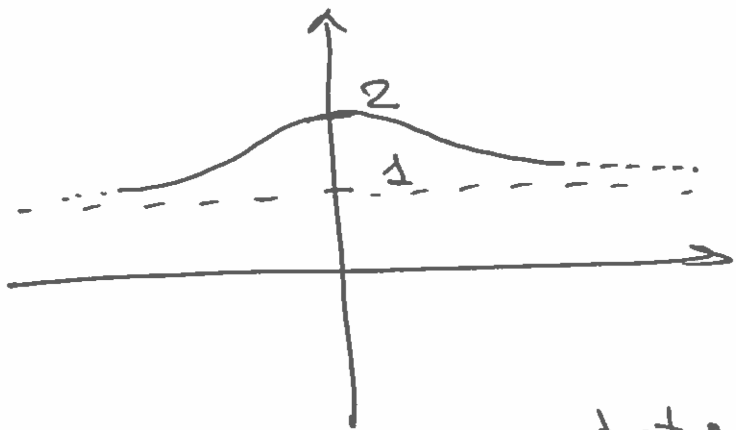
Advanced Mathematical Methods for Engineers (1)

March 18, 2018 (Appello Straordinario)

1) The function $f(x,y) = |y|(1-y)\frac{x^3}{1+x^4}$ is $C^0(\mathbb{R}^2;\mathbb{R})$ and locally Lip. in y uniformly in x $\Rightarrow$ $\exists!$ local solution in $\mathcal{U}(0)$.

$y \equiv 1$ and $y \equiv 0$ are the only constant solutions of the ODE $\Rightarrow$ since $y_0 = 2 > 1$ $\Rightarrow$ $y(x) > 1$ $\forall x \in \operatorname{dom}(y)$.

Moreover since the solution y is increasing for $x < 0$ and decreasing for $x > 0$ $\Rightarrow$ the set of maxima is $\{x = 0\}$ $\Rightarrow$ the qualitative graph is



By using the asymptots theorem we can infer that $\lim_{x\to\pm\infty} y(x) = 1$ and $\operatorname{dom} y = \mathbb{R}$

Moreover we can compute explicitely the solution by using the Bernoulli's method.

Set $z = \frac{1}{y}$ and then z solves the following Cauchy problem:

$$\begin{cases} z' + p(x)\, z = p(x) \\ z(0) = \frac{1}{2} \end{cases}$$

where $p(x) = \dfrac{x^3}{1+x^4}$.

Hence we get $z(x) = \dfrac{1}{(1+x^4)^{1/4}} \left(-\frac{1}{2} + (1+x^4)^{1/4} \right)$

and so $y(x) = \dfrac{2\sqrt[4]{1+x^4}}{-1+2\,(1+x^4)^{1/4}}$, $\operatorname{dom} y \equiv \mathbb{R}$

2) Let's consider first the homogeneous equation, then the characteristic equation is: $t^2 + 2\lambda t + 2\lambda^2 = 0$

which for $\lambda \neq 0$ has solutions $t_{1,2} = -\lambda \pm \lambda i$.

Moreover a particular solution of the non-hom. equation is $y_p(x) = \frac{1}{\lambda^2} x + 0$

hence we get

$$y(x) = e^{-\lambda x}\left(c_1 \cos\lambda x + c_2 \operatorname{sen}\lambda x\right) + \frac{x}{\lambda^2}$$ 3

Imposing the boundary conditions we get

$y(0) = c_1 = 0$

$y(\pi) = e^{-\lambda\pi}\left(c_1\cos\lambda\pi + c_2\operatorname{sen}\lambda\pi\right) + \frac{\pi}{\lambda^2} = \frac{\pi}{4}$

If $\lambda \neq k$ $(k \in \mathbb{Z} \setminus \{0\})$ $\Longrightarrow$ we have a unique solution

$$y(x) = e^{\lambda(\pi - x)}\,\frac{\pi(\lambda^2-4)}{4\lambda^2 \operatorname{sen}\lambda\pi}\operatorname{sen}\lambda x + \frac{\pi}{\lambda^2}x$$

If $\lambda = \pm 2$ $\Longrightarrow$ we have ∞-many solutions $\quad y(x) = c_2\, e^{\pm 2x}\operatorname{sen}(\mp 2x) + \frac{x}{4}$

If $\lambda = k$ $(k \in \mathbb{Z} \setminus \{0, \pm 2\})$ $\Longrightarrow$ we have <u>no</u> solutions.

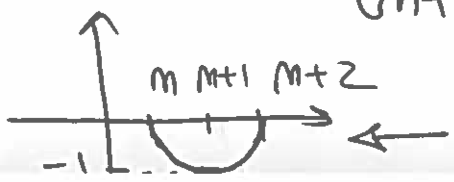
3) Let us consider $f_m(x) = \frac{2}{(m+2)^3}\left((x-m)(x-m-2)\right)\cdot \chi_{(m,m+2)}(x)$

Then we can compute

$$\lim_{m\to\infty}\int_0^{+\infty} f_n(x)\,dx.$$

Notice that $|f_n(x)| \leq \frac{2}{(m+2)^3}$ $\forall x$

because



$(x-m)(x-m-2)$ on $(m, m+2)$

and so

in $\mathcal{U}(+\infty)$ $\quad |f_n(x)| \leq \frac{2}{x^3}$ for n suff. large

in $\mathcal{U}(0^+)$ $\quad |f_n(x)| \leq 2$

and so, by the Lebesgue dominated converge theorem, we can pass to the limit $n\to\infty$ under integral and since $f_n(x) \to 0$ pointwise

$$\Rightarrow \lim_{n\to\infty} \int_0^{+\infty} f_n(x)\,dx = 0 .$$

4) Let's define $v = u'$ $\Rightarrow$ we search for $v \in \mathcal{D}'(\mathbb{R})$: $(x^3-8)v = \delta_0'$

since $x=2$ is a root of order 1 of $x^3-8=0$ $\Rightarrow$ $v = c_1 \delta_{x=2} + v_p$

and since $\operatorname{supp} \delta_0' = \{0\}$ $\Rightarrow$ we can formally divide by x^3-8 $(\neq 0 \text{ in } 0)$ and we get $v_p = \frac{\delta_0'}{x^3-8}$ and

$\forall \varphi \in \mathcal{D}(\mathbb{R})$ we have

$$< \frac{\delta_0'}{x^3-8}, \varphi > = < \delta_0', \frac{\varphi(x)}{x^3-8} >$$

$$= -\left(\frac{\varphi(x)}{x^3-8}\right)'\Big|_{x=0} = +\frac{\varphi'(0)}{8} + \frac{3\varphi(0)x^2}{(x^3-8)^2}\Big|_{x=0}$$

$$\Rightarrow \quad <v_p, \varphi> = <-\frac{\delta_0'}{8}, \varphi> \ \forall \varphi$$

$$\Rightarrow \; v = c_1 \delta_2 - \frac{\delta_0'}{8} \; \Rightarrow$$

$$u(x) = c_1 H(x-2) - \frac{\delta_0}{8} + c_2 \,, \quad c_1, c_2 \in \mathbb{R}$$

where H is the Heaviside function.