symmetric (i.e. $\vec{c} = \vec{b} = \vec{0}$, M symmetric). Moreover, if Ω is smooth and M is Lipschitz continuous, we get $u \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^s(\Omega))$, only with $s \in (1,3/2)$. We do not have here the regularity results for elliptic equations.

Application 2. Let Ω be an open bounded subset of $\mathbb{R}^n$ with *sufficiently regular* boundary Γ. Denote by ν the outward normal derivative to the boundary Γ and by Δ and ∇ the spatial Laplacian and gradient, respectively. Then, we can consider the following fourth order problem. Let V be a closed subspace of $H^2(\Omega)$, such that $H^2_0(\Omega) \subset V$ and consider the following fourth order $H^2(\Omega)$-elliptic bilinear form

$$a(u,v) = \int_\Omega \left[\Delta u \Delta v + \nabla u \cdot \nabla v + uv\right]\, dx\,. \tag{2.41}$$

If $V = H^2_0(\Omega)$, then, we recover the following *Cauchy-Dirichlet* fourth order problem

$$\begin{cases} \partial_t u + \Delta^2 u - \Delta u + u = f & \text{in } \Omega \times (0,T)\,, \\ u(0) = u_0 \quad \text{in } \Omega, \quad u = 0, \quad \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma \times (0,T)\,. \end{cases}$$

Using the previous results, we get existence and uniqueness of solutions in case $f \in L^2(\Omega \times (0,T))$ and $u_0 \in L^2(\Omega)$. Moreover, $u_0 \in H^2_0(\Omega)$, then, we can apply the regularity result for elliptic equations (in case Ω is sufficiently regular) finding $u \in L^2(0,T;H^4(\Omega)) \cap H^1(0,T;L^2(\Omega))$.

We can also treat, with these results, elliptic equations with dynamic boundary conditions, i.e. the following types of systems

$$\begin{cases} -\Delta u = f & \text{in } \Omega \times (0,T)\,, \\ u(0) = w_0 \quad \text{on } \Gamma, \quad \partial_t u + \frac{\partial u}{\partial \nu} = g & \text{on } \Gamma \times (0,T)\,. \end{cases} \tag{2.42}$$

If we make the following change of variables $w := u_{|\Gamma}$, once we have find w, we can recover u by solving the following elliptic problem

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega\,, \\ u = w & \text{on } \Gamma\,. \end{cases} \tag{2.43}$$

We choose $V = H^{1/2}(\Gamma)$, $H = L^2(\Gamma)$, $(u,v) = \int_\Gamma uv\,ds$. We introduce the linear and continuous operator $\mathcal{R} : H^{1/2}(\Gamma) \to H^1(\Omega)$, such that $(\mathcal{R}w)_{|\Gamma} = w$ for all $w \in H^{1/2}(\Gamma)$. Then, we can re-write problem (2.42) as

$$\begin{cases} \partial_t w + \dfrac{\partial \mathcal{R}w}{\partial \nu} = g & \text{on } \Gamma \times (0,T)\,, \\ w(0) = w_0 \quad \text{on } \Gamma\,. \end{cases} \tag{2.44}$$

In order to find the corresponding variational formulation, we can test by a generic test function $v \in H^{1/2}(\Gamma)$, and, if we call formally $a(w,v) = \int_\Gamma \frac{\partial \mathcal{R}w}{\partial \nu} v\,ds$, integrating by parts in time, we get

$$\begin{aligned} a(w,v) &= \int_\Gamma \frac{\partial \mathcal{R}w}{\partial \nu} v\,ds = \int_\Omega (\nabla \mathcal{R}w)(\nabla \mathcal{R}v)\,dx + \int_\Omega (\Delta \mathcal{R}w)\mathcal{R}v\,dx \\ &= \int_\Omega (\nabla \mathcal{R}w)(\nabla \mathcal{R}v)\,dx\,. \end{aligned}$$

We can verify now that a is a bilinear, continuous, and weakly coercive form, i.e.

$$\begin{aligned} &|a(w,v)| \le \|\mathcal{R}w\|_{H^1(\Omega)} \|\mathcal{R}v\|_{H^1(\Omega)} \le \|\mathcal{R}\|_{\mathcal{L}(H^{1/2}(\Gamma);H^1(\Omega))} \|w\|_{H^{1/2}(\Gamma)} \|v\|_{H^{1/2}(\Gamma)}\,, \\ &a(v,v) + \|v\|^2_{L^2(\Gamma)} \ge \alpha_p \|\mathcal{R}v\|^2_{H^1(\Omega)} \ge \alpha \|v\|^2_{H^{1/2}(\Gamma)}\,, \end{aligned}$$

where we have used the Poincarè inequality. We are in the position of applying our existence theorem: if $g \in L^2(0,T;H^{-1/2}(\Gamma))$, $w_0 \in L^2(\Gamma)$, then there exists a unique solution $w \in L^2(0,T;H^{1/2}(\Gamma)) \cap H^1(0,T;H^{-1/2}(\Gamma))$ and $w \in C^0([0,T];L^2(\Gamma))$. Hence, $u = \mathcal{R}w \in L^2(0,T;H^1(\Omega)) \cap C^0([0,T];L^2(\Omega))$. The bilinear form a is also symmetric. Finally, we can apply also our regularity result in case $w_0 \in H^{1/2}(\Gamma)$, getting $\partial_t w \in L^2(\Gamma \times (0,T))$, $\partial_t u \in L^2(\Omega \times (0,T))$.

