Weierstrass Institute for
Applied Analysis and Stochastics

New results on Cahn-Hilliard-Navier-Stokes systems with nonlocal interactions

Sergio Frigeri
ERC Group “Entropy Formulation of Evolutionary Phase Transitions”

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- An isothermal model for the flow of a **mixture of two**
  - viscous
  - incompressible
  - Newtonian fluids
  - of equal density
- Avoid problems related to interface singularities

  $\Longrightarrow$ use a **diffuse interface model**

  $\Longrightarrow$ the classical sharp interface replaced by a **thin interfacial region**
- A partial mixing of the macroscopically immiscible fluids is allowed

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- The original idea of diffuse interface model for fluids: HOHENBERG and HALPERIN, '77

  $\Longrightarrow$ **H-model**

  Later, GURTIN ET AL., '96: continuum mechanical derivation based on microforces
- Models of two-phase or two-component fluids are receiving growing attention (e.g., ABELS, BOYER, GARCKE, GRÜN, GRASSELLI, LOWENGRUB, TRUSKINOVSKI, ...)

In $\Omega \times (0,\infty)$, $\Omega \subset \mathbb{R}^d$, $d = 2,3$

$$\boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla\pi = \mu\nabla\varphi + \boldsymbol{v}$$

$$\mathrm{div}(\boldsymbol{u}) = 0$$

$$\varphi_t + \boldsymbol{u}\cdot\nabla\varphi = \mathrm{div}\,(m(\varphi)\nabla\mu)$$

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- μ: **chemical potential** (Cahn-Hilliard), first variation of the (total Helmholtz) free energy

$$E(\varphi) = \int_\Omega \left(\frac{\epsilon}{2} |\nabla \varphi|^2 + \frac{1}{\epsilon} F(\varphi) \right) dx$$

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- F double-well potential: Helmholtz free energy density
  - Singular

$$F(s) = -\frac{\theta_c}{2}s^2 + \frac{\theta}{2}\big((1+s)\log(1+s) + (1-s)\log(1-s)\big)$$

    for all $s \in (-1,1)$, with $0 < \theta < \theta_c$
  - Regular

$$F(s) = (1-s^2)^2 \qquad \forall s \in \mathbb{R}$$

- **Nonlocal free energy** rigorously justified by Giacomin and Lebowitz ('97 & '98) as macroscopic limit of microscopic phase segregation models

$$E(\varphi) = \frac{1}{4}\int_\Omega\int_\Omega J(x-y)(\varphi(x)-\varphi(y))^2 dxdy + \int_\Omega F(\varphi(x))dx$$

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- First analytical results on nonlocal CH: Giacomin & Lebowitz '97 and '98; Gajewski '02; Gajewski & Zacharias '03
- Several other contributions on nonlocal Allen-Cahn equations and phase-field systems (notably by Bates et al. and Sprekels et al.)

$$\begin{aligned}
&\varphi_t + \boldsymbol{u}\cdot\nabla\varphi = \text{div}\,(m(\varphi)\nabla\mu)\\
&\mu = a\varphi - J*\varphi + F'(\varphi)\\
&\boldsymbol{u}_t - 2\text{div}(\nu(\varphi)D\boldsymbol{u}) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla\pi = \mu\nabla\varphi + \boldsymbol{v}\\
&\text{div}(\boldsymbol{u}) = 0
\end{aligned}$$

subject to

$$\frac{\partial\mu}{\partial n} = 0 \qquad \boldsymbol{u} = 0 \qquad \text{on} \quad \partial\Omega\times(0,\infty)$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \qquad \varphi(0) = \varphi_0 \qquad \text{in} \quad \Omega$$

- Mass is conserved

$$\overline{\varphi(t)} := |\Omega|^{-1}\int_\Omega \varphi(x,t)dx = \overline{\varphi}_0$$

- **Constant mobility+ regular potential**
  - $\exists$ **global weak sols in 2D-3D** (Colli, F. & Grasselli, J. Math. Anal. Appl. '12)
  - global attractor in 2D and trajectory attractor in 3D (F. & Grasselli, J. Dynam Differential Equations '12)
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- **Degenerate mobility+ singular potential**
  - ∃ and regularity of global weak sols in 2D-3D, global attractor in 2D (F., Grasselli & Rocca, Nonlinearity ’15)

More recent results

- **Constant mobility+ regular or singular potential & degenerate mobility + singular potential**
  - **Uniqueness of global weak sols in 2D**

- **Constant mobility, nonconstant viscosity +regular potential**
  - ∃ global unique strong sols in 2D, regularity of global attractor in 2D, convergence to equilibria of weak sols in 2D
  - weak-strong uniqueness in 2D
  - Connectedness and regularity of global attractor, ∃ exponential attractor in 2D.

Last results in: F., Gal & Grasselli, WIAS Preprint ’14

Theorem (Colli, F. & Grasselli '12)

Assume $J \in W^{1,1}(\mathbb{R}^d)$ *and that* $\boldsymbol{v} \in L^2(0,T;H^1_{div}(\Omega)')$, $\boldsymbol{u}_0 \in L^2_{div}(\Omega)^d$, $\varphi_0 \in L^2(\Omega)$ *with* $F(\varphi_0) \in L^1(\Omega)$. *Then,* $\forall T > 0\ \exists$ *a weak sol* $[\boldsymbol{u}, \varphi]$ *on* $[0,T]$ *s.t.*

$$\boldsymbol{u} \in L^\infty(0,T;L^2_{div}(\Omega)^d) \cap L^2(0,T;H^1_{div}(\Omega)^d), \qquad \boldsymbol{u}_t \in L^{4/d}(0,T;H^1_{div}(\Omega)')$$
$$\varphi \in L^\infty(0,T;L^4(\Omega)) \cap L^2(0,T;H^1(\Omega)), \qquad \varphi_t \in L^2(0,T;H^1(\Omega)')$$
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which satisfies the energy inequality (identity if $d = 2$*)*

$$\mathcal{E}(\boldsymbol{u}(t),\varphi(t)) + \int_0^t (\nu\|\nabla\boldsymbol{u}(\tau)\|^2 + \|\nabla\mu(\tau)\|^2)d\tau \leq \mathcal{E}(\boldsymbol{u}_0,\varphi_0) + \int_0^t \langle \boldsymbol{v}(\tau), \boldsymbol{u}(\tau)\rangle d\tau$$

for all $t > 0$, *where we have set*

$$\mathcal{E}(\boldsymbol{u}(t),\varphi(t)) = \frac{1}{2}\|\boldsymbol{u}(t)\|^2 + \frac{1}{4}\int_\Omega\int_\Omega J(x-y)(\varphi(x,t)-\varphi(y,t))^2 dxdy + \int_\Omega F(\varphi(t))$$

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- **The nonlocal term implies that φ is not as regular as for the standard (local) CHNS system**: $\varphi \in L^2(H^1)$ (nonlocal), instead of $\varphi \in L^\infty(H^1)$ (local) $\Longrightarrow$ regularity results and uniqueness of weak sols in 2D difficult issues

- We need stronger assumptions on J. In particular $J \in W^{2,1}(\mathbb{R}^2)$ or J **admissible**

Definition (J. Bedrossian, N. Rodríguez & A. Bertozzi '11)

A kernel $J \in W^{1,1}_{loc}(\mathbb{R}^2)$ is admissible if the following conditions are satisfied:

(A1) $J \in C^3(\mathbb{R}^d \setminus \{0\})$;

(A2) J *is radially symmetric,* $J(x) = \tilde{J}(|x|)$ *and* $\tilde{J}$ *is non-increasing;*

