Adv. Math. Meth.

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1) $f(x,y) = x^5 \cdot (e^{4-y^2}-1) \in C^\infty(\mathbb{R}^2)$

and $\forall\ x \in [-a,a]$, $\forall y \in \mathbb{R}$ we have

$|f(x,y)| \leq |x^5||e^{4-y^2}-1| \leq a^5 (e^4+1)$
$\leq C$

So f is sublinear $\forall\ [a,a] \times \mathbb{R}$

$\Rightarrow \exists!$ global solution $y: \mathbb{R} \to \mathbb{R}$

$\forall (x_0, y_0) \in \mathbb{R}^2$

$e^{4-y^2}-1 = 0 \iff 4-y^2=0 \iff$

$y = \pm 2 \Rightarrow y = \pm 2$ are constant

solutions for $y_0 = \pm 2$ respectively

In virtue of uniqueness of solutions, the other solutions (for other values y_0) cannot intersect them.

The stationary points are for $x=0$.

$y' > 0$ for $x > 0$ and $-2 < y < 2$

and for $x < 0$ and

$y < -2$ or $y > 2$

For monotonicity, $\exists \lim\limits_{x \to \pm\infty} y(x)$

If it is finite then it is $l = \pm 2$.

So for $y_0 > -2$ we get $\lim\limits_{x \to \pm\infty} y(x) = 2$

and for $y_0 < -2$ we get

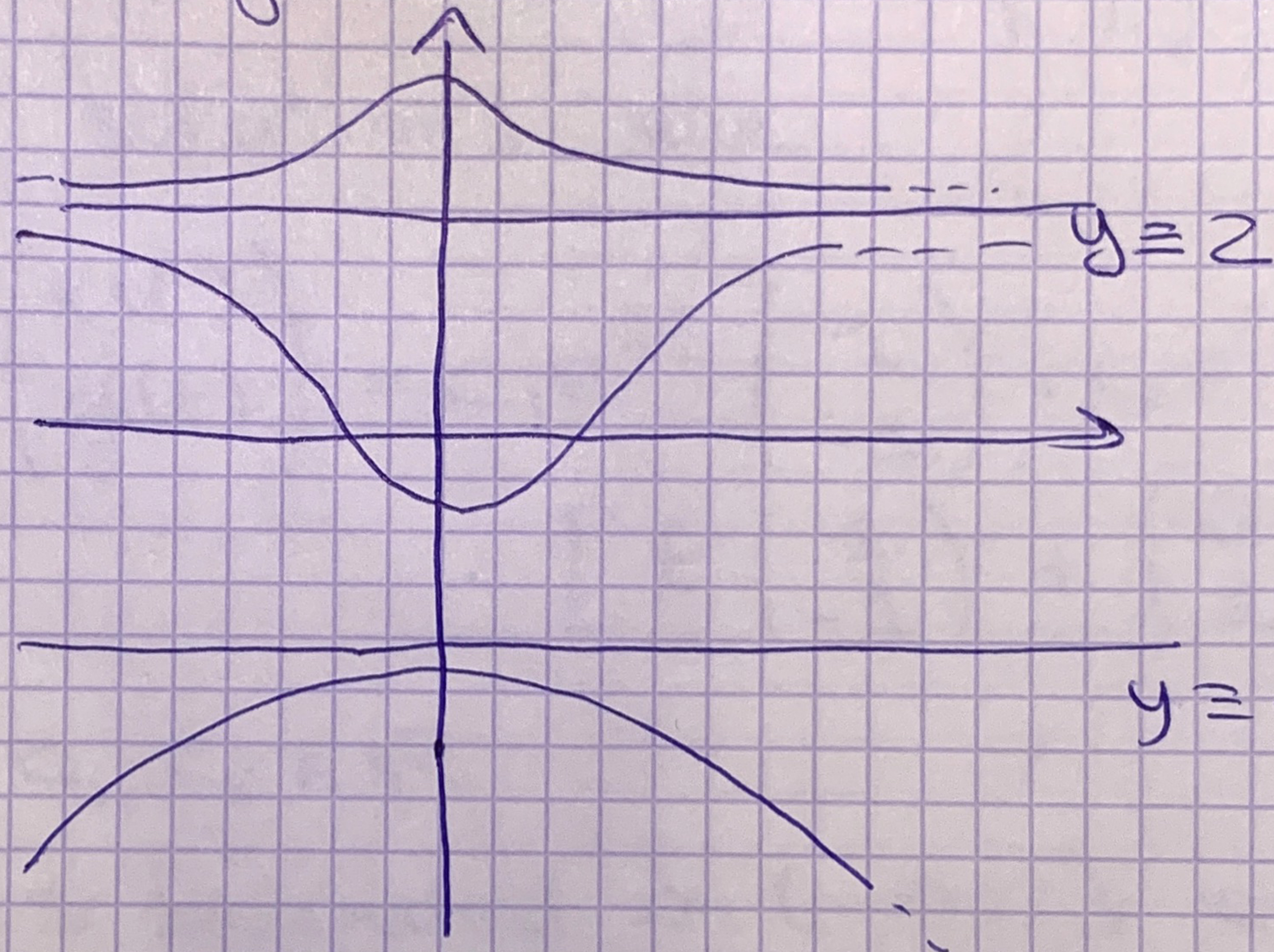
$$\lim_{x \to \pm\infty} y(x) = -\infty .$$