Chapter 3

Abstract linear hyperbolic equations

Let us consider the following Cauchy-Dirichlet problem for the wave equation

$$\begin{aligned} & u_{tt} - c^2\Delta u = f \quad \text{in } \Omega\times(0,T)\,, \\ & u(\cdot,0) = u_0, \quad u_t(\cdot,0) = v_0 \quad \text{in } \Omega\,, \\ & u = 0 \quad \text{on } \partial\Omega\times(0,T)\,. \end{aligned}$$

We consider the function

$$u:\, t\mapsto u(t)\in H_0^1(\Omega)(=V)\,.$$

Then, given $f\in L^2(0,T;L^2(\Omega))$, $u_0\in H_0^1(\Omega)$, $v_0\in L^2(\Omega)$, we search for $u\in L^2(0,T;H_0^1(\Omega)$, u'', $\Delta u\in L^2(0,T;H^{-1}(\Omega))$. We consider the operator $A\,:\,H_0^1(\Omega)\to H^{-1}(\Omega)$ such that

$$\langle Au, v\rangle = \int_\Omega \nabla u\nabla v\,dx \quad \forall u,\, v\in H_0^1(\Omega)\,.$$

We define u be a *weak solution* if $u\in L^2(0,T;H_0^1(\Omega)$, $u''\in L^2(0,T;H^{-1}(\Omega))$, $u'\in L^2(0,T;L^2(\Omega))$ and

$$\begin{aligned} & \langle u''(t), v\rangle + \langle Au(t), v\rangle = \langle f(t), v\rangle \quad \forall v\in H_0^1(\Omega) \quad \text{and for a.e. } t\in[0,T]\,, \\ & u(0) = u_0, \quad u'(0) = v_0\,. \end{aligned}$$

We will show in Thm. 3.1.7 that $u\in C^0([0,T];V)\cap C^1([0,T];H)$ and this will give the meaning to the initial conditions and we can interpret the equation in $\mathcal{D}'(0,T)$ as

$$\frac{d^2}{dt^2}\langle u(t), v\rangle + \langle Au(t), v\rangle = \langle f(t), v\rangle \quad \text{for all } v\in V, \quad \text{in } \mathcal{D}'(0,T)\,,$$

indeed we have

$$\int_0^T \langle u''(t), v\rangle \varphi(t)\, dt = \langle \int_0^T u''(t)\varphi(t)\, dt, v\rangle$$
$$= \langle \int_0^T u(t)\varphi''(t)\, dt, v\rangle = \int_0^T \langle u(t), v\rangle \varphi''(t)\, dt\,.$$

In the following section we will give an abstract formulation of this hyperbolic problem and we will give proper existence and uniqueness results.

3.1 Abstract formulation

Let (V, H, V') be an Hilbert triplet with V separable. We would like to find a suitable notion of solution for the following abstract Cauchy problem:

$$u''(t) + Au(t) = f(t) \quad \text{in } V' \text{ and a.e. in } (0, T)\,, \tag{3.1}$$
$$u(\cdot, 0) = u_0\,, \tag{3.2}$$
$$u'(\cdot, 0) = v_0\,, \tag{3.3}$$

where $A : V \to V'$ is a linear, continuous, symmetric and weakly coercive operator (cf. (2.13)), u_0 and v_0 are initial data and f is a known source.

Before stating the main results, let us state some preliminary results.

Lemma 3.1.1. *If W is an Hilbert space and b is a bilinear, symmetric form on $W \times W$, then the following formula holds true*

$$2b(u - v, u) = b(u - v, u - v) + b(u, u) - b(v, v), \quad \forall u,\, v \in W\,.$$

Proof. Obvious, by definition of b.