(A3) $\tilde{J}''(r)$ *and* $\tilde{J}'(r)/r$ *are monotone on* $(0, r_0)$ *for some* $r_0 > 0$;

(A4) $|D^3 J(x)| \leq C_d |x|^{-d-1}$ *for some* $C_d > 0$

Newtonian and Bessel kernels are admissible for all $d \geq 2$

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Lemma (J. Bedrossian, N. Rodríguez & A. Bertozzi '11)

Let J be admissible and $\chi = \nabla J * \psi$. Then, for all $p \in (1, \infty)$, there exists $C_p > 0$ such that

$$\|\nabla \chi\|_{L^p(\Omega)} \leq C_p \|\psi\|_{L^p(\Omega)}$$

Theorem (F., Grasselli & Krejčí '13)

Assume that $J \in W^{2,1}(\mathbb{R}^2)$ or J admissible and that

$$\boldsymbol{v} \in L^2(0,T;L^2_{div}(\Omega)^2) \qquad \boldsymbol{u}_0 \in H^1_{div}(\Omega)^2 \qquad \varphi_0 \in H^2(\Omega)$$

Then, $\forall T > 0\ \exists$ **unique** *strong sol $[\boldsymbol{u},\varphi]$ on $[0,T]$ s.t.*

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Only recently (F., Gal & Grasselli, WIAS Preprint '14) we included

- **Nonconstant viscosity**

$$\nu = \nu(\varphi), \qquad \nu \text{ loc. Lipschitz on } \mathbb{R}, \qquad 0 < \nu_1 \leq \nu(\varphi) \leq \nu_2$$

How to handle with nonconstant viscosity to get regularity results?

- We cannot rely on NS regularity in 2D to get $\boldsymbol{u} \in L^2\big(0,T;H^2(\Omega)^2\big)$. Indeed

$$\varphi \text{ weak sol }, \quad \boldsymbol{u} \in H^2(\Omega)^2 \cap H^1_{div}(\Omega)^2 \Longrightarrow \text{div}(\nu(\varphi)D\boldsymbol{u}) \in L^{2-\epsilon}(\Omega)^2$$

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- Approach: (nonloc CH)$\times \mu_t$ and avoid the use of the H^2- norm of $\boldsymbol{u}$. We deduce

$$\frac{d}{dt}\|\nabla\mu\|^2 + c_0\|\varphi_t\|^2 \le Q(R)\big(\|\boldsymbol{u}\|^2\|\nabla\boldsymbol{u}\|^2\big)\|\nabla\varphi\|^2 + c\|\boldsymbol{u}\|^2\|\nabla\boldsymbol{u}\|^2\|\nabla\varphi\|^2 + Q(R)$$

$$+ c\big(\|\nabla a\|^2_{L^\infty(\Omega)} + Q(R)\big)\|\nabla\varphi\|^2 + Q(R)\sum_{i,j=1}^{2}\|\partial^2_{ij}a\|^2$$

$$+ c\sum_{i,j=1}^{2}\|\partial_i(\partial_j J * \varphi)\|^2 + c\|J\|^2_{W^{1,1}(\mathbb{R}^2)}\|\varphi_t\|^2_{H^1(\Omega)'} \qquad \|\varphi\|_{L^\infty(Q)} \le R$$

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- Second step: (NS)$\times \boldsymbol{u}_t$, integrate by parts in time to get

$$\frac{1}{2}\|\boldsymbol{u}_t\|^2 + \frac{d}{dt}\int_\Omega \nu(\varphi)|D\boldsymbol{u}|^2 + b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{u}_t) \leq \frac{1}{2}\|l\|^2 + \int_\Omega |D\boldsymbol{u}|^2\nu'(\varphi)\varphi_t$$

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$$\frac{d}{dt}\int_\Omega \nu(\varphi)|D\boldsymbol{u}|^2 + \frac{1}{8}\|\boldsymbol{u}_t\|^2$$
$$\leq Q(R,\|\varphi_0\|_V,\|\boldsymbol{u}_0\|)\Big(\|l\|^2 + ((\|\boldsymbol{u}\|^2 + \|\boldsymbol{u}\|^{p-2})\|\nabla\boldsymbol{u}\|^2)\|D\boldsymbol{u}\|^2$$
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Exploiting the regularity obtained at previous step

$$\Longrightarrow \boldsymbol{u} \in L^\infty(0,T;H^1_{div}(\Omega)^2) \cap L^2(0,T;H^2(\Omega)^2) \qquad \boldsymbol{u}_t \in L^2(0,T;L^2_{div}(\Omega)^2)$$

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and then also

$$\varphi_t\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega)) \qquad \varphi\in L^\infty(0,T;H^2(\Omega))$$

Relevant case: mobility m degenerates at ± 1 and singular double-well potential F on $(-1, 1)$ (e.g. logarithmic like).

- $\varphi-$dependent mobility in the original derivation of CH eq. (J.W. Cahn & J.E. Hilliard, 1971). Thermodynamically reasonable choice

$$m(\varphi) = k(1 - \varphi^2)$$

- Key assumption (cf. [Elliot & Garcke '96], [Gajewski & Zacharias '03], [Giacomin & Lebowitz '97,'98])

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- local CH eq. : Elliot & Garcke '96 (∃), Schimperna & Zelik '13 (∃, asymptotic behavior, separation)
- nonlocal CH eq.: Giacomin & Lebowitz '97,'98, Gajewski & Zacharias '03 (∃ and uniqueness), Londen & Petzeltová '11, '11 (conv. to eq., separation)
- local CHNS: Boyer '99 (∃), Abels, Depner & Garcke '13 (unmatched densities, ∃)

We are not able to control $\nabla\mu$ in some L^p space; hence we reformulate the definition of weak sol in such a way that μ does not appear any more.

Notion of weak sol

Let $\boldsymbol{u}_0 \in L^2_{div}(\Omega)^d$, $\varphi_0 \in L^2(\Omega)$ with $F(\varphi_0) \in L^1(\Omega)$ and $0 < T < +\infty$ be given. $[\boldsymbol{u}, \varphi]$ is a weak sol on $[0, T]$ if

- $\boldsymbol{u}$, φ satisfy

$$\boldsymbol{u} \in L^\infty(0,T; L^2_{div}(\Omega)^d) \cap L^2(0,T; H^1_{div}(\Omega)^d), \qquad \boldsymbol{u}_t \in L^{4/d}(0,T; H^1_{div}(\Omega)')$$
$$\varphi \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega)), \qquad \varphi_t \in L^2(0,T; H^1(\Omega)')$$

and

$$\varphi \in L^\infty(Q_T) \qquad |\varphi(x,t)| \leq 1 \quad \text{a.e. } (x,t) \in Q_T := \Omega \times (0,T)$$

- $\forall \psi \in H^1(\Omega)$, $\forall \boldsymbol{v} \in H^1_{div}(\Omega)^d$ and for a.e. $t \in (0,T)$

$$\langle \varphi_t, \psi \rangle + \int_\Omega \big(m(\varphi)F''(\varphi) + m(\varphi)a\big)\nabla\varphi \cdot \nabla\psi + \int_\Omega m(\varphi)(\varphi\nabla a - \nabla J * \varphi) \cdot \nabla\psi = (\boldsymbol{u}\varphi, \nabla\psi)$$
$$\langle \boldsymbol{u}_t, \boldsymbol{v} \rangle + \nu(\nabla\boldsymbol{u}, \nabla\boldsymbol{v}) + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) = \big((a\varphi - J * \varphi)\nabla\varphi, \boldsymbol{v}\big) + \langle \boldsymbol{h}, \boldsymbol{v} \rangle$$
$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \qquad \varphi(0) = \varphi_0$$

