A nonlinear degenerating PDE system for phase transitions in thermoviscoelastic materials

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joint work with Riccarda Rossi (University of Brescia, Italy)

Plan of the Talk

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- Related open problems

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with a proper choice of our **internal energy functional** (depending on the state variables) and of **the pseudo-potential of dissipation** (depending on the dissipative variables).

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$$\vartheta_t + \chi_t \vartheta - \Delta \vartheta = g \qquad \text{(I)}$$

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$$\mathbf{u}_{tt} - \operatorname{div}\,((1-\chi)\varepsilon(\mathbf{u}) + \chi\varepsilon(\mathbf{u}_t)) = \mathbf{f} \qquad \text{(III)}$$

which describes a **phase transition phenomenon for a two-phase viscoelastic system**, occupying a bounded domain $\Omega \subseteq \mathbb{R}^N$, $N = 1, 2, 3$, during a time interval $[0, T]$.

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[THM. 1] **Local in time well-posedness** for a suitable formulation of (I–III)+I.C.+B.C. in the 3D (in space) setting

[THM. 2] **Global in time well-posedness** in the 1D setting

The free-energy functional

We take into account of elasticity effects by choosing

$$\Psi(\vartheta, \varepsilon(\mathbf{u}), \chi, \nabla\chi) = c_V\vartheta(1 - \log\vartheta) - \frac{\lambda}{\vartheta_c}(\vartheta - \vartheta_c)\chi + \frac{(1-\chi)\varepsilon(\mathbf{u})\mathcal{R}_e\varepsilon(\mathbf{u})}{2} + W(\chi) + \frac{\nu}{2}|\nabla\chi|^2$$

- $\varepsilon(\mathbf{u})$ the linearized symmetric strain tensor, namely $\varepsilon_{ij}(\mathbf{u}) := (u_{i,j} + u_{j,i})/2$, $i, j = 1, 2, 3$
- $(1-\chi)$ the local proportion of the non viscous phase, e.g. the solid phase in solid-liquid phase transitions
- $\mathcal{R}_e$ a symmetric positive definite elasticity tensor (set $\mathcal{R}_e \equiv \mathbb{I}$)
- c_V, ϑ_c , λ and $\nu(> 0)$ the specific heat, the equilibrium temperature, the latent heat of the system, and the interfacial energy coefficient (set $c_V = \nu = \lambda/\vartheta_c = 1$)
- $W(\chi) + (\nu/2)|\nabla\chi|^2$ a mixture or interaction free-energy

The Pseudo-Potential of dissipation

Following the line of [Moreau, '71], we include dissipation by means of the following functional

$$\underline{\Phi(\chi_t, \varepsilon(\mathbf{u}_t), \nabla\vartheta)} = \frac{1}{2}|\chi_t|^2 + \frac{\chi}{2}\varepsilon(\mathbf{u}_t)\mathcal{R}_v\varepsilon(\mathbf{u}_t) + \frac{|\nabla\vartheta|^2}{2\vartheta},$$

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where

- $\mathcal{R}_v$ is a symmetric and positive definite viscosity matrix (set $\mathcal{R}_v \equiv \mathbb{I}$);
- χ represents the local proportion of the viscous phase, e.g. the liquid phase in solid-liquid phase transitions;
- all physical parameters have been set equal to 1

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a and b sufficiently regular functions

$a(\chi) + b(\chi) = 1$ for all $\chi \in (0, 1)$

$a(\chi) \to 0$ for $\chi \nearrow 1$, $a(\chi) \to 1$ for $\chi \searrow 0$, and, conversely, $b(\chi) \to 1$ for $\chi \nearrow 1$, $b(\chi) \to 0$ for $\chi \searrow 0$.

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For simplicity we shall confine our analysis to the meaningful case in which $a(\chi) = 1 - \chi$ and $b(\chi) = \chi$.

The equation of macroscopic motion

The **equation of macroscopic motion** is the following stress-strain relation, taking into account of accelerations:

$$\boxed{\mathbf{u}_{tt} - \operatorname{div}\sigma = \mathbf{f} \quad \text{in } \Omega \times (0, T)}$$

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However, our analysis carries over to other kinds of boundary conditions on $\mathbf{u}$ like a *pure traction* problem or a *displacement-traction* problem.

