Analysis of a nonlinear degenerating PDE system for phase transitions in thermoviscoelastic materials

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joint work with R. Rossi (Brescia, Italy)

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Plan of the Talk

- ♢ Frémond’s model of phase transitions

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- Related open problems

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The analysis of the initial boundary-value problem for the following PDE system in $\Omega \times (0, T)$:

$$\vartheta_t + \chi_t \vartheta - \Delta\vartheta = g \qquad \text{(I)}$$

$$\chi_t - \Delta\chi + W'(\chi) = \vartheta - \vartheta_c + \frac{|\varepsilon(\mathbf{u})|^2}{2} \qquad \text{(II)}$$

$$\mathbf{u}_{tt} - \operatorname{div}\left((1-\chi)\varepsilon(\mathbf{u}) + \chi\varepsilon(\mathbf{u}_t)\right) = \mathbf{f} \qquad \text{(III)}$$

which describes a phase transition phenomenon for a two-phase viscoelastic system, occupying a bounded domain $\Omega \subseteq \mathbb{R}^N$, $N = 1, 2, 3$, during a time interval $[0, T]$.

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Our results

- ◊ Local in time well-posedness for a suitable formulation of (I–III)+I.C.+B.C.
- ◊ Global in time well-posedness for the corresponding isothermal problem, i.e. for (II–III)+I.C.+B.C. in case $\vartheta \equiv \vartheta_c$, being ϑ_c the equilibrium temperature

The free-energy functional

$$\Psi(\vartheta, \varepsilon(\mathbf{u}), \chi, \nabla\chi) = c_V \vartheta(1 - \log\vartheta) - \frac{\lambda}{\vartheta_c}(\vartheta - \vartheta_c)\chi + \frac{(1-\chi)\varepsilon(\mathbf{u})\mathcal{R}_e\varepsilon(\mathbf{u})}{2} + W(\chi) + \frac{\nu}{2}|\nabla\chi|^2$$

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- $\varepsilon(\mathbf{u})$ is the linearized symmetric strain tensor, namely $\varepsilon_{ij}(\mathbf{u}) := (u_{i,j} + u_{j,i})/2$, $i, j = 1, 2, 3$
- $(1 - \chi)$ represents the local proportion of the non viscous phase, e.g. the solid phase in solid-liquid phase transitions
- $\mathcal{R}_e$ is a symmetric positive definite elasticity tensor (set $\mathcal{R}_e \equiv \mathbb{I}$)
- c_V, ϑ_c , λ and $\nu(> 0)$ are the specific heat, the equilibrium temperature, the latent heat of the system, and the interfacial energy coefficient (set $c_V = \nu = \lambda/\vartheta_c = 1$)
- the term $W(\chi) + (\nu/2)|\nabla\chi|^2$ is a *mixture or interaction free-energy*

The Pseudo-Potential of dissipation

Following the line of [MOREAU, '71], we include dissipation by means of the following functional

$$\Phi(\chi_t, \varepsilon(\mathbf{u}_t), \nabla\vartheta) = \frac{1}{2}|\chi_t|^2 + \frac{\chi}{2}\varepsilon(\mathbf{u}_t)\mathcal{R}_v\varepsilon(\mathbf{u}_t) + \frac{|\nabla\vartheta|^2}{2\vartheta},$$

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where

- for the sake of simplicity, and without any loss of generality all physical parameters have been set equal to 1
- $\mathcal{R}_v$ is a symmetric and positive definite viscosity matrix (set $\mathcal{R}_v \equiv \mathbb{I}$)
- χ represents the local proportion of the viscous phase, e.g. the liquid phase in solid-liquid phase transitions

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- a and b sufficiently regular functions
- $a(\chi) + b(\chi) = 1$ for all $\chi \in (0, 1)$
- $a(\chi) \to 0$ for $\chi \nearrow 1$, $a(\chi) \to 1$ for $\chi \searrow 0$, and, conversely, $b(\chi) \to 1$ for $\chi \nearrow 1$, $b(\chi) \to 0$ for $\chi \searrow 0$.