Theorem (F., Grasselli & Rocca '15)

Let $M \in C^2(-1,1)$ *s.t.* $m(s)M''(s) = 1$, $M(0)= M'(0) = 0$. *Let*

$$\boldsymbol{u}_0 \in L^2_{div}(\Omega)^d, \quad \varphi_0 \in L^\infty(\Omega), \quad F(\varphi_0) \in L^1(\Omega), \quad M(\varphi_0) \in L^1(\Omega)$$

Then, $\forall T > 0$ $\exists$ *a weak sol* $z := [\boldsymbol{u}, \varphi]$ *on* $[0,T]$ *s.t.* $\overline{\varphi(t)} = \overline{\varphi}_0$ $\forall t \in [0,T]$. *In addition,* z *satisfies the energetic inequality (identity if* $d = 2$*)*

$$\frac{1}{2}\left(\|\boldsymbol{u}(t)\|^2 + \|\varphi(t)\|^2\right) + \int_0^t \int_\Omega \big(m(\varphi)F''(\varphi) + am(\varphi)\big)|\nabla\varphi|^2 + \nu \int_0^t \|\nabla \boldsymbol{u}\|^2 \leq \frac{1}{2}\|\boldsymbol{u}_0\|^2$$
$$+ \frac{1}{2}\|\varphi_0\|^2 + \int_0^t \int_\Omega (a\varphi - J * \varphi)\boldsymbol{u} \cdot \nabla\varphi + \int_0^t \int_\Omega m(\varphi)(\nabla J * \varphi - \varphi\nabla a) \cdot \nabla\varphi + \int_0^t \langle \boldsymbol{h}, \boldsymbol{u}\rangle$$

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Remark: a comparison with the constant mobility case

- Condition $|\overline{\varphi}_0| < 1$ not required (different weak sol formulation w.r.t. the case of const. mob.)
- Therefore, if F is bdd (e.g. F is the log pot) and at $t = 0$ the fluid is in a pure phase, i.e. $\varphi_0 = 1$ a.e. in Ω, and furthermore $\boldsymbol{u}_0 = \boldsymbol{u}(0)$ is given in $L^2(\Omega)^d_{div}$, then the couple $\boldsymbol{u} = \boldsymbol{u}(x,t)$, $\varphi = \varphi(x,t) = 1$ a.e. in Ω, a.a. t, where $\boldsymbol{u}$ is a sol of NS with non-slip b.c. explicitly satisfies the weak formulation.

Theorem (F., Grasselli & Rocca '15)

Let φ_0 be s.t.

$$F'(\varphi_0) \in L^2(\Omega)$$

Then, $\exists$ weak sol $z = [\boldsymbol{u}, \varphi]$ that also satisfies

$$\mu \in L^\infty(0,T;L^2(\Omega)) \qquad \nabla\mu \in L^2(0,T;L^2(\Omega)^d)$$

As a consequence, $z = [\boldsymbol{u}, \varphi]$ also satisfies the weak formulation and the energy inequality (identity for $d = 2$) of the non degenerate mobility case

$$\mathcal{E}(\boldsymbol{u}(t),\varphi(t)) + \int_0^t (\nu\|\nabla\boldsymbol{u}(\tau)\|^2 + \|\sqrt{m(\varphi)}\nabla\mu(\tau)\|^2)d\tau \leq \mathcal{E}(\boldsymbol{u}_0,\varphi_0) + \int_0^t \langle\boldsymbol{v}(\tau),\boldsymbol{u}(\tau)\rangle d\tau$$

for all $t > 0$, where we have set

$$\mathcal{E}(\boldsymbol{u}(t),\varphi(t)) = \frac{1}{2}\|\boldsymbol{u}(t)\|^2 + \frac{1}{4}\int_\Omega\int_\Omega J(x-y)(\varphi(x,t)-\varphi(y,t))^2 dxdy + \int_\Omega F(\varphi(t))$$

Constant mobility + regular potentials

Theorem (F., Gal & Grasselli '14)

Let $\boldsymbol{u}_0 \in L^2_{div}(\Omega)^2$, $\varphi_0 \in L^2(\Omega)$ *with* $F(\varphi_0) \in L^1(\Omega)$. *Then,* $\exists$ *a* **unique** *weak sol* $[\boldsymbol{u}, \varphi]$ *corresponding to* $[\boldsymbol{u}_0, \varphi_0]$

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- **Uniqueness of sol and $\exists$ of the global attractor for the local CH with degenerate mobility are open issues**

Consequences

- the nonlocal CHNS system generates a **semigroup** $S(t)$ of *closed* operators: $[\boldsymbol{u}(t), \varphi(t)] = S(t)[\boldsymbol{u}_0, \varphi_0]$ on the (metric) phase-space

$$\mathcal{X}_\eta = L^2_{div}(\Omega)^2 \times \mathcal{Y}_\eta \quad \mathcal{Y}_\eta = \{\varphi \in L^2(\Omega) : F(\varphi) \in L^1(\Omega), |\bar{\varphi}| \leq \eta\}$$

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- The global attractor in $\mathcal{X}_\eta$ for $S_\eta(t)$ is **connected**
- **Smoothing property** for the difference of two sols in $L^2_{div}(\Omega)^2 \times L^2(\Omega)$

Theorem (F., Gal & Grasselli '14)

For every $\eta \geq 0$ the dynamical system $\big(\mathcal{X}_\eta, S(t)\big)$ possesses an **exponential attractor** $\mathcal{M}_\eta$*, i.e., a compact set in $\mathcal{X}_\eta$ s.t.*

(i) *Positively invariance: $S(t)\mathcal{M} \subset \mathcal{M}\ \forall t \geq 0$*

(ii) *Finite dimensionality: $\dim_F \mathcal{M} < \infty$*

(iii) *Exponential attraction: $\exists J : \mathbb{R}^+ \to \mathbb{R}^+$ increasing and $\kappa > 0$ s.t., $\forall R > 0$ and $\forall \mathcal{B} \subset \mathcal{X}_\eta$ with $\sup_{z\in\mathcal{B}} \mathbf{d}_{\mathcal{X}_\eta}(z, 0) \leq R$ there holds*

$$\textit{dist}(S(t)\mathcal{B}, \mathcal{M}) \leq J(R)e^{-\kappa t}$$

Constant mobility+regular potential

Problem **(CP)**: minimize the **cost functional**

$$J(y,\boldsymbol{v}) := \frac{\beta_1}{2}\|\boldsymbol{u}-\boldsymbol{u}_Q\|^2_{L^2(Q)^2} + \frac{\beta_2}{2}\|\varphi-\varphi_Q\|^2_{L^2(Q)} + \frac{\beta_3}{2}\|\boldsymbol{u}(T)-\boldsymbol{u}_\Omega\|^2 + \frac{\beta_4}{2}\|\varphi(T)-\varphi_\Omega\|^2 + \frac{\gamma}{2}\|\boldsymbol{v}\|^2_{L^2(Q)^2}$$

where $y := [\boldsymbol{u},\varphi]$ solves

(nlocCHNS)