Lemma 3.1.2. *Let $\beta,\, \gamma \geq 0$, $\{S_m\}$ be a sequence such that $S_1 \leq \beta$, $S_m \leq \beta + \gamma \sum_{k=1}^{m-1} S_k$ for every $m \geq 2$. Then $S_m \leq \beta(1 + \gamma)^{m-1}$, for every $m \geq 2$.*

Proof. For $m = 1$ it is already true. Let it be true for $k \leq m$ and prove it for $m + 1$. We have

$$S_{m+1} \leq \beta + \gamma \sum_{k=1}^{m} S_k \leq \beta + \beta\gamma \sum_{k=1}^{m} (1 + \gamma)^{k-1}$$
$$\leq \beta + \beta\gamma \left(\frac{1 - (1 + \gamma)^m}{1 - (1 + \gamma)} \right) = \beta(1 + \gamma)^m\,.$$

Lemma 3.1.3. *Let A, $B \in \mathbb{R}$, $h \in L^1(0,T)$, $h \geq 0$, a.e. in $(0,T)$, $R \in L^2(0,T)$ such that*

$$R^2(t) \leq A^2 + B^2 \int_0^t R^2(\tau)\,d\tau + \int_0^t h(\tau)\,R(\tau)\,d\tau \quad \text{for all } t \in [0,T]$$

Then

$$R^2(t) \leq \left(2A^2 + \frac{1}{2}\|h\|^2_{L^1(0,T)}\right) \mathrm{e}^{2B^2 t} \quad \text{for all } t \in [0,T]\,.$$

Proof. The proof is based on [2, Lemmas A.4, A.5, p.157].

Let us start now with an existence result for *strong* solutions under strong assumption on the data. Then, we will use this result in order to prove existence of weak solutions.

Theorem 3.1.4. *Let $A : V \to V'$ be a linear, continuous, symmetric and weakly coercive operator (cf. (2.13)), $u_0 \in V$, $v_0 \in V$, $f \in W^{2,1}(0,T;V') + H^1(0,T;H)$ with $f(0) - Au_0 \in H$. Then, there exists at least a solution u to problem (3.1–3.3) such that $u \in W^{2,\infty}(0,T;H) \cap W^{1,\infty}(0,T;V)$.*

Proof. We will use here a time discretization scheme. Fix $n \in \mathbb{N}$ (which we will take sufficiently large in the following) and split the interval $[0,T]$ in n subintervals of length $\tau = T/n$. Let $f = f_1 + f_2$, where $f_1 \in W^{2,1}(0,T;V')$, $f_2 \in H^1(0,T;H)$. Since both f_1 and f_2 are continuous in time, we can put

$$f_1^i := f_1(\tau i), \quad f_2^i := f_2(\tau i), \quad f^i = f_1^i + f_2^i\,.$$

Let us rewrite here our problem.

$$v'(t) + Au(t) = f(t), \quad v = u' \quad \text{in } V' \text{ and a.e. in } (0,T)\,, \tag{3.4}$$

$$u(\cdot,0) = u_0\,, \tag{3.5}$$

$$v(\cdot,0) = v_0\,, \tag{3.6}$$

We consider then the following discretized problem.
Problem (P_τ). Find two vectors $(u^0, u^1, \ldots, u^n) \in V^{n+1}$, $(v^0, v^1, \ldots, v^n) \in V^{n+1}$ such that $u^0 = u_0$, $v^0 = v_0$ and (for $i = 1, \ldots, n$)

$$\begin{cases} v^i = \dfrac{u^i - u^{i-1}}{\tau}\,, \\ \dfrac{v^i - v^{i-1}}{\tau} + Au^i = f^i\,. \end{cases}$$

Existence of a unique solution for Problem (P_τ).

We know u^0 and v^0, suppose to know u^{i-1} and v^{i-1} we want now to find u^i such that

$$\frac{u^i}{\tau} + \tau A u^i = \tau f^i + v^{i-1} + \frac{u^{i-1}}{\tau}.$$

Since A is weakly coercive, then $\frac{I}{\tau} + \tau A$ is an isomorphism, indeed

$$\begin{aligned}\tau\langle Au, u\rangle + \frac{1}{\tau}(u,u) &\geq \tau L\|u\|^2 - \tau\lambda|u|^2 + \frac{1}{\tau}|u|^2 \\ &\geq \tau L\|u\|^2 ,\end{aligned}$$

if $1/\tau \geq \tau\lambda$, i.e. $\tau \leq \sqrt{1/\lambda}$, i.e. if n is sufficiently large. Hence, the bilinear form $(\tau A + I/\tau)$ is $V-$elliptic and so there exists a unique u^i solution to Problem (P_τ).

A priori estimates.

Introduce the auxiliary vector $(z^0, \dots, z^n) \in V^{n+1}$ such that

$$z^0 = z^1 := f(0) - Au_0, \quad z^i = \frac{v^i - v^{i-1}}{\tau}, \quad i = 2, \dots, n\,.$$

Take, moreover, the piecewise constant functions

$$\overline{u}_\tau(t) = u^i, \quad \overline{v}_\tau(t) = v^i, \quad \overline{z}_\tau(t) = z^i, \quad \overline{f}_\tau(t) = f^i\,, \quad \text{for } t \in ((i-1)\tau, i\tau]$$

and the linear interpolants of the constants u^i, v^i, z^i

$$\begin{aligned}\hat{u}_\tau &= u^1 + \frac{u^1 - u^{i-1}}{\tau}(t - i\tau), \quad \hat{v}_\tau = v^1 + \frac{v^1 - v^{i-1}}{\tau}(t - i\tau), \\ \hat{z}_\tau &= z^1 + \frac{z^1 - z^{i-1}}{\tau}(t - i\tau), \quad \text{for } t \in ((i-1)\tau, i\tau]\,.\end{aligned}$$

Notice that they coincide with the constants u^i and v^i at times τi.