The equation of microscopic motion

If the volume amount of mechanical energy provided by the external actions is zero, the **generalized principle of virtual power by [Frémond, '02]** gives

$$\boxed{B - \operatorname{div} \mathbf{H} = 0 \quad \text{in } \Omega \times (0,T), \qquad \mathbf{H} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega \times (0,T)}$$

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we derive **the phase equation**

$$\boxed{\chi_t - \Delta\chi + W'(\chi) = \vartheta - \vartheta_c + \frac{|\varepsilon(\mathbf{u})|^2}{2}}$$

coupled with the B.C. $\partial_{\mathbf{n}}\chi = 0$ on $\partial\Omega \times (0,T)$.

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where $\widehat{\gamma} \in \mathrm{C}^2([0,1])$ and

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- We also set $\boxed{\gamma := \widehat{\gamma}'}$, so that we have $\boxed{W' = \beta + \gamma}$.

The internal energy balance

$$e_t + \operatorname{div} \mathbf{q} = g + \sigma : \varepsilon(\mathbf{u}_t) + B\chi_t + \mathbf{H} \cdot \nabla\chi_t \quad \text{in } \Omega \times (0, T)$$

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Small perturbation assumption (cf. [Germain, '73]): we get rid of the higher order dissipative terms on the right-hand side - smaller w.r.t. the other terms -

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$$\vartheta_t + \chi_t \vartheta - \Delta \vartheta = g \quad \text{in } \Omega \times (0, T)$$

which is our internal energy equation.

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- It is sufficient to note that the internal energy balance can be expressed in terms of the entropy $s = -\frac{\partial \Psi}{\partial \vartheta}$ in this way:

$$\vartheta\left(s_t - \operatorname{div}\left(\frac{\mathbf{q}}{\vartheta}\right) - \frac{g}{\vartheta}\right) = \sigma^{\mathrm{d}} : \varepsilon(\mathbf{u}_t) + B^{\mathrm{d}}\chi_t - \frac{\mathbf{q}}{\vartheta} \cdot \nabla\vartheta,$$

B^{d} being the dissipative part of B

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  B^{d} being the dissipative part of B
- The right-hand side turns out to be non negative because $(\sigma^{\mathrm{d}}, B^{\mathrm{d}}, -\mathbf{q}/\vartheta) \in \partial\Phi(\mathbf{u}_t, \chi_t, \nabla\vartheta)$, and Φ is convex in all of its variables

The thermodynamical consistency

Our model complies with the Second Principle of Thermodynamics: in fact, the following form of the Clausius-Duhem inequality

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- Therefore, the Clausius-Duhem inequality ensues from the positivity of ϑ

Formulation of the Problem

Find functions $\vartheta, \chi : \Omega \times [0, T] \to \mathbb{R}$ such that

$$\chi(x,t) \in \mathrm{dom}(W) \text{ and } \vartheta(x,t) > 0 \text{ a.e. in } \Omega \times (0,T)$$

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and the boundary conditions:

$$\partial_{\mathbf{n}}\vartheta = 0, \quad \partial_{\mathbf{n}}\chi = 0, \quad \mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega \times (0,T)\,. \tag{B.C.}$$

Main mathematical difficulties

The **degenerating** character of equation

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- The nonlinear term $W'(\chi)$ and the quadratic terms $\frac{|\varepsilon(\mathbf{u})|^2}{2}$ and $\chi_t\,\vartheta$ occurring in (EQ1)–(EQ2) give a strongly **nonlinear character** to the system

Our results

[THM. 1] **Local (in time) well-posedness** result for this problem in the spatially 3D setting

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[Thm. 2] **Global well-posedness** result for this system in the 1D case

The literature: $[\chi+\vartheta]$-equations

- So far Frémond's models of phase change do not take into account the different properties of the viscous and elastic parts of the system (cf., e.g., Colli, Bonfanti, Luterotti, Schimperna, Stefanelli).

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- Recent discussions with E. Feireisl and H. Petzeltová: introduce a weaker notion of solution (satisfying an entropy inequality and the total energy conservation).

The literature: [ϑ+u]-equations

- Frémond thermoviscoelastic system not subject to a phase transition has been tackled in [Bonetti, Bonfanti, '03]: a linear viscoelastic equation for $\mathbf{u}$ and an internal energy balance for ϑ are considered

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- Due to the highly nonlinear character of the system, only a local well-posedness result is available in the 3D case
- However, in this framework no degeneracy of the elliptic operator in the **u**-equation is allowed

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  - the coefficients in the u-equation vanish only as $\chi \searrow 0$, contrary to the twofold degeneracy of our equation (EQ3)
- Local well-posedness results are proved for the resulting PDE system in [Bonetti, Schimperna, Segatti, '05].