$$\begin{aligned}
&\boldsymbol{u}_t - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla\pi = \mu\nabla\varphi + \boldsymbol{v}\\
&\varphi_t + \boldsymbol{u}\cdot\nabla\varphi = \Delta\mu\\
&\mu = a\varphi - J*\varphi + F'(\varphi)\\
&\text{div}(\boldsymbol{u}) = 0\\
&\partial_{\boldsymbol{n}}\mu = 0 \qquad \boldsymbol{u} = 0 \qquad \text{on } \partial\Omega\\
&\boldsymbol{u}(0) = \boldsymbol{u}_0 \qquad \varphi(0) = \varphi_0
\end{aligned}$$

and the external body force density $\boldsymbol{v}$, which plays the role of the **control**, belongs to a suitable closed, bounded and convex subset of the **space of controls**

$$\mathcal{V} := L^2\big(0,T;L^2_{div}(\Omega)^2\big)$$

- Introducing the space

$$\mathcal{H} := \left[L^\infty(0,T;H^1_{div}(\Omega)^2) \cap L^2(0,T;H^2(\Omega)^2)\right] \times L^\infty(0,T;H^2(\Omega))$$

then, the **control-to-state map**

$$S : \mathcal{V} \to \mathcal{H}, \qquad \boldsymbol{v} \in \mathcal{V} \mapsto S(\boldsymbol{v}) := y := [\boldsymbol{u}, \varphi] \in \mathcal{H}$$

where $y := [\boldsymbol{u}, \varphi]$ is the unique strong sol to Problem **(nloc CHNS)** corresponding to $\boldsymbol{v} \in \mathcal{V}$ and to fixed initial data $\boldsymbol{u}_0 \in H^1_{div}(\Omega)^2$, $\varphi_0 \in H^2(\Omega)$, is well defined

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- Set of **admissible controls**

$$\mathcal{V}_{ad} := \left\{\boldsymbol{v} \in \mathcal{V} : \ v_{a,i}(x,t) \le v_i(x,t) \le v_{b,i}(x,t), \ \text{a.e. } (x,t) \in Q, \ \ i = 1,2\right\}$$

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- Introducing the **reduced cost functional** $f(\boldsymbol{v}) := J\big(S(\boldsymbol{v}), \boldsymbol{v}\big)$, for all $\boldsymbol{v} \in \mathcal{V}$, then

$$\textbf{(CP)} \iff \min_{\boldsymbol{v} \in \mathcal{V}_{ad}} f(\boldsymbol{v})$$

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Theorem

*Problem **(CP)** admits a sol* $\overline{\boldsymbol{v}} \in \mathcal{V}_{ad}$, *with associated state* $\overline{y} := [\overline{\boldsymbol{u}}, \overline{\varphi}] := S(\overline{\boldsymbol{v}})$

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$$\boldsymbol{\xi}_t - \nu\Delta\boldsymbol{\xi} + (\overline{\boldsymbol{u}}\cdot\nabla)\boldsymbol{\xi} + (\boldsymbol{\xi}\cdot\nabla)\overline{\boldsymbol{u}} + \nabla\widetilde{\pi} = \big(a\eta - J*\eta + F''(\overline{\varphi})\eta\big)\nabla\overline{\varphi} + \overline{\mu}\nabla\eta + \boldsymbol{h}$$

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$$\text{div}(\boldsymbol{\xi}) = 0$$

$$\boldsymbol{\xi} = 0, \qquad \frac{\partial}{\partial\boldsymbol{n}}\big(a\eta - J*\eta + F''(\overline{\varphi})\eta\big) = 0 \qquad \text{on } \Sigma := \partial\Omega\times(0,T)$$

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Proposition

For every $\boldsymbol{h}\in\mathcal{V}$ the linearized problem above has a unique sol satisfying

$$\boldsymbol{\xi}\in C\big([0,T];L^2_{div}(\Omega)^2\big)\cap L^2\big(0,T;H^1_{div}(\Omega)^2\big), \quad \eta\in C([0,T];L^2(\Omega))\cap L^2(0,T;H^1(\Omega))$$

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Remark. States $\overline{y} = [\overline{\boldsymbol{u}}, \overline{\varphi}]$ need to be **strong sols** to **(nloc CHNS)**

Differentiability of the control-to-state operator. Set

$$\mathcal{Z} := \big[C\big([0,T];L^2_{div}(\Omega)^2\big) \cap L^2\big(0,T;H^1_{div}(\Omega)^2\big)\big] \times \big[C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))\big]$$

Theorem

The control-to-state operator $S:\mathcal{V}\to\mathcal{Z}$ is Frechét differentiable on $\mathcal{V}$ and the Frechét derivative $S'(\overline{\boldsymbol{v}})\in\mathcal{L}(\mathcal{V},\mathcal{Z})$ is given by

$$S'(\overline{\boldsymbol{v}})\boldsymbol{k} = [\boldsymbol{\xi}^{\boldsymbol{k}},\eta^{\boldsymbol{k}}], \qquad \forall \boldsymbol{k}\in\mathcal{V},$$

where $[\boldsymbol{\xi}^{\boldsymbol{k}},\eta^{\boldsymbol{k}}]$ is the unique sol to the linearized system at $[\overline{\boldsymbol{u}},\overline{\varphi}] = S(\overline{\boldsymbol{v}})$ and corresponding to $\boldsymbol{k}\in\mathcal{V}$

Key tool for the proof: stability estimates

Lemma (Stability estimate I — F., Gal & Grasselli '14)

Let $\boldsymbol{u}_{0i} := \boldsymbol{u}_i(0) \in H^1_{div}(\Omega)^2$, $\varphi_{0i} := \varphi_i(0) \in H^2(\Omega)$, $\boldsymbol{v}_i \in L^2(0,T;L^2_{div}(\Omega)^2)$ *and let* $[\boldsymbol{u}_i, \varphi_i]$ *be the corresponding (unique) strong sols,* $i = 1, 2$*. Then, we have*

$$\|\boldsymbol{u}_2 - \boldsymbol{u}_1\|^2_{L^\infty(0,T;L^2_{div}(\Omega)^2)} + \|\boldsymbol{u}_2 - \boldsymbol{u}_1\|^2_{L^2(0,T;H^1_{div}(\Omega)^2)} + \|\varphi_2 - \varphi_1\|^2_{L^\infty(0,T;L^2(\Omega))}$$

$$+ \|\varphi_2 - \varphi_1\|^2_{L^2(0,T;H^1(\Omega))} \leq \Lambda_1 \big(\|\boldsymbol{u}_{20} - \boldsymbol{u}_{10}\|^2 + \|\varphi_{20} - \varphi_{10}\|^2 + \|\boldsymbol{v}_2 - \boldsymbol{v}_1\|^2_{\mathcal{V}}\big)$$

where

$$\Lambda_1 = \Lambda_1\big(\|\nabla \boldsymbol{u}_{01}\|, \|\varphi_{01}\|_{H^2(\Omega)}, \|\boldsymbol{v}_1\|_{\mathcal{V}}, \|\nabla \boldsymbol{u}_{02}\|, \|\varphi_{02}\|_{H^2(\Omega)}, \|\boldsymbol{v}_2\|_{\mathcal{V}}\big)$$

Key tool for the proof: stability estimates

Lemma (Stability estimate I — F., Gal & Grasselli '14)

