Second-Order Degenerate Differential Equations in Banach Spaces

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Abstract

In this talk we will extend our previous results and solve the problem not only for first-order differential equations but also for second-order differential equations in time that reduced to weakly parabolic systems.
Consider the following problem:

$$\frac{d}{dt}(Mu) + Lu = f(t)z\,, \qquad 0 \le t \le \tau\,, \tag{1}$$

$$(Mu)(0) = Mu_0\,, \tag{2}$$

$$\Phi[Mu(t)] = g(t)\,, \qquad 0 \le t \le \tau\,, \tag{3}$$

where L, M are two closed linear operators with $D(L) \subseteq D(M)$, L being invertible, $\Phi \in X^*$, $g \in C^{1+\theta}([0,\tau];\mathbb{R})$ for $\theta \in (0,1)$ and M may have no bounded inverse.

The main assumption here is:

$$\|M(\lambda M + L)^{-1}\|_{\mathcal{L}(X)} \le c(1+|\lambda|)^{-\beta}\,, \quad \forall \lambda \in \Sigma_\alpha\,,$$

or, equivalently, (where $T = ML^{-1}$)

$$\|L(\lambda M + L)^{-1}\|_{\mathcal{L}(X)} = \|(\lambda T + I)^{-1}\|_{\mathcal{L}(X)} \le c(1+|\lambda|)^{1-\beta}\,, \quad \forall \lambda \in \Sigma_\alpha\,,$$

where

$$\Sigma_\alpha = \{\lambda \in \mathbb{R} :\ Re\lambda \ge -c(1+|Im\lambda|)^\alpha\}\ ,$$

$c > 0$, $\alpha, \beta \in (0,1)$, $0 < \beta \le \alpha \le 1$, $\alpha + \beta > 3/2$, $2 - \alpha - \beta < \theta < \alpha + \beta - 1$, $z = Tz^*$ and $Lu_0 = Tv^*$. Then we show that problem (1)-(3) admits a unique global solution

$$(u, f) \in C^\theta([0,\tau], D(L)) \times C^\theta([0,\tau];\mathbb{R})$$

provided that $\Phi[z] \neq 0$ and $\Phi[Mu_0] = g(0)$.

To find similar results for second order degenerate problem we consider the following system:

$$\frac{d}{dt}\big(My'\big) + Ly' + Ky = f(t)z\,, \quad 0 \le t \le \tau\,, \tag{4}$$
$$y(0) = y_0\,, \tag{5}$$
$$My'(0) = My_1\,, \tag{6}$$
$$\Phi[My(t)] = g(t)\,, \quad 0 \le t \le \tau\,, \tag{7}$$

with the compatibility relations

$$\Phi[My_0] = g(0)\,, \tag{8}$$
$$\Phi[My_1] = g'(0)\,, \tag{9}$$
$$\Phi[z] \neq 0\,, \tag{10}$$

where $D(L) \subseteq D(M) \cap D(K)$, $0 \in \rho(L)$, $\|u\|_{D(L)} = \|Lu\|$,
$\|M(\lambda M+L)^{-1}\| \le \dfrac{C}{(1+|\lambda|)^{\beta}}, \quad \mathrm{Re}(\lambda) \ge c(1+|\mathrm{Im}(\lambda)|)^{\alpha}, \quad \alpha+\beta > 1.$

Let $y' = w$, then the system (4)-(7) is equivalent to:

$$\frac{d}{dt}\begin{bmatrix} 1 & 0 \\ 0 & M \end{bmatrix}\begin{bmatrix} y(t) \\ w(t) \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ K & L \end{bmatrix}\begin{bmatrix} y(t) \\ w(t) \end{bmatrix} = f(t)\begin{bmatrix} 0 \\ z \end{bmatrix},$$
$$\begin{bmatrix} 1 & 0 \\ 0 & M \end{bmatrix}\begin{bmatrix} y(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & M \end{bmatrix}\begin{bmatrix} y_0 \\ y_1 \end{bmatrix},$$
$$\Psi\left(\begin{bmatrix} 1 & 0 \\ 0 & M \end{bmatrix}\begin{bmatrix} y(t) \\ w(t) \end{bmatrix}\right) = \Phi[Mw(t)] = g'(t)\,.$$

where the linear functional $\Psi : D(L) \times D(L) \to \mathbb{R}$ is defined by:

$$\Psi\left(\begin{bmatrix} y(t) \\ w(t) \end{bmatrix}\right) = \Phi[w(t)]\,.$$

Using the previous results we can show that problem (4)-(7) has a unique strict global solution (y, f) such that $y' \in C^{\theta}([0,\tau]; D(L))$, $(My')' \in C^{\theta}([0,\tau]; X)$ and $f \in C^{\theta}([0,\tau]; \mathbb{R})$.