Hence, we can rewrite Problem (P_τ) as: find $\overline{u}_\tau, \hat{u}_\tau, \overline{v}_\tau, \hat{v}_\tau, \overline{z}_\tau \in L^1(0,T;V)$ such that

$$\overline{z}_\tau(t) + A\overline{u}_\tau(t) = \overline{f}_\tau(t)\,, \tag{3.7}$$

$$\overline{v}_\tau(t) = \hat{u}'_\tau(t), \quad \overline{z}_\tau(t) = \hat{v}'_\tau(t)\,, \tag{3.8}$$

$$\hat{u}_\tau(0) = u_0, \quad \hat{v}_\tau(0) = v_0\,, \tag{3.9}$$

where $t \in (0,T)$. We would like to find now estimates on $\overline{u}_\tau$, $\hat{u}_\tau, \dots$ and pass to the limit as $\tau \to 0$ in (3.7–3.9), recovering a solution to our problem.

Take the difference between (3.7) written for the index i and (3.7) written for the index $i-1$ finding

$$z^i - z^{i-1} + \tau A v^i = f^i - f^{i-1}.$$

Sum to both sides $\lambda\tau v^i$ (cf. (2.13)) and multiply the resulting equation by z^i getting

$$\begin{aligned}&\frac{1}{2}|z^i|^2 - \frac{1}{2}|z^{i-1}|^2 + \frac{1}{2}|z^i - z^{i-1}|^2 + \frac{1}{2}a_\lambda(v^i, v^i) \\ &- \frac{1}{2}a_\lambda(v^{i-1}, v^{i-1}) + \frac{\tau^2}{2}a_\lambda(z^i, z^i) = \langle f^i - f^{i-1} + \lambda\tau v^i, z^i\rangle\,,\end{aligned} \tag{3.10}$$

where $a_\lambda(u,v) := \langle Au, v\rangle + \lambda(u,v)$. Here we have used Lemma 3.1.1 with $b = a_\lambda$. Indeed, we have

$$a_\lambda(v^i, v^i - v^{i-1}) = \frac{1}{2}a_\lambda(\tau z^i, \tau z^i) + \frac{1}{2}a_\lambda(v^i, v^i) - \frac{1}{2}a_\lambda(v^{i-1}, v^{i-1})\,.$$

Summing up the equality (3.10) for $i = 1, \ldots m$, we get

$$S_m = \frac{1}{2}|z^0|^2 + \frac{1}{2}a_\lambda(v^0, v^0) + \sum_{i=1}^{m} \tau\langle\frac{f^i - f^{i-1}}{\tau} + \lambda v^i, z^i\rangle\,,$$

where

$$S_m = \frac{1}{2}|z^m|^2 + \frac{1}{2}\sum_{i-1}^{m}|z^i - z^{i-1}|^2 + \frac{1}{2}a_\lambda(v^m, v^m) + \sum_{i=1}^{m}\frac{\tau^2}{2}a_\lambda(z^i, z^i)\,.$$

Now, $|z^0|^2 = |Au_0 - f(0)|^2 \le C$, $\quad \frac{1}{2}a_\lambda(v^0, v^0) \le C\|v_0\|^2 \le C$, with C independent of n. Hence, denoting by C_i some constants independent of n, we have

$$S_m \le C_1 + \sum_{i=1}^{m}\tau\langle\frac{f^i - f^{i-1}}{\tau} + \lambda v^i, z^i\rangle, \quad m \ge 1\,.$$

Now we need to estimate the term $\sum_{i=1}^{m}\tau\langle\frac{f^i - f^{i-1}}{\tau} + \lambda v^i, z^i\rangle$. Use the notation $g_k^i := \frac{f_k^i - f_k^{i-1}}{\tau}$, with $k = 1, 2$ and write down

$$\sum_{i=1}^{m}\tau\langle\frac{f^i - f^{i-1}}{\tau} + \lambda v^i, z^i\rangle \le R_1(m) + R_2(m) + R_3(m)\,,$$