Let $\boldsymbol{u}_{0i} := \boldsymbol{u}_i(0) \in H^1_{div}(\Omega)^2$, $\varphi_{0i} := \varphi_i(0) \in H^2(\Omega)$, $\boldsymbol{v}_i \in L^2(0,T;L^2_{div}(\Omega)^2)$ *and let* $[\boldsymbol{u}_i, \varphi_i]$ *be the corresponding (unique) strong sols,* $i = 1, 2$*. Then, we have*

$$\|\boldsymbol{u}_2 - \boldsymbol{u}_1\|^2_{L^\infty(0,T;L^2_{div}(\Omega)^2)} + \|\boldsymbol{u}_2 - \boldsymbol{u}_1\|^2_{L^2(0,T;H^1_{div}(\Omega)^2)} + \|\varphi_2 - \varphi_1\|^2_{L^\infty(0,T;L^2(\Omega))}$$
$$+ \|\varphi_2 - \varphi_1\|^2_{L^2(0,T;H^1(\Omega))} \leq \Lambda_1\big(\|\boldsymbol{u}_{20} - \boldsymbol{u}_{10}\|^2 + \|\varphi_{20} - \varphi_{10}\|^2 + \|\boldsymbol{v}_2 - \boldsymbol{v}_1\|^2_{\mathcal{V}}\big)$$

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Remak. To prove Frechét differentiability of $S : \mathcal{V} \to \mathcal{Z}$ we need an improved stability estimate

Key tool for the proof: stability estimates

Lemma (Stability estimate II)

Let $\boldsymbol{u}_{0i} := \boldsymbol{u}_i(0) \in H^1_{div}(\Omega)^2$, $\varphi_{0i} := \varphi_i(0) \in H^2(\Omega)$, $\boldsymbol{v}_i \in L^2(0,T;L^2_{div}(\Omega)^2)$ *and let* $[\boldsymbol{u}_i, \varphi_i]$ *be the corresponding (unique) strong sols,* $i = 1, 2$*. Then, we have*

$$\|\boldsymbol{u}_2 - \boldsymbol{u}_1\|^2_{L^\infty(0,T;L^2_{div}(\Omega)^2)} + \|\boldsymbol{u}_2 - \boldsymbol{u}_1\|^2_{L^2(0,T;H^1_{div}(\Omega)^2)} + \|\varphi_2 - \varphi_1\|^2_{L^\infty(0,T;H^1(\Omega))}$$
$$+ \|\varphi_2 - \varphi_1\|^2_{L^2(0,T;H^2(\Omega))} \leq \Lambda_2 \big(\|\boldsymbol{u}_{20} - \boldsymbol{u}_{10}\|^2 + \|\varphi_{20} - \varphi_{10}\|^2_{H^1(\Omega)} + \|\boldsymbol{v}_2 - \boldsymbol{v}_1\|^2_{\mathcal{V}} \big)$$

where

$$\Lambda_2 = \Lambda_2 \big(\|\nabla \boldsymbol{u}_{01}\|, \|\varphi_{01}\|_{H^2(\Omega)}, \|\boldsymbol{v}_1\|_{\mathcal{V}}, \|\nabla \boldsymbol{u}_{02}\|, \|\varphi_{02}\|_{H^2(\Omega)}, \|\boldsymbol{v}_2\|_{\mathcal{V}} \big)$$

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$$+ \|\varphi_2 - \varphi_1\|^2_{L^2(0,T;H^2(\Omega))} \leq \Lambda_2\big(\|\boldsymbol{u}_{20} - \boldsymbol{u}_{10}\|^2 + \|\varphi_{20} - \varphi_{10}\|^2_{H^1(\Omega)} + \|\boldsymbol{v}_2 - \boldsymbol{v}_1\|^2_{\mathcal{V}}\big)$$

where

$$\Lambda_2 = \Lambda_2\big(\|\nabla \boldsymbol{u}_{01}\|, \|\varphi_{01}\|_{H^2(\Omega)}, \|\boldsymbol{v}_1\|_{\mathcal{V}}, \|\nabla \boldsymbol{u}_{02}\|, \|\varphi_{02}\|_{H^2(\Omega)}, \|\boldsymbol{v}_2\|_{\mathcal{V}}\big)$$

Sketch of the proof of differentiability of $S : \mathcal{V} \to \mathcal{Z}$. Let $\overline{\boldsymbol{v}} \in \mathcal{V}$ be fixed, $\overline{y} := [\overline{\boldsymbol{u}}, \overline{\varphi}] = S(\overline{\boldsymbol{v}})$, and consider a perturbation $\boldsymbol{h} \in \mathcal{V}$. Set

$$y^{\boldsymbol{h}} := [\boldsymbol{u}^{\boldsymbol{h}}, \varphi^{\boldsymbol{h}}] := S(\overline{\boldsymbol{v}} + \boldsymbol{h})$$
$$\boldsymbol{p}^{\boldsymbol{h}} := \boldsymbol{u}^{\boldsymbol{h}} - \overline{\boldsymbol{u}} - \boldsymbol{\xi}^{\boldsymbol{h}}, \qquad q^{\boldsymbol{h}} := \varphi^{\boldsymbol{h}} - \overline{\varphi} - \eta^{\boldsymbol{h}}$$

- Then, $\boldsymbol{p}^{\boldsymbol{h}}$, $q^{\boldsymbol{h}}$ solve

$$\begin{aligned}
&\boldsymbol{p}_t - \nu\Delta\boldsymbol{p} + (\boldsymbol{p}\cdot\nabla)\overline{\boldsymbol{u}} + (\overline{\boldsymbol{u}}\cdot\nabla)\boldsymbol{p} + \big((\boldsymbol{u}^{\boldsymbol{h}} - \overline{\boldsymbol{u}})\cdot\nabla\big)(\boldsymbol{u}^{\boldsymbol{h}} - \overline{\boldsymbol{u}}) + \nabla\pi^{\boldsymbol{h}} \\
&= a(\varphi^{\boldsymbol{h}} - \overline{\varphi})\nabla(\varphi^{\boldsymbol{h}} - \overline{\varphi}) - \big(J*(\varphi^{\boldsymbol{h}} - \overline{\varphi})\big)\nabla(\varphi^{\boldsymbol{h}} - \overline{\varphi}) + (aq - J*q)\nabla\overline{\varphi} \\
&+ (a\overline{\varphi} - J*\overline{\varphi})\nabla q + \big(F'(\varphi^{\boldsymbol{h}}) - F'(\overline{\varphi})\big)\nabla(\varphi^{\boldsymbol{h}} - \overline{\varphi}) + F'(\overline{\varphi})\nabla q \\
&+ \big(F'(\varphi^{\boldsymbol{h}}) - F'(\overline{\varphi}) - F''(\overline{\varphi})\eta^{\boldsymbol{h}}\big)\nabla\overline{\varphi} \qquad (0.1)
\end{aligned}$$

$$\begin{aligned}
&q_t + (\boldsymbol{u}^{\boldsymbol{h}} - \overline{\boldsymbol{u}})\cdot\nabla(\varphi^{\boldsymbol{h}} - \overline{\varphi}) + \boldsymbol{p}\cdot\nabla\overline{\varphi} + \overline{\boldsymbol{u}}\cdot\nabla q \\
&= \Delta\big(aq - J*q + F'(\varphi^{\boldsymbol{h}}) - F'(\overline{\varphi}) - F''(\overline{\varphi})\eta^{\boldsymbol{h}}\big) \qquad (0.2)
\end{aligned}$$