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For simplicity we shall confine our analysis to the meaningful case in which $a(\chi) = 1 - \chi$ and $b(\chi) = \chi$.

The equation of macroscopic motion

The equation of macroscopic motion is

$$\mathbf{u}_{tt} - \operatorname{div} \sigma = \boldsymbol{f} \quad \text{in } \Omega \times (0, T)$$

where **f** stands for the exterior volume force and σ is the stress tensor. Using the constitutive law

$$\sigma = \sigma^{nd} + \sigma^{d} = \frac{\partial \Psi}{\partial \varepsilon(\mathbf{u})} + \frac{\partial \Phi}{\partial \varepsilon(\mathbf{u}_t)},$$

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$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega \times (0, T)\,.$$

However, our analysis carries over to other kinds of boundary conditions on $\mathbf{u}$ like a *pure traction* problem or a *displacement-traction* problem.

The equation of microscopic motion

If the volume amount of mechanical energy provided by the external actions is zero, the generalized principle of virtual power by [FRÉMOND, '02] gives

$$\boxed{B - \operatorname{div} \mathbf{H} = 0 \quad \text{in } \Omega \times (0, T)}$$

where B and $\mathbf{H}$ represent the internal microscopic forces responsible for the mechanically induced heat sources. From the constitutive relations

$$B = \frac{\partial \Psi}{\partial \chi} + \frac{\partial \Phi}{\partial \chi_t} = -\vartheta + \vartheta_c - \frac{|\varepsilon(\mathbf{u})|^2}{2} + W'(\chi) + \chi_t$$

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$$\boxed{\chi_t - \Delta \chi + W'(\chi) = \vartheta - \vartheta_c + \frac{|\varepsilon(\mathbf{u})|^2}{2}}$$

coupled with the B.C. $\partial_{\mathbf{n}} \chi = 0$ on $\partial \Omega \times (0, T)$.

The internal energy balance

It reads

$$e_t + \operatorname{div} \mathbf{q} = g + \sigma : \varepsilon(\mathbf{u}_t) + B\chi_t + \mathbf{H} \cdot \nabla \chi_t \quad \text{in } \Omega \times (0, T)$$

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$$\vartheta_t + \chi_t \vartheta - \Delta \vartheta = g + \chi |\varepsilon(\mathbf{u}_t)|^2 + |\chi_t|^2 \quad \text{in } \Omega \times (0,T).$$

Now, we can get rid of the higher order dissipative terms on the right-hand side

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by means of the small perturbation assumption (cf. [GERMAIN, '73]) and

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$$\vartheta_t + \chi_t\vartheta - \Delta\vartheta = g \quad \text{in } \Omega\times(0,T)$$

which is our internal energy equation.

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- ♢ It is sufficient to note that the internal energy balance can be expressed in terms of the entropy $s = -\frac{\partial \Psi}{\partial \vartheta}$ in this way:

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- The right-hand side turns out to be non negative because $(\sigma^{\text{d}}, B^{\text{d}}, -\mathbf{q}/\vartheta) \in \partial\Phi(\mathbf{u}_t, \chi_t, \nabla\vartheta)$, and Φ is convex in all of its variables

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- Therefore, the Clausius-Duhem inequality ensues from the positivity of ϑ

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We shall assume that the potential W is given by

$$W = \widehat{\beta} + \widehat{\gamma}\,,$$

where $\widehat{\gamma} \in \mathrm{C}^2([0,1])$ and

$$\overline{\mathrm{dom}(\widehat{\beta})} = [0,1]\,, \quad \widehat{\beta} : \mathrm{dom}(\widehat{\beta}) \to \mathbb{R} \text{ is proper, l.s.c., convex,}$$
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- Since $\chi \in (0,1)$, β is a single-valued operator
- We also set $\boxed{\gamma := \widehat{\gamma}'}$, so that we have $\boxed{W' = \beta + \gamma}$.