where

$$
\begin{aligned}
R_1(m) &= \sum_{i=1}^{m}(g_1^i, v^i - v^{i-1}) = (g^m, v^m) + \sum_{i=1}^{m-1}(g_1^i, v^i) - (g_1^1, v^0) - \sum_{i=2}^{m}(g_1^i, v^{i-1}) \\
&= (g^m, v^m) - (g_1^1, v^0) - \sum_{i=2}^{m}\langle \frac{g_1^i - g_1^{i-1}}{\tau}, v^{i-1}\rangle \\
&\leq \sigma \max_{1\leq i\leq m} \|v^i\|^2 + C_2(\sigma)\,,
\end{aligned}
$$

and

$$
\begin{aligned}
R_2(m) &= \sum_{i=1}^{m}\tau|g_2^i||z^i| \leq \frac{1}{2}\sum_{i=1}^{m}\tau|z^i|^2 + \frac{1}{2}\|f_2'\|^2_{L^2(0,T;H)}\,, \\
R_3(m) &= \sum_{i=1}^{m}\lambda\tau|v^i||z^i| \leq C_3\sum_{i=1}^{m}\tau\|v^i\|^2 + \lambda\sum_{i=1}^{m}\tau|z^i|^2
\end{aligned}
$$

for every $\sigma > 0$ which we will choose properly. The estimate for R_1 has been obtained using the fact that

$$
\begin{aligned}
\|g_1^i\|_* &\leq \frac{1}{\tau}\int_{(i-1)\tau}^{i\tau}\|f_1'(t)\|_*\,dt \leq \|f_1'\|_{C^0([0,T];V')}\,, \\
\left\|\frac{g_1^i - g_1^{i-1}}{\tau}\right\|_* &\leq \frac{1}{\tau}^2\left\|\int_{(i-1)\tau}^{i\tau}(f_1'(t) - f_1'(t-\tau))\,dt\right\|_* \\
&\leq \frac{1}{\tau}^2\int_{(i-1)\tau}^{i\tau}\int_{t-\tau}^{t}\|f_1''(s)\|_*\,ds\,dt \\
&\leq \frac{1}{\tau}\|f_1\|_{L^1((i-1)\tau,i\tau;V')}\,.
\end{aligned}
$$

Let us denote by

$$
N_m = \frac{1}{2}\left(|z^m|^2 + \sum_{i=1}^{m}|z^i - z^{i-1}|^2\right) + \frac{L}{2}\left(\|v^m\|^2 + \sum_{i=1}^{m}\tau^2\|z^i\|^2\right)\,.
$$

Choosing properly σ, we get

$$
\begin{aligned}
|R_1(m)| &\leq \frac{1}{2}\max_{1\leq i\leq m} N_i\,, \\
|R_2(m)| &\leq C\left(1 + \sum_{i=1}^{m}\tau N_i\right)\,, \\
|R_3(m)| &\leq C\sum_{i=1}^{m}\tau N_i\,.
\end{aligned}
$$

Hence, if $k \le m$, we have

$$\begin{aligned} N_k &\le C_4 \left(1 + \sum_{i=1}^{k} \tau N_i\right) + \frac{1}{2} \max_{1\le i\le k} N_i \\ &\le C_4 \left(1 + \sum_{i=1}^{m} \tau N_i\right) + \frac{1}{2} \max_{1\le i\le m} N_i \,. \end{aligned}$$

Taking the maximum over $1 \le k \le m$, we obtain

$$\frac{1}{2} N_m \le \frac{1}{2} \max_{1\le i\le m} N_i \le C_4 \left(1 + \sum_{i=1}^{m} \tau N_i\right) .$$

Taking then $\tau \le 1/(4C_4)$, we have

$$\frac{1}{2} N_m \le 4C_4 \left(1 + \sum_{i=1}^{m-1} \tau N_i\right) \quad \text{for every } m \ge 2$$

and $N_1 \le 4C_4$.

Applying now the discrete Gronwall Lemma 3.1.2 with $\beta = 4C_4$, $\gamma = 4C_4\tau$, we obtain

$$\begin{aligned} N_m &\le 4C_4 \left(1 + \frac{4C_4T}{n}\right)^{m-1} \\ &\le 4C_4 \left(\left(1 + \frac{4C_4T}{n}\right)^{\frac{n}{4C_4T}}\right)^{4C_4T} \\ &\le 4C_4 \mathrm{e}^{4C_4T} \,. \end{aligned}$$

Hence, for τ sufficiently small (n sufficiently large), we get the following estimate

$$\|\overline{z}_\tau\|^2_{L^\infty(0,T;H)} + \tau\|\hat{z}'_\tau\|^2_{L^2(0,T;H)} + \|\overline{v}_\tau\|^2_{L^\infty(0,T;V)} + \tau\|\hat{v}'_\tau\|^2_{L^2(0,T;V)} \le C_5 \,. \tag{3.11}$$