- Then, $\boldsymbol{p}^{\boldsymbol{h}}, q^{\boldsymbol{h}}$ solve

$$\begin{aligned} &\boldsymbol{p}_t - \nu\Delta\boldsymbol{p} + (\boldsymbol{p}\cdot\nabla)\overline{\boldsymbol{u}} + (\overline{\boldsymbol{u}}\cdot\nabla)\boldsymbol{p} + \big((\boldsymbol{u}^{\boldsymbol{h}} - \overline{\boldsymbol{u}})\cdot\nabla\big)(\boldsymbol{u}^{\boldsymbol{h}} - \overline{\boldsymbol{u}}) + \nabla\pi^{\boldsymbol{h}} \\ &= a(\varphi^{\boldsymbol{h}} - \overline{\varphi})\nabla(\varphi^{\boldsymbol{h}} - \overline{\varphi}) - \big(J * (\varphi^{\boldsymbol{h}} - \overline{\varphi})\big)\nabla(\varphi^{\boldsymbol{h}} - \overline{\varphi}) + (aq - J * q)\nabla\overline{\varphi} \\ &+ (a\overline{\varphi} - J * \overline{\varphi})\nabla q + \big(F'(\varphi^{\boldsymbol{h}}) - F'(\overline{\varphi})\big)\nabla(\varphi^{\boldsymbol{h}} - \overline{\varphi}) + F'(\overline{\varphi})\nabla q \\ &+ \big(F'(\varphi^{\boldsymbol{h}}) - F'(\overline{\varphi}) - F''(\overline{\varphi})\eta^{\boldsymbol{h}}\big)\nabla\overline{\varphi} \end{aligned} \tag{0.1}$$

$$\begin{aligned} &q_t + (\boldsymbol{u}^{\boldsymbol{h}} - \overline{\boldsymbol{u}})\cdot\nabla(\varphi^{\boldsymbol{h}} - \overline{\varphi}) + \boldsymbol{p}\cdot\nabla\overline{\varphi} + \overline{\boldsymbol{u}}\cdot\nabla q \\ &= \Delta\big(aq - J * q + F'(\varphi^{\boldsymbol{h}}) - F'(\overline{\varphi}) - F''(\overline{\varphi})\eta^{\boldsymbol{h}}\big) \end{aligned} \tag{0.2}$$

- Let us test (0.1) by $\boldsymbol{p}$ in $L^2_{div}(\Omega)^2$ and (0.2) by q in $L^2(\Omega)$. After some technical arguments we are led to

$$\frac{d}{dt}\big(\|\boldsymbol{p}^{\boldsymbol{h}}\|^2 + \|q^{\boldsymbol{h}}\|^2\big) + \nu\|\nabla\boldsymbol{p}^{\boldsymbol{h}}\|^2 + c_0\|\nabla q^{\boldsymbol{h}}\|^2 \leq \alpha(t)\|\boldsymbol{p}^{\boldsymbol{h}}\|^2 + \overline{\Gamma}\|q^{\boldsymbol{h}}\|^2 + \beta_{\boldsymbol{h}}(t)$$

$$\overline{\Gamma} = \overline{\Gamma}\big(\|\nabla\boldsymbol{u}_0\|, \|\varphi_0\|_{H^2(\Omega)}, \|\overline{\boldsymbol{v}}\|_{\mathcal{V}}\big)$$

and $\alpha, \beta_{\boldsymbol{h}} \in L^1(0,T)$ given by

$$\alpha := \overline{\Gamma}\big(1 + \|\overline{\boldsymbol{u}}\|^2_{H^2(\Omega)^2}\big)$$

$$\beta_{\boldsymbol{h}} := \overline{\Gamma}\big(\|\boldsymbol{u}^{\boldsymbol{h}} - \overline{\boldsymbol{u}}\|^2 \|\nabla(\boldsymbol{u}^{\boldsymbol{h}} - \overline{\boldsymbol{u}})\|^2 + \|\varphi^{\boldsymbol{h}} - \overline{\varphi}\|^2 \|\varphi^{\boldsymbol{h}} - \overline{\varphi}\|^2_{H^1(\Omega)}$$
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- Thanks to Stability estimate II we have

$$\int_0^T \beta_{\boldsymbol{h}}(t)dt \leq \overline{\Gamma}\|\boldsymbol{h}\|^4_{\mathcal{V}}$$

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$$\|\boldsymbol{p}^{\boldsymbol{h}}\|^2_{L^\infty(0,T;L^2_{div}(\Omega)^2)} + \nu\|\boldsymbol{p}^{\boldsymbol{h}}\|^2_{L^2(0,T;H^1_{div}(\Omega)^2)} + \|q^{\boldsymbol{h}}\|^2_{L^\infty(0,T;L^2(\Omega))}$$
$$+ c_0\|q^{\boldsymbol{h}}\|^2_{L^2(0,T;H^1(\Omega))} \leq \overline{\Gamma}\|\boldsymbol{h}\|^4_{\mathcal{V}}$$

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$$+ c_0\|q^{\boldsymbol{h}}\|^2_{L^2(0,T;H^1(\Omega))} \leq \overline{\Gamma}\|\boldsymbol{h}\|^4_{\mathcal{V}}$$

$$\Longrightarrow \frac{\|S(\overline{\boldsymbol{v}} + \boldsymbol{h}) - S(\overline{\boldsymbol{v}}) - [\boldsymbol{\xi}^{\boldsymbol{h}}, \eta^{\boldsymbol{h}}]\|_{\mathcal{Z}}}{\|\boldsymbol{h}\|_{\mathcal{V}}} \leq \overline{\Gamma}\|\boldsymbol{h}\|_{\mathcal{V}} \to 0 \quad \text{as } \boldsymbol{h} \to 0 \text{ in } \mathcal{V}$$

Remark. The weaker differentiability property of the control-to-state map from $\mathcal{V}$ with values in

$$\left[C\big([0,T];L^2_{div}(\Omega)^2\big)\cap L^2\big(0,T;H^1_{div}(\Omega)^2\big)\right]\times\left[C([0,T];H^1(\Omega)')\cap L^2(0,T;L^2(\Omega))\right]$$

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easier to establish: test (0.2) by $(-\Delta)^{-1}q$ and use only Stability estimate I

Nevertheless, with this weaker differentiability we get necessary conditions for existence of the optimal control for the control problem associated to the “incomplete"cost functional

$$J(y,\boldsymbol{v}):=\frac{\beta_1}{2}\|\boldsymbol{u}-\boldsymbol{u}_Q\|^2_{L^2(Q)^2}+\frac{\beta_2}{2}\|\varphi-\varphi_Q\|^2_{L^2(Q)}+\frac{\beta_3}{2}\|\boldsymbol{u}(T)-\boldsymbol{u}_\Omega\|^2 + \frac{\gamma}{2}\|\boldsymbol{v}\|^2_{L^2(Q)^2}$$

If $\overline{\boldsymbol{v}} \in \mathcal{V}_{ad}$ is an optimal control for Problem **(CP)**, then

$$f'(\overline{\boldsymbol{v}})(\boldsymbol{v} - \overline{\boldsymbol{v}}) \geq 0 \qquad \forall \boldsymbol{v} \in \mathcal{V}_{ad}$$

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But

$$f'(\boldsymbol{v}) = J'_y\big(S(\boldsymbol{v}), \boldsymbol{v}\big)S'(\boldsymbol{v}) + J'_{\boldsymbol{v}}\big(S(\boldsymbol{v}), \boldsymbol{v}\big)$$

and hence the Frechét differentiability result for $S : \mathcal{V} \to \mathcal{Z}$ yields

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and hence the Frechét differentiability result for $S : \mathcal{V} \to \mathcal{Z}$ yields