Problem (P)

Find functions $\vartheta, \chi : \Omega \times [0, T] \to \mathbb{R}$ *such that*

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and the boundary conditions:

$$\partial_{\mathbf{n}}\vartheta = 0, \quad \partial_{\mathbf{n}}\chi = 0, \quad \mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega \times (0,T)\,. \qquad \text{(B.C.)}$$

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- The former is due to the presence of the terms $(1-\chi)$ and χ in front of the elasticity and viscosity contributions: such terms vanish as $\chi \nearrow 1$ and $\chi \searrow 0$, making the related elliptic operator degenerate
- The nonlinear term $W'(\chi)$ and the quadratic terms $\frac{|\varepsilon(\mathbf{u})|^2}{2}$ and $\chi_t\,\vartheta$ occurring in (EQ1)–(EQ2) give a strongly nonlinear character to the system

Our results

◇ We prove a *local well-posedness* result for Problem (P)

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- We prove a *local well-posedness* result for Problem (P)
- Only under the assumption that $\vartheta \equiv \vartheta_c$, we obtain a *global well-posedness* result for isothermal system

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$$\mathbf{u}_{tt} - \operatorname{div}\left((1-\chi)\varepsilon(\mathbf{u}) + \chi\varepsilon(\mathbf{u}_t)\right) = \mathbf{f} \quad \text{in } \Omega \times (0,T)$$

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- A global existence result has been proved for (a generalization of) (EQ1)+(EQ2) in the 1D case

The literature: [ϑ+**u**]-equations

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- Due to the highly nonlinear character of the system, only a local well-posedness result is available in the 3D case
- However, in this framework no degeneracy of the elliptic operator in the **u**-equation is allowed

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- The authors Bonetti, Bonfanti, Schimperna, Segatti, ... address Frémond models for damaging phenomena. The variable χ stands for the local proportion of damaged material: $\chi \in [0, 1]$, $\chi = 0$ when the body is completely damaged and $\chi = 1$ in the damage-free case

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- On the other hand the equation for $\mathbf{u}$ displays a different degeneracy in the elliptic operators: their coefficients vanish only as $\chi \searrow 0$, contrary to the twofold degeneracy of equation (EQ3)
- *Local* well-posedness results are proved for the resulting PDE system.

Hypothesis 1

Assume that the potential W is given by

$$W = \widehat{\beta} + \widehat{\gamma}\,, \text{ where } \widehat{\gamma} \in \mathrm{C}^2([0,1]) \text{ and}$$

$$\overline{\mathrm{dom}(\widehat{\beta})} = [0,1]\,, \widehat{\beta} : \mathrm{dom}(\widehat{\beta}) \to \mathbb{R} \text{ is proper, l.s.c., convex,}$$

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Suppose that

$g \in H^1(0,T;L^2(\Omega)), \quad g(x,t) \geq 0$ for a.e. $(x,t) \in \Omega \times (0,T)$

$\mathbf{f} \in L^2(0,T;L^2(\Omega))$

$\vartheta_0 \in H^2_N(\Omega)$ and $\min_{x \in \overline{\Omega}} \vartheta_0(x) > 0, \quad \chi_0 \in H^2_N(\Omega)$

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Assume that χ_0 is "separated from the potential barriers":

$$\min_{x \in \overline{\Omega}} \chi_0(x) > 0,$$

$$\max_{x \in \overline{\Omega}} \chi_0(x) < 1.$$

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- The separation conditions on χ_0 and the assumptions on $\beta \Longrightarrow \widehat{\beta}(\chi_0), \ \beta(\chi_0) \in L^\infty(\Omega)$

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- In this framework it would not be necessary any longer to require $\widehat{\beta}$ to have a bounded domain.