By comparison, we also deduce

$$\begin{aligned} &\|\hat{u}_\tau\|_{L^\infty(0,T;V)} \le \|u_0\| + T\|\hat{u}'_\tau\|_{L^\infty(0,T;V)} \le \|u_0\| + T\|\overline{v}_\tau\|_{L^\infty(0,T;V)} \le C_6, \qquad (3.12) \\ &\|\hat{v}_\tau\|_{L^\infty(0,T;V)} \le \max_{1\le i\le n} \left(\|v^i\| + \|v^{i-1}\|\right) \le 2\|\overline{v}_\tau\|_{L^\infty(0,T;V)} + \|v_0\| \le C_7, \\ &\|\hat{v}_\tau\|_{W^{1,\infty}(0,T;H)} \le \|\hat{v}_\tau\|_{L^\infty(0,T;H)} + \|\overline{z}_\tau\|_{L^\infty(0,T;H)} \le C_8 \,. \end{aligned}$$

Moreover, we also get

$$\begin{aligned}
&\|\overline{f}_\tau - f\|_{L^1(0,T;V')} = \sum_{i=1}^{n} \int_{(i-1)\tau}^{i\tau} \left\| \int_t^{i\tau} f'(s)\, ds \right\|_* dt \le \tau \|f'\|_{L^1(0,T;V')} \le \tau C_9\,,\\
&\|\overline{u}_\tau - \hat{u}_\tau\|_{L^\infty(0,T;V)} \le \tau \|\overline{v}_\tau\|_{L^\infty(0,T;V)} \le \tau C_{10},\\
&\|\overline{v}_\tau - \hat{v}_\tau\|_{L^\infty(0,T;H)} \le \tau \|\overline{z}_\tau\|_{L^\infty(0,T;H)} \le \tau C_{11}\,. \qquad (3.13)
\end{aligned}$$

Hence, for a subsequence of $n \to \infty$ or $\tau \to 0$, we get the following convergences

$$\begin{aligned}
&\hat{u}_\tau \to u \quad \text{weakly star in } W^{1,\infty}(0,T;V)\,,\\
&\overline{u}_\tau \to \tilde{u} \quad \text{weakly star in } L^\infty(0,T;V)\,,\\
&\hat{v}_\tau \to v \quad \text{weakly star in } W^{1,\infty}(0,T;H)\,,\\
&\overline{v}_\tau \to \tilde{v} \quad \text{weakly star in } L^\infty(0,T;V)\,,\\
&\overline{z}_\tau \to z \quad \text{weakly star in } L^\infty(0,T;H)\,.
\end{aligned}$$

Moreover, due to (3.13), we also have

$$\overline{u}_\tau - \hat{u}_\tau \to 0 \quad \text{in } L^\infty(0,T;V) \quad \overline{v}_\tau - \hat{v}_\tau \to 0 \quad \text{in } L^\infty(0,T;H)\,,$$

from which we deduce $u = \tilde{u}$ and $v = \tilde{v}$. In order to pass to the limit in PROBLEM (P_τ), let us note that

$$A\overline{u}_\tau \to Au \quad \text{weakly in } L^2(0,T;V')\,.$$

Using the assumptions on f together with (3.13), we can also pass to the limit in $\overline{f}_\tau$; indeed, we have

$$\|\overline{f}_\tau\|_{L^\infty(0,T;V')+L^\infty(0,T;H)} \le C_{12} \left(\|f_1\|_{C^0([0,T];V')} + \|f_2\|_{C^0([0,T];H)} \right)\,.$$

Finally, we can pass to the limit in the initial conditions since $W^{1,\infty}(0,T;V) \subseteq C^0([0,T];V)$ and so

$$\hat{u}_\tau(0) \to u(0) \quad \text{weakly in } V\,,$$

and so we recover $u(0) = u_0$. Analogously, since $W^{1,\infty}(0,T;H) \subseteq C^0([0,T];H)$, we recover the condition $v(0) = v_0$. Hence u is the desired solution to our problem. This concludes the proof of Thm. 3.1.4.

Let us prove now the following uniqueness and continuous dependence of the solution with respect to the data result.

Theorem 3.1.5. *Let $f = f_1 + f_2$, $f_1 \in W^{1,1}(0,T;V')$, $f_2 \in L^1(0,T;H)$, $u_0 \in V$, $v_0 \in H$, $A : V \to V'$ be a linear, continuous, weakly coercive and symmetric operator, u be a solution to (3.1–3.3). Then, there exists a positive constant $C = C(T,\lambda,L,M)$ such that*

$$\|u\|_{C^1([0,T];H)} + \|u\|_{C^0([0,T];V)} \le C\left(\|u_0\| + |v_0| + \|f_1\|_{W^{1,1}(0,T;V')} + \|f_2\|_{L^1(0,T;H)}\right) .$$