Corollary

*Let $\overline{\boldsymbol{v}} \in \mathcal{V}_{ad}$ be an optimal control for Problem **(CP)** with associated state $\overline{y} = [\overline{\boldsymbol{u}}, \overline{\varphi}] := S(\overline{\boldsymbol{v}})$. Then*

$$\beta_1 \int_0^T \int_\Omega (\overline{\boldsymbol{u}} - \boldsymbol{u}_Q) \cdot \boldsymbol{\xi}^{\boldsymbol{h}} + \beta_2 \int_0^T \int_\Omega (\overline{\varphi} - \varphi_Q)\eta^{\boldsymbol{h}} + \beta_3 \int_\Omega (\overline{\boldsymbol{u}}(T) - \boldsymbol{u}_\Omega) \cdot \boldsymbol{\xi}^{\boldsymbol{h}}(T)$$
$$+ \beta_4 \int_\Omega (\overline{\varphi}(T) - \varphi_\Omega)\eta^{\boldsymbol{h}}(T) + \gamma \int_0^T \int_\Omega \overline{\boldsymbol{v}} \cdot (\boldsymbol{v} - \overline{\boldsymbol{v}}) \geq 0 \qquad \forall \boldsymbol{v} \in \mathcal{V}_{ad}$$

where $[\boldsymbol{\xi}^{\boldsymbol{h}}, \eta^{\boldsymbol{h}}]$ is the unique sol to the linearized system corresponding to $\boldsymbol{h} = \boldsymbol{v} - \overline{\boldsymbol{v}}$

Aim: eliminate $\boldsymbol{\xi}^{\boldsymbol{h}}, \eta^{\boldsymbol{h}}$ from the previous inequality. Hence, introduce the **adjoint system**

$$\begin{aligned}
\widetilde{\boldsymbol{p}}_t =& -\nu\Delta\widetilde{\boldsymbol{p}} - (\overline{\boldsymbol{u}}\cdot\nabla)\widetilde{\boldsymbol{p}} + (\widetilde{\boldsymbol{p}}\cdot\nabla^T)\overline{\boldsymbol{u}} + \widetilde{q}\nabla\overline{\varphi} - \beta_1(\overline{\boldsymbol{u}} - \boldsymbol{u}_Q)\\
\widetilde{q}_t =& -(a\Delta\widetilde{q} + \nabla J \dot{*} \nabla\widetilde{q} + F''(\overline{\varphi})\Delta\widetilde{q}) - \overline{\boldsymbol{u}}\cdot\nabla\widetilde{q}\\
&- \big(a\widetilde{\boldsymbol{p}}\cdot\nabla\overline{\varphi} - J*(\widetilde{\boldsymbol{p}}\cdot\nabla\overline{\varphi}) + F''(\overline{\varphi})\widetilde{\boldsymbol{p}}\cdot\nabla\overline{\varphi}\big) + \widetilde{\boldsymbol{p}}\cdot\nabla\overline{\mu} - \beta_2(\overline{\varphi} - \varphi_Q)\\
&\text{div}(\widetilde{\boldsymbol{p}}) = 0\\
&\widetilde{\boldsymbol{p}} = 0, \qquad \frac{\partial\widetilde{q}}{\partial\boldsymbol{n}} = 0 \quad \text{on } \Sigma\\
&\widetilde{\boldsymbol{p}}(T) = \beta_3(\overline{\boldsymbol{u}}(T) - \boldsymbol{u}_\Omega), \qquad \widetilde{q}(T) = \beta_4(\overline{\varphi}(T) - \varphi_\Omega)
\end{aligned}$$

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$$\widetilde{q}_t = -\left(a\Delta\widetilde{q} + \nabla J \dot{*} \nabla\widetilde{q} + F''(\overline{\varphi})\Delta\widetilde{q}\right) - \overline{\boldsymbol{u}}\cdot\nabla\widetilde{q}$$
$$- \left(a\widetilde{\boldsymbol{p}}\cdot\nabla\overline{\varphi} - J*(\widetilde{\boldsymbol{p}}\cdot\nabla\overline{\varphi}) + F''(\overline{\varphi})\widetilde{\boldsymbol{p}}\cdot\nabla\overline{\varphi}\right) + \widetilde{\boldsymbol{p}}\cdot\nabla\overline{\mu} - \beta_2(\overline{\varphi} - \varphi_Q)$$

$$\text{div}(\widetilde{\boldsymbol{p}}) = 0$$

$$\widetilde{\boldsymbol{p}} = 0, \qquad \frac{\partial\widetilde{q}}{\partial\boldsymbol{n}} = 0 \quad \text{on } \Sigma$$

$$\widetilde{\boldsymbol{p}}(T) = \beta_3(\overline{\boldsymbol{u}}(T) - \boldsymbol{u}_\Omega), \qquad \widetilde{q}(T) = \beta_4(\overline{\varphi}(T) - \varphi_\Omega)$$

Proposition

The adjoint system has a unique weak sol $\widetilde{\boldsymbol{p}}, \widetilde{q}$ satisfying

$$\widetilde{\boldsymbol{p}} \in C\left([0,T]; L^2_{div}(\Omega)^2\right) \cap L^2\left(0,T; H^1_{div}(\Omega)^2\right), \quad \widetilde{q} \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))$$

Theorem

Let $\overline{\boldsymbol{v}} \in \mathcal{V}_{ad}$ *be an optimal control for Problem* ***(CP)*** *with associated state* $\overline{y} = [\overline{\boldsymbol{u}}, \overline{\varphi}] = S(\overline{\boldsymbol{v}})$ *and adjoint state* $[\widetilde{\boldsymbol{p}}, \widetilde{q}]$*. Then*

$$\gamma \int_0^T \int_\Omega \overline{\boldsymbol{v}} \cdot (\boldsymbol{v} - \overline{\boldsymbol{v}}) + \int_0^T \int_\Omega \widetilde{\boldsymbol{p}} \cdot (\boldsymbol{v} - \overline{\boldsymbol{v}}) \geq 0 \qquad \forall \boldsymbol{v} \in \mathcal{V}_{ad}$$

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- The system **(nloc CHNS)**, written for $[\overline{\boldsymbol{u}}, \overline{\varphi}]$, the adjoint system and the above variational inequality form together the first order necessary optimality conditions

Theorem

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- The system **(nloc CHNS)**, written for $[\overline{\boldsymbol{u}}, \overline{\varphi}]$, the adjoint system and the above variational inequality form together the first order necessary optimality conditions
- Since $\mathcal{V}_{ad}$ is a nonempty, closed and convex subset of $L^2(Q)^2$, then the above variational inequality with $\gamma > 0$ is equivalent to

$$\overline{\boldsymbol{v}} = P_{\mathcal{V}_{ad}}\Big(\Big\{ -\frac{\widetilde{\boldsymbol{p}}}{\gamma} \Big\}\Big)$$

where $P_{\mathcal{V}_{ad}}$ is the orthogonal projector in $L^2(Q)^2$ onto $\mathcal{V}_{ad}$

(In progress, jointly with E. Rocca & J. Sprekels)

(In progress, jointly with E. Rocca & J. Sprekels)

- Minimize the cost functional $J(y, \boldsymbol{v})$ where $y := [\boldsymbol{u}, \varphi]$ solves

$$\boldsymbol{u}_t - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla\pi = \mu\nabla\varphi + \boldsymbol{v}$$
$$\varphi_t + \boldsymbol{u}\cdot\nabla\varphi = \mathrm{div}(m(\varphi)\nabla\mu)$$
$$\mu = a\varphi - J*\varphi + F'(\varphi)$$
$$\mathrm{div}(\boldsymbol{u}) = 0$$
$$\partial_{\boldsymbol{n}}\mu = 0 \qquad \boldsymbol{u} = 0 \qquad \text{on } \partial\Omega$$
$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \qquad \varphi(0) = \varphi_0$$

and the control $\boldsymbol{v}$ belongs to a suitable closed, bounded and convex subset of the space of controls $\mathcal{V} := L^2\big(0, T; L^2_{div}(\Omega)^2\big)$

(In progress, jointly with E. Rocca & J. Sprekels)

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$$
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- Aim: first order necessary conditions for existence of optimal control

(In progress, jointly with E. Rocca & J. Sprekels)

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\end{aligned}
$$

  and the control $\boldsymbol{v}$ belongs to a suitable closed, bounded and convex subset of the space of controls $\mathcal{V} := L^2\big(0, T; L^2_{div}(\Omega)^2\big)$

- Aim: first order necessary conditions for existence of optimal control

- First prove existence of strong solutions in 2D (i.e., extend the regularity result in F., Grasselli & Krejčí '13 to the case of the nonlocal CHNS system in 2D with deg. mob.+ sing. pot.)