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Under Hypothesis 1, there exist $\widehat{T} \in (0, T]$, $\sigma > 0$, *and a unique triplet* $(\vartheta, \chi, \mathbf{u})$ *with the regularity*

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By [Baiocchi, '67] this yield ϑ, $\chi \in \mathrm{C}^1([0, \widehat{T}]; L^2(\Omega))$, $\mathbf{u} \in \mathrm{C}^1([0, \widehat{T}]; H^1_0(\Omega))$.

The case $\vartheta \equiv \vartheta_c$. The PDE system

Problem (ISO)

Find functions $\chi : \Omega \times [0, T] \to \mathbb{R}$, $\chi \in \mathrm{dom}(\beta)$ *a.e.*, $\mathbf{u} : \Omega \times [0, T] \to \mathbb{R}^3$, *and* $\xi \in \beta(\chi)$ *a.e.*

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Suppose that, beside the conditions

$\overline{\mathrm{dom}(\widehat{\beta})} = [0,1]\,, \quad \widehat{\beta} : \mathrm{dom}(\widehat{\beta}) \to \mathbb{R}$ is proper, l.s.c., convex,

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$$\lim_{x \to 0^+} \beta^0(x) = -\infty\,,$$

where for all $r \in \mathrm{dom}(\beta)$ $\beta^0(r)$ denotes the element of minimal norm in $\beta(r)$.

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Finally, suppose that

$$\mathbf{f} \in L^2(0,T;L^2(\Omega)),$$
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- We do not need in this case the assumption $\widehat{\beta} \in C^{1,1}_{\mathrm{loc}}(0,1)$ and so $\partial\widehat{\beta}$ has to be regarded as a truly multivalued nonlinearity
- The coercivity condition on β rules out the case in which $\widehat{\beta}$ is the indicator function of $[0,1]$, but is fulfilled in the case of the logarithmic potential:
    - $\widehat{\beta}(r) = r\ln(r) + (1-r)\ln(1-r)$, for $r \in (0,1)$

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$$\chi \in H^2(0,T;H^1(\Omega)').$$

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Proof of Thm. 1. First step.

◇ Following the approach of [BONETTI, SCHIMPERNA, SEGATTI, '05], we fix a constant $\sigma \in (0,1)$ such that

$$\sigma \leq \frac{2}{3} \min \left\{ \min_{x \in \overline{\Omega}} \chi_0(x), 1 - \max_{x \in \overline{\Omega}} \chi_0(x) \right\},$$

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- Hence, we consider the PDE system where (EQ3) is replaced by

$$\mathbf{u}_{tt} - \operatorname{div} \left(T_\sigma(1-\chi)\varepsilon(\mathbf{u}) + T_\sigma(\chi)\varepsilon(\mathbf{u}_t) \right) = \mathbf{f}.$$

We shall prove the existence of a local-in-time solution to this system by a Schauder fixed point argument

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- Assumptions $\min_{x\in\overline{\Omega}}\chi_0(x)>0$, $\max_{x\in\overline{\Omega}}\chi_0(x)<1$ $\Longrightarrow$ χ *locally* stays away from both the potential barriers $\Longrightarrow$ the coefficients of $\varepsilon(\mathbf{u}_t)$ and $\varepsilon(\mathbf{u})$ do not degenerate $\Longrightarrow$ Problem (P) is (locally) well-posed

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- We could dispense assumption $\max_{x\in\overline{\Omega}}\chi_0(x)<1$ by slightly strengthening our assumptions on W (this would exclude, e.g., the logarithmic potential from our analysis).

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Open problems

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- In this regard, in the future it would be interesting to study the long-time behavior of our system in the isothermal case
- Finally, it would be interesting to study the control problem associated, e.g., to Problem (ISO), i.e. to find a volume force input $\mathbf{f}$ such that the resulting strain $\mathbf{u}$ and phase parameter χ match the desired strain and phase profiles $\mathbf{u}_d$ and χ_d, respectively