Proof. Test (3.1) by u', integrate over $(0,t)$, $t \in (0,T]$, getting

$$\begin{aligned}
&\frac{1}{2}|u'(t)|^2 + \frac{L}{2}\|u(t)\|^2 \le \frac{1}{2}|v_0|^2 + \frac{1}{2}(\lambda + M)\|u_0\|^2 + \lambda\int_0^t (u(s),u'(s))\,ds \\
&+\|f_1\|_*\|u(t)\| + \|f_1(0)\|_*\|u_0\| + \int_0^t \|f_1'(s)\|_*\|u(s)\|\,ds + \int_0^t |f_2(s)||u'(s)|^2\,ds \\
&\le C_1\left(1 + \int_0^t |u(s)|^2\,ds + \int_0^t |u'(s)|^2\,ds\right) + \int_0^t \|f_1'(s)\|_*\|u(s)\|\,ds \\
&+\int_0^t |f_2(s)||u'(s)|^2\,ds + \frac{L}{4}\|u(t)\|^2 + C_2\left(\|f_1(t)\|_*^2 + \|f_1(0)\|_*\|u_0\|\right) .
\end{aligned}$$

Using now the Gronwall Lemma 3.1.3, we get the desired estimate.

Remark 3.1.6. Let us notice that there is a gap between the regularity of the existence and the uniqueness theorems. Hence, we will prove in the following Theorem 3.1.7 the existence of a *weak* solution in the regularity framework of the uniqueness theorem.

Theorem 3.1.7. *Let $u_0 \in V$, $v_0 \in H$, $f \in W^{1,1}(0,T;V') + L^1(0,T;H)$, $A : V \to V'$ be a linear, continuous, symmetric and weakly coercive operator (cf. (2.13)). Then, there exists a unique $u \in C^1([0,T];H) \cap C^0([0,T];V)$ solution of (3.1–3.3).*

Proof. The idea is to approximate these data with more regular ones. Apply Theorem 3.1.4 and prove that the solution of the approximating problem $\{u_n\}$ is a Cauchy sequence in $C^0([0,T];V)$ as well as its derivative $\{u_n'\}$ is a Cauchy sequence in $C^0([0,T];H)$. Then, we will pass to the limit in the approximating problem (or in its integrated in time version) obtaining the desired *weak* solution.

The main problem consists in the approximation of the data. In the assumptions of Theorem 3.1.4, we had $u_0 \in V$, $v_0 \in V$ and $f(0) - Au_0 \in H$. Hence, in particular, we need $u_0 \in \mathcal{D}(A;H)(=\{u_0 \in V : Au_0 \in H\})$. Hence, we search for $u_{0n} \in \mathcal{D}(A;H)$ for all n such that $u_{0n} \to u_0 \in V$. This is possible because $\mathcal{D}(A;H)$ is densely embedded into V. Indeed it is sufficient to take $u_{0n} \in V$ solution of (cf. 2.13)

$$u_{0n} + \frac{1}{n}(\lambda u_{0n} + Au_{0n}) = u_0 . \tag{3.14}$$

Let us call $J_\lambda := \lambda I + A$. J_λ is $V-$elliptic and so the bilinear form a_λ associated defines a scalar product in V equivalent to the usual one and J_λ is an isomorphism isometric of V on V' (if we take in V such a scalar product). Hence we have

$$\langle J_\lambda u, v\rangle = ((u,v)) = ((Ju, Jv))_* \quad \forall u,\, v \in V\,.$$

Testing now (3.14) by $J_\lambda u_{0n}$, we get

$$\|u_{0n}\|^2 + \frac{1}{n}|J_\lambda u_{0n}|^2 = \langle J_\lambda u_{0n}, u_0\rangle = ((u_{0n}, u_0)) \le \|u_{0n}\|\|u_0\|\,.$$

Hence $\|u_{0n}\| \le \|u_0\|$. Since u_{0n} is bounded in V independently of n, we also have that u_{0n} converges weakly to $\tilde u$ in V. But $\frac{1}{n}J_\lambda u_{0n}$ converges strongly to 0 in V' and weakly to $J\tilde u$ in V'. We can pass to the limit in (3.14) getting $\tilde u = u_0$. Hence, $u_{0n} \to u_0$ weakly in V and $\|u_{0n}\| \le \|u_0\|$, and so $u_{0n} \to u_0$ strongly in V because

$$\|u_{0n} - u_0\|^2 = \|u_{0n}\|^2 + \|u_0\|^2 - 2((u_{0n}, u_0)) \le 2\|u_0\|^2 - 2((u_{0n}, u_0))\,,$$

passing to the limit as $n \to \infty$, we get 0 on the right hand side, and so we conclude the proof of the strong convergence.

