Advanced Mathematical Methods for Engineers

February 6, 2018 Solutions

1) $$y' - \frac{1}{x-1}\, y - x y^2 = 0$$

is a Bernoulli-type equation where the coefficient $P(x) = -\frac{1}{x-1} \in C^0(\mathbb{R}\setminus\{1\})$

If $a=0$ then the solution is $y \equiv 0$

$f(x,y) = \frac{1}{x-1}\, y + x y^2$ is in $C^0((-\infty,1)\times\mathbb{R})$ and so we have existence of local solutions. And so we have $f_y \in C^0((-\infty,1)\times\mathbb{R})$ and so we also have existence and uniqueness of local solutions.

If $a > 0 \Rightarrow y > 0$ in $\mathcal{U}(0)$

If $a < 0 \Rightarrow y < 0$ in $\mathcal{U}(0)$. If $a \neq 0$

Taking $z(x) = (y(x))^{-1}$ in $\mathcal{U}(0)$ we obtain

$$z(x) = e^{-\int_0^x \frac{1}{t-1}\,dt}\left(\frac{1}{a} + \int_0^x -t\, e^{\int_0^t \frac{1}{s-1}\,ds}\,dt\right)$$

$$= \frac{1}{1-x}\left(\frac{1}{a} + \int_0^x -t((1-t))\,dt\right)$$

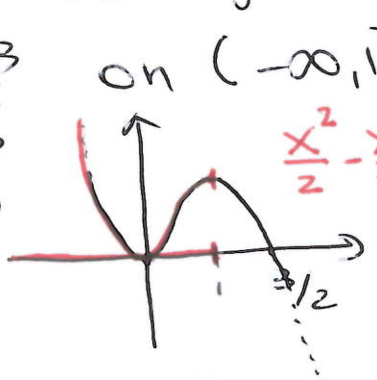
$$= \frac{1}{1-x}\left(\frac{1}{a} + \left(-\frac{x^2}{2} + \frac{x^3}{3}\right)\right)$$

Then $y(x) = (z(x))^{-1}$ is defined on $(-\infty,1)$

iff $\frac{1}{a} \neq \frac{x^2}{2} - \frac{x^3}{3}$ on $(-\infty,1)$

iff $\frac{1}{a} < 0$ i.e. $a < 0$

Hence $a \in (-\infty, 0]$.



$\frac{x^2}{2} - \frac{x^3}{3}$ on $(-\infty,1)$

2) We have the following eigenvalues of the matrix of coefficients:

$$A = \begin{pmatrix} -3 & \alpha^2 \\ 1 & -3 \end{pmatrix}$$

$$\det \begin{pmatrix} -3-\lambda & \alpha^2 \\ 1 & -3-\lambda \end{pmatrix} = 0 = (-3-\lambda)^2 - \alpha^2$$

iff $-3-\lambda = \pm |\alpha|$ i.e.

$$\lambda_1 = -3-|\alpha| < 0 \quad \forall \alpha$$

$$\lambda_2 = -3+|\alpha| < 0 \quad \text{iff } |\alpha| < 3$$

Hence, we have that $(0,0)$ is asymptotically stable iff $|\alpha| < 3$

3) u is periodic of period 2. In $(-1,1)$ it is

$$u(t) = \begin{cases} 1 & \text{if } |t| \leq 1/2 \\ -1 & \text{otherwise} \end{cases}$$

Hence, by periodicity we get

$$\frac{d}{dt}u = \sum_{k \in \mathbb{Z}} 2\delta(t-2k+1/2) - 2\delta(t-2k-1/2) = 2 Ш_2(t+1/2) - 2Ш_2(t-1/2)$$

and so

$$\frac{d^2}{dt^2}u = \sum_{k \in \mathbb{Z}} 2\delta'(t-2k+1/2) - 2\delta'(t-2k-1/2)$$

Continuation of ex. 3)

$$\frac{d}{dt} v = \mathrm{rect}(t/2) - 2\delta(t+1) - 2\delta(t-1)$$

Hence

$$\frac{d^2}{dt^2} v = \delta(t+1) - \delta(t-1) - 2\delta'(t+1) - 2\delta'(t-1)$$

where $\mathrm{rect}(t/2) = \begin{cases} 1 & \text{if } |t| \le 1 \\ 0 & \text{if } |t| > 1 \end{cases}$

4) We search for

$$u(x,t) = X(x)T(t)$$

Then the PDE becomes:

$$T'X = TX'' + 2TX'$$ hence

$$\frac{T'}{T}(t) = \frac{X'' + 2X'}{X}(x)$$ and so

$$\frac{T'}{T} = \frac{X'' + 2X'}{X} = \lambda$$

for some constant λ.

Hence we have

$$\begin{cases} X''(x) + 2X'(x) = \lambda X(x), & x \in (0, \ell) \\ X(0) = X(\ell) = 0 \end{cases}$$

$$T'(t) = \lambda T(t), \quad t > 0.$$

We need $\lambda + 1 < 0$ and we get

$$X(x) = e^{-x} \operatorname{sen}\left(\sqrt{|1+\lambda|}\, x \right)$$

with $\sqrt{|1+\lambda|}\, e = n\pi$ and so

$$\lambda = -\left(\frac{n\pi}{e}\right)^2 - 1$$

and so

$$u_n(x,t) = c_n e^{-\left(\left(\frac{n\pi}{e}\right)^2+1\right)t} e^{-x} \operatorname{sen}\left(\frac{n\pi}{e} x\right)$$

We search now for a solution

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\left(\left(\frac{n\pi}{e}\right)^2+1\right)t} e^{-x} \operatorname{sen}\left(\frac{n\pi}{e} x\right)$$

satisfying $u(x,0) = 2e^{-x} \operatorname{sen}\left(\frac{3\pi x}{e}\right)$.

In order to have

$$2e^{-x} \operatorname{sen}\left(\frac{3\pi x}{e}\right) = \sum_{n=1}^{\infty} c_n e^{-x} \operatorname{sen}\left(\frac{n\pi}{e} x\right)$$

we need $c_3 = 2$, $c_n = 0$ for $n \neq 3$

And so, the solution is

$$u(x,t) = 2 e^{-\left(\left(\frac{3\pi}{e}\right)^2+1\right)t} e^{-x} \operatorname{sen}\left(\frac{3\pi}{e} x\right)$$