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(ii) the data satisfy

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(iii) the datum χ_0 is "separated from the potential barriers"

$$\min_{x\in\overline{\Omega}} \chi_0(x) > 0,$$

$$\max_{x\in\overline{\Omega}} \chi_0(x) < 1.$$

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✓ In this framework it would not be necessary any longer to require $\widehat{\beta}$ to have a bounded domain.

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& \cap L^\infty(0, \widehat{T}; H^2_N(\Omega)) \hookrightarrow \mathrm{C}^1([0, \widehat{T}]; L^2(\Omega)), \\
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& \hookrightarrow \mathrm{C}^1([0, \widehat{T}]; H^1_0(\Omega)),
\end{aligned}$$

solving Problem (P) on the interval $(0, \widehat{T})$, and fulfilling

$$\min_{x \in \overline{\Omega}} \vartheta(x, t) > 0 \quad \forall\, t \in [0, \widehat{T}],$$

$$0 < \sigma \leq \chi(x, t) \leq 1 - \sigma < 1 \quad \forall\, (x, t) \in \Omega \times (0, \widehat{T}).$$

Under the additional assumption of Lipschitz continuity of β on $[\rho, 1)$, the solution triple $(\vartheta, \chi, \mathbf{u})$ depends continuously on the initial data and on g and $\mathbf{f}$ in a proper sense.

First step.

- Following the approach of [BONETTI, SCHIMPERNA, SEGATTI, '05], we fix a constant $\sigma \in (0,1)$ such that

$$\sigma \leq \frac{2}{3} \min \left\{ \min_{x \in \overline{\Omega}} \chi_0(x), 1 - \max_{x \in \overline{\Omega}} \chi_0(x) \right\},$$

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- Hence, we consider the PDE system where (EQ3) is replaced by

$$\mathbf{u}_{tt} - \operatorname{div}\left(T_\sigma(1-\chi)\varepsilon(\mathbf{u}) + T_\sigma(\chi)\varepsilon(\mathbf{u}_t)\right) = \mathbf{f}.$$

We shall prove the existence of a local-in-time solution to this truncated system by a Schauder fixed point argument

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- Together with the assumption that $\widehat{\beta} \in C^{1,1}_{\mathrm{loc}}(0,1)$ (e.g., for the logarithmic potential and for the indicator function), the local (in time) inequality $\chi \le 1-\sigma < 1$ implies enhanced regularity on χ needed to prove compactness of the Schauder operator

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$$\mathbf{f} \in L^2(0, T; L^2(\Omega)),$$
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(iv) the datum χ_0 is "separated from 0-barrier" of the potential

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- ✓ The coercivity condition on β rules out the case in which $\widehat{\beta}$ is the indicator function of $[0,1]$, but is fulfilled, e.g., in the case of the logarithmic potential:

$$\widehat{\beta}(r) = r\ln(r) + (1-r)\ln(1-r), \quad \text{for } r \in (0,1)$$

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Then, the triple $(\vartheta, \chi, \mathbf{u})$ is the **unique** solution to our 1D problem and χ has the further regularity

$$\chi \in H^2(0,T; H^1(\Omega)')\,.$$

An idea of the proof of Thm. 2.

- **First step: existence of a local solution.** Fix $\delta > 0$ such that

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☞ *Global* separation inequalities of the same kind as our:

❶ with a similar comparison technique by [MIRANVILLE, ZELIK, '04] for the viscous Cahn-Hilliard equation with a logarithmic potential

❷ and by [HORN, SPREKELS, ZHENG, '96] for the Penrose-Fife model by means of a Moser iteration scheme

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- It is possible, e.g., to get **global existence** and some results on the long-time behaviour of solutions (existence of the ω-limit of trajectories) to the following **3D isothermal system** (e.g. $\vartheta \equiv \vartheta_c$ during the evolution)

$$\chi_t + A\chi + W'(\chi) = \frac{|\varepsilon(\mathbf{u})|^2}{2} \tag{1iso}$$

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- For (1iso)-(2iso) it would be interesting to study the convergence of the whole trajectories to stationary states by means , e.g., of Łojasiewicz-Simon techniques.