Moreover, always using density results, we take, for all n, $v_{0n} \in V$ such that $v_{0n} \to v_0$ in H, $f_{1n} \in W^{2,1}(0,T;V')$ such that $f_{1n}(0) \in H$, $f_{1,n} \to f_1$ in $W^{1,1}(0,T;V')$, $f_{2n} \in C^1([0,T];H)$ such that $f_{2n} \to f_2$ in $L^1(0,T;H)$. We can take f_1, $f_{01n} \in H$ such that $f_{01n} \to f_1(0)$ in V' and define $f_{1n} := f_{01n} + \int_0^t g_n(s)\,ds$ with $g_n \to f_1'$ in $L^1(0,T;V')$, $g_n \in W^{1,1}(0,T;V')$. Regarding f_2, instead, we can take the prolongation at 0 of f_2 outside $(0,T)$, take the convolution with a regularizing sequence and truncate on $[0,T]$ (cf. [1, p. 111–115]).

This concludes the idea of the proof.

The nonlinear case. Let us consider the following nonlinear hyperbolic problem

$$\begin{aligned} &u_{tt} - \Delta u = f(x,t)\beta(u,u_t) \quad \text{in } \Omega\times(0,T)\,, \\ &u = 0 \quad \text{on } \partial\Omega\times(0,T)\,, \\ &u(\cdot,0) = u_0, \quad u_t(\cdot,0) = v_0 \quad \text{in } \Omega\,, \end{aligned} \tag{3.15}$$

where $\beta : \mathbb{R}^2 \to \mathbb{R}$ is Lipschitz continuous with respect to both arguments, i.e.

$$|\beta(u_1,v_1) - \beta(u_2,v_2)| \le \Lambda_\beta\{|u_1-u_2| + |v_1-v_2|\} \quad \text{for all } (u_i,v_i), \quad i = 1,2\,,$$

and $f \in L^1(0,T;L^\infty(\Omega))$. Hence, we take $V = H_0^1(\Omega)$ (consider the homogeneous Cauchy-Dirichlet problem), $A : H_0^1(\Omega) \to H^{-1}(\Omega)$ defined as $\langle Au, v\rangle = \int_\Omega \nabla u \nabla v$ for all $u, v \in H_0^1(\Omega)$. Then, for a suitable regular Ω A is H_0^1-elliptic, due to the Poincaré inequality.

Then, we try to apply a fixed point argument. Take $w \in C^1([0,T];H)$ and notice that $\beta(w,w_t) \in C^0([0,T];H)$, hence $f(x,t)\beta(w(x,t),w_t(x,t)) \in L^1(0,T;H)$. Consider then the following

PROBLEM (P_w). Find u such that

$$\begin{aligned}
&u_{tt} - \Delta u = f\beta(w,w_t) \quad \text{in } \Omega \times (0,T)\,,\\
&u = 0 \quad \text{on } \partial\Omega \times (0,T)\,,\\
&u(\cdot,0) = u_0, \quad u_t(\cdot,0) = v_0 \quad \text{in } \Omega\,,
\end{aligned}$$

with $u_0 \in H$, $v_0 \in V$, $f \in L^1(0,T;L^\infty(\Omega))$. Due to Thm. 3.1.7, there exists a unique u solution to PROBLEM (P_w). Let us call Ψ the map going from a closed ball of $C^1([0,T];H)$ into the same ball in $C^1([0,T];H) \cap C^0([0,T];V)$ (for T sufficiently small) which maps w into the solution u of PROBLEM (P_w). We want to show now that Ψ is a contraction in case T is small. Indeed we have the estimates

$$\|u\|_{C^1([0,T];H)} + \|u\|_{C^0([0,T];V)} \le C\left(\|u_0\| + |v_0| + \|f\beta(w,w_t)\|_{L^1(0,T;H)}\right).$$

and

$$\begin{aligned}
&\|u_1 - u_2\|_{C^1([0,T];H)} + \|u_1 - u_2\|_{C^0([0,T];V)}\\
&\le C\Lambda_\beta \int_0^t \left(\int_\Omega |f(x,s)|^2 \left(|w_1 - w_2| + |w_1' - w_2'|\right)^2 (x,s)\,dx\right)^{1/2} ds\\
&\le \sqrt{2}C\Lambda_\beta \int_0^t \|f(s)\|_{L^\infty(\Omega)} \left(|(w_1 - w_2)(s)| + |(w_1' - w_2')(s)|\right) ds\\
&\le \sqrt{2}C\Lambda_\beta \int_0^t \|f(s)\|_{L^\infty(\Omega)} \|w_1 - w_2\|_{C^1([0,T];H)}\,.
\end{aligned}$$

Using the absolute continuity of the integral, we have that there exists $\delta > 0$ such that if $|I| \le \delta$ then

$$\int_I \|f(s)\|_{L^\infty(\Omega)}\,ds \le \frac{1}{2\sqrt{2}C\Lambda_\beta}\,.$$

Then, applying the contraction Theorem on the time interval $[0,\delta]$, we get the existence of a unique solution u on $[0,\delta]$. Taking then $u(\delta)$ as datum for the problem, we can continue the same procedure on $[\delta,2\delta]$, etc... till we arrive to the whole $[0,T]$.

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