Identification of time-dependent kernels in phase-field problems of hyperbolic type

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We recover the two (smooth) unknown convolution kernels $k, h : [0,T] \to \mathbf{R}$, depending on time only, entering the following phase-field system coupling two integro-differential equations, where $\vartheta, \chi : Q_T := \Omega \times (0,T) \to \mathbf{R}$ stand, respectively, for the temperature and the order parameter:

$$\begin{aligned}
&D_t(\vartheta + \ell\chi) - k * \Delta\vartheta = f \quad \text{in } Q_T,\\
&D_t\chi + h * \Big[\Delta^2\chi - \Delta\beta(\chi) + \ell\Delta\vartheta\Big] = 0 \quad \text{in } Q_T,\\
&\vartheta(0) = \vartheta_0, \quad \chi(0) = \chi_0 \quad \text{in } \Omega,\\
&D_\nu\vartheta = D_\nu\chi = D_\nu\Delta\chi = 0 \quad \text{on } \partial\Omega \times (0,T),
\end{aligned}$$

under the additional information on the temperature

$$\Phi_i[\vartheta(t)] = g_i(t) \qquad \text{for } t \in [0,T],\ i = 1,2.$$

Here $\beta \in C^5(\overline{\Omega})$, while T and Ω denote, respectively, the final time of the phase transition process and a connected, bounded, and sufficiently smooth subset of $\mathbf{R}^n$, $n = 1,2,3$. Finally, ν and $*$ denote, respectively, the outward unit vector to $\partial\Omega$ and the standard convolution (in time) operator, i.e. $(l * \rho)(t) := \int_0^t l(t-s)\rho(s)\,ds$.
We stress that the differential operators - of order two and four, in space, respectively - appear under the integral sign, only. Assuming that the initial values $k(0)$ and $h(0)$ are *positive*, by differentiating in time the two equations, one can see that both of them are actually of *hyperbolic type*.
Our basic result will be concerned with the *local in time* existence and uniqueness of the solution (ϑ, χ, h, k) to our identification problem when $n = 2,3$ as well as a *global in time* existence and uniqueness when $n = 1$, provided some some monotonicity is assumed for β. As a byproduct, we will derive a *global* well-posedness for the solution (ϑ, χ) of the direct problem, the continuous dependence involving not only the standard data f, ϑ_0, χ_0, but also the kernels h, k.

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