Advanced Mathematical Methods for Engineers

June 26, 2018 — Solutions

1) Consider $y_0'' - 4y_0' + 5y_0 = 0$.

1.1) The polynomial associated is $\lambda^2 - 4\lambda + 5 = p(\lambda)$ which has roots $\lambda_{1,2} = 2 \pm i$ $\Rightarrow$

$$y_0(t) = e^{2t}(c_1 \,\text{sen}\, t + c_2 \cos t)$$

1.2) In order to find one particular solution we need to find particular solutions for

(a) $y'' - 4y' + 5y = e^{2t}$

(b) $y'' - 4y' + 5y = e^{2t}\cos t$

(c) $y'' - 4y' + 5y = 5t^2$

and then sum them up.

For (a), we search for $y_{p_a}(t) = a e^{2t}$ and substituting in (a) we get $a = 1$ $\Rightarrow$

$$y_{p_a}(t) = e^{2t}$$

For (b), since $e^{2t}\cos t$ is solution of the homogeneous eq., then the equation (b) has a particular solution of the form

$$y_{p_b}(t) = t e^{2t}(b_1 \,\text{sen}\, t + b_2 \cos t).$$

Substituting in (b), we get

$$y_{p_b}(t) = t \frac{\text{sen}\, t}{2} e^{2t}.$$

For the equation (c), we search for a solution of the form $y_{p_c}(t) = c_1 + c_2 t + c_3 t^2$ and substituting we get

$$y_{p_c}(t) = \frac{22}{25} + \frac{8}{5}t + t^2$$

Then one particular solution of (1) is

$$y_p(t) = e^{2t} + \frac{1}{2} t \operatorname{sen} t e^{2t} + \frac{22}{25} + \frac{8}{5}t + t^2.$$

Note that to this y_p we can always add combinations (linear) of $e^{2t}\operatorname{sen} t$ and $e^{2t}\cos t$ and we still get a particular solution of (1).

2) The characteristic polynomial is

$$0 = \det(A - \lambda I) = \lambda^2 - (\alpha+1)\lambda + \alpha + \alpha^2 + 2$$

and $\Delta = -3\alpha^2 - 2\alpha - 7 < 0 \quad \forall \alpha \in \mathbb{R}$

$$\Rightarrow \quad \lambda_{1,2} = \frac{1+\alpha \pm \sqrt{-3\alpha^2 - 2\alpha - 7}}{2}$$

are always complex conjugate.

2.1) The solutions are bounded on $[0, +\infty)$ if $\underbrace{\operatorname{Re}(\lambda_1) = \operatorname{Re}(\lambda_2)}_{= \frac{1+\alpha}{2}} \leq 0$

$$\Leftrightarrow \quad \alpha \leq -1$$

Indeed, in this case, if $\underline{v}_1 + i\underline{v}_2$ is the eigenvector associated to λ_1, the solutions are

$$c_1 e^{(\mathrm{Re}\lambda_1)t}(\underline{v}_1 \cos t - \underline{v}_2 \sin t) + c_2 e^{(\mathrm{Re}\lambda_1)t}(\underline{v}_1 \sin t + \underline{v}_2 \cos t)$$

$c_1, c_2 \in \mathbb{R}$.

2.2) When $\mathrm{Re}\,\lambda_1 = 0 \implies$ the solutions are bounded $\forall t \in \mathbb{R}$.

So if $\alpha \leq -1$ all solutions are bounded on $[0,+\infty)$ $(\forall t \in [0,+\infty))$

if $\alpha = -1$ all solutions are bounded in the whole $\mathbb{R}$ $(\forall t \in \mathbb{R})$

3) 3.1) For $x \to +\infty$ $f_n(x) \sim \frac{1}{x^n}$

and so $\forall n \geq 2$ it is integrable

For x in bdd intervals it is continuous and so Lebesgue-integrable

3.2) $$f_n(x) \xrightarrow[n\to\infty]{} \begin{cases} 1/2 & x \in (0,1) \\ 1/3 & x = 1 \\ 0 & x > 1 \end{cases} =: f(x)$$

Since $f_n(x) \leq g(x) := \begin{cases} 1/2 & x \in (0,1] \\ \frac{1}{x^2+2} & x > 1 \end{cases}$

$g \in L^1(0,+\infty)$

Then, we can apply Lebesgue theorem (dominated convergence theorem) and we get

$$\lim_{n\to\infty} \int_0^{+\infty} f_n(x)\,dx = \int_0^{+\infty} f(x)\,dx = \int_0^1 \frac{1}{2}\,dx = \frac{1}{2}$$

4) 4.1)

$$< f'(-x), v(x) > = < f'(x), v(-x) > =$$

$$= - < f(x), (v(-x))' > = < f(x), v'(-x) >$$

$$\underset{f \text{ is even}}{=} < f(-x), v'(x) > = < f(x), v'(x) >$$

$$= - < f'(x), v(x) >$$

4.2) $\forall v$ test function, since f is even, we get:

$$< f(x), v(x) > = < f(-x), v(x) > = < f(x), v(-x) > \underset{(*)}{=}$$

If v is odd, then

$$\underset{(*)}{=} - < f(x), v(x) > \Rightarrow$$

$$< f(x), v(x) > = - < f(x), v(x) > \Rightarrow$$

$$< f(x), v(x) > = 0$$