Moreover $y'' = 5x^4 (e^{4-y^2} - 1) +$

$$+ x^5 \left(-2y\, e^{4-y^2} \cdot x^5 \cdot (e^{4-y^2} - 1) \right)$$

$$= (e^{4-y^2} - 1) x^4 \left(5 - 2y x^5 e^{4-y^2} \right)$$

$y'' \neq 0$ if $y < 0$ $(y \neq 2)$, so we do not have flexes under x axis



2) a) $A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$

$$0 = \det(A - \lambda Id) = (3-\lambda)(1-\lambda) + 1$$
$$= (\lambda - 2)^2$$

$\lambda = 2$ has multiplicity 2

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff$$

$\Leftrightarrow \quad x+y=0 \quad \Rightarrow \quad \underline{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\Rightarrow \quad \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{2t} \underline{u} + c_2 e^{2t} [t \underline{u} + \underline{v}]$$

where $\underline{v}$ solves $(A - 2I)\underline{v} = \underline{u}$

which means $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\Leftrightarrow \quad x+y=1$, e.g. $\underline{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\Rightarrow$ the solutions are

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{2t} \cdot \left[t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$$

$c_1, c_2 \in \mathbb{R}$

b) e^{2t} is bounded on $(-\infty, 0]$ and $e^{2t} \cdot t$ is also bounded on $(-\infty, 0]$, and so all solutions are bounded on $(-\infty, 0]$.

3) $\forall x > 0$ and $n \to \infty$, we have

$$f_n(x) = x^n e^{-nx} \longrightarrow 0$$

and $f_n \in L^1(0, +\infty)$ because f_n is bounded and for n fixed and

$x \to +\infty$, $f_n(x) \to 0$ exponentially, i.e. faster than every negative power of x.

To show $f_n(x) \leq f_1(x)$ we compute

$$\frac{d}{dn}\left(x^n e^{-nx}\right) = x^n e^{-nx} \log(x e^{-x}) =$$
$$= x^n e^{-nx} (\log x - x) < 0$$

$\forall x > 0$ and $n = 1, 2, 3, \dots \Rightarrow$
$f_n(x) \leq f_1(x)$.

We can apply now dominated Lebesgue convergence theorem with $g(x) = f_1(x)$ and so $\lim_{n \to \infty} \int_0^{+\infty} x^n e^{-nx}\, dx =$

$$= \int_0^{+\infty} \left(\lim_{n\to\infty} x^n e^{-nx}\right) dx = 0$$

4) $f(t,x) = tx$ is bounded on $[0,\pi] \times [0,\pi]$

$\Rightarrow \exists!$ of a solution $u \in C^0([0,\pi] \times [0,\pi])$

Consider first the homogeneous problem and use separation of variables to find

$$\begin{cases} v''(x) - \lambda v(x) = 0 \\ v'(0) = v'(\pi) = 0 \end{cases}$$

We know that:

$\lambda_k = -k^2$ are the eigenvalues and the eigenvectors are $v_k(x) = \cos kx$

The candidate solution is

$$u(x,t) = \sum_{k=0}^{\infty} c_k(t)\, v_k(x)$$

and we impose:

$$u_t - u_{xx} = \sum [c_k'(t) + k^2 c_k(t)]\, v_k(x)$$
$$= tx$$

and

$$u(x,0) = \sum_{k=0}^{\infty} c_k(0)\, v_k(x) = 1$$

Developing $f(x) = x$ in cosinus-series we get

$$x = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\cos(2k+1)x}{(2k+1)^2}$$

Then we get

$$c_0'(t) = \frac{\pi}{2} t \qquad c_0(0) = 1$$

$$c_{2k}'(t) + 4k^2 c_{2k}(t) = 0 \qquad c_k(0) = 0 \ \forall k \geq 1$$

$$c_{2k+1}'(t) + (2k+1)^2 c_{2k+1}(t) = -\frac{4}{\pi} \frac{1}{(2k+1)^2} t$$

$$c_{2k+1}(0) = 0 \quad k \geq 0$$

Solving we get

$$c_0(t) = \frac{\pi}{4} t^2 + 1$$

$$c_{2k}(t) = 0 \qquad \forall k \geq 1$$

$$c_{2k+1}(t) = -\frac{4}{\pi (2k+1)^4} \left[t + \frac{1}{(2k+1)^2} \left(e^{-(2k+1)^2 t} - 1 \right) \right]$$

$\forall k \geq 0$

Hence the solution is

$$u(x,t) = \frac{\pi}{4} t^2 + 1 + \sum_{k=0}^{+\infty} C_{2k+1}(t) \cos(2k+1)x$$