(In progress)

Singular potential, nondegenerate mobility

$$(\rho\boldsymbol{u})_t + \mathrm{div}(\rho\boldsymbol{u}\otimes\boldsymbol{u}) - \nu\Delta\boldsymbol{u} + \nabla\pi + \mathrm{div}(\boldsymbol{u}\otimes\widetilde{\boldsymbol{J}}) = \mu\nabla\varphi$$
$$\mathrm{div}(\boldsymbol{u}) = 0$$
$$\varphi_t + \boldsymbol{u}\cdot\nabla\varphi = \mathrm{div}(m(\varphi)\nabla\mu)$$
$$\mu = a\varphi - J*\varphi + F'(\varphi)$$
$$\widetilde{\boldsymbol{J}} := -\beta m(\varphi)\nabla\mu, \qquad \beta = (\widetilde{\rho}_2 - \widetilde{\rho}_1)/2$$

where

$$\rho(\varphi) = \frac{1}{2}(\widetilde{\rho}_2 + \widetilde{\rho}_1) + \frac{1}{2}(\widetilde{\rho}_2 - \widetilde{\rho}_1)\varphi$$

and where $\widetilde{\rho}_1, \widetilde{\rho}_2 > 0$ are the specific constant mass densities of the unmixed fluids.

The above system endowed with

$$\boldsymbol{u} = 0, \qquad \frac{\partial\mu}{\partial\boldsymbol{n}} = 0, \qquad \text{on } \Gamma := \partial\Omega$$
$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \qquad \varphi(0) = \varphi_0$$

- The previous system is the nonlocal version of the model derived by Abels, Garcke and Grün. In that case

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- $\exists$ weak sol derived by means of an implicit time discretization scheme (singular potential+non degenerate mobility)

- $\exists$ weak sol also derived for the case of degenerate mobility and regular potential

Theorem

Let $\boldsymbol{u}_0 \in L^2_{div}(\Omega)^d$, $\varphi_0 \in L^\infty(\Omega)$ *such that* $F(\varphi_0) \in L^1(\Omega)$ *and* $|\overline{\varphi}_0| < 1$. *Then,* $\forall T > 0 \, \exists$ *weak sol* $[\boldsymbol{u}, \varphi]$ *s.t.*

$$\boldsymbol{u} \in L^\infty(0,T; L^2_{div}(\Omega)^d) \cap L^2(0,T; H^1_{div}(\Omega)^d)$$
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$$\mu = a\varphi - J * \varphi + F'(\varphi) \in L^2(0,T; H^1(\Omega))$$

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and satisfying the following energy inequality

$$\int_\Omega \frac{1}{2}\rho(\varphi(t))|\boldsymbol{u}(t)|^2 + E(\varphi(t)) + \nu \int_0^t \|\nabla \boldsymbol{u}\|^2 d\tau + \int_0^t \left\|\sqrt{m(\varphi)}\nabla\mu\right\|^2 d\tau$$
$$\leq \int_\Omega \frac{1}{2}\rho(\varphi_0)\boldsymbol{u}_0^2 + E(\varphi_0), \qquad \forall t \in [0,T]$$

where

$$E(\varphi) = \frac{1}{4}\int_\Omega\int_\Omega J(x-y)\big(\varphi(x) - \varphi(y)\big)^2 dxdy + \int_\Omega F(\varphi)$$

Approach by Abels, Depner and Garcke hard to be directly applied here. Indeed

- the proof of Abels, Depner and Garcke (cf. fixed point argument for $\exists$ sol of the time-discrete problem) exploits the possibility of inverting the relation $\mu = -\Delta\varphi + F_0'(\varphi)$, where $F_0(s) = F(s) + \frac{k}{2}s^2$ is convex.

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Strategy of the proof I

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Idea: replace $\rho(\varphi)$ by a fixed smooth extension $\widetilde{\rho}(\varphi)$ from $[-1,1]$ onto $\mathbb{R}$ satisfying

$$0 < \rho_* \leq \widetilde{\rho}(s) \leq \rho^*, \qquad |\widetilde{\rho}^{(k)}(s)| \leq R_k, \qquad \forall s \in \mathbb{R}, \quad k = 1,2,$$
$$\widetilde{\rho}(s) = \rho(s), \qquad \forall s \in [-1,1]$$

with ρ_*, ρ^*, R_1, R_2 fixed positive constants.

Another difficulty: in this case the (formal) energy identity becomes

$$\frac{d}{dt}\Big(\int_\Omega \frac{1}{2}\widetilde{\rho}(\varphi)\boldsymbol{u}^2 + E_\epsilon(\varphi)\Big) + \nu\|\nabla \boldsymbol{u}\|^2 + \big\|\sqrt{m(\varphi)}\nabla\mu\big\|^2 = \frac{1}{2}\int_\Omega \widetilde{\rho}''(\varphi)m(\varphi)(\nabla\varphi\cdot\nabla\mu)\boldsymbol{u}^2$$

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Still another difficulty: how can we pass to the limit in this nasty nonlinear term?

Idea: introduce suitable regularizing terms (depending on another approx. parameter $\delta > 0$). In particular: $\delta A^3\boldsymbol{u}$ in the modified momentum balance equation and $-\delta\Delta\varphi$ in the chemical potential

Summing up, our approach consists in proving existence of a weak sol by approximating the nonlocal CHNS system with unmatched densities with a two-parameter family of problems $\mathbf{P}_{\epsilon,\delta}$

$$(\widetilde{\rho}\boldsymbol{u})_t + \mathrm{div}(\widetilde{\rho}\boldsymbol{u}\otimes\boldsymbol{u}) - \nu\Delta\boldsymbol{u} + \delta A^3\boldsymbol{u} + \nabla\pi + \mathrm{div}(\boldsymbol{u}\otimes\widetilde{\boldsymbol{J}}) + \frac{1}{2}\widetilde{\rho}''m(\varphi)(\nabla\varphi\cdot\nabla\mu)\boldsymbol{u} = \mu\nabla\varphi$$

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Remark: the regularizing term $\delta A^3\boldsymbol{u}$ should actually be introduced in the variational formulation of the momentum balance eq. (with test funct. $\boldsymbol{w}\in D(A^{3/2})$)

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- in the limit $|\varphi| < 1$. In particular, in the limit as $\epsilon\to 0$ the bad nonlinear term in the momentum balance eq. vanishes

Step I: Problem $\mathbf{P}_{\epsilon,\delta}$ has a sol (ϵ, δ fixed)

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- (Mod. mom. bal. eq.)$\times \boldsymbol{u}$ in $L^2_{div}(\Omega)^d$+(nonlocal CH)$\times\mu$ in $L^2(\Omega)$ to get

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- Interpolation, Gagliardo-Nirenberg ($\boldsymbol{u}\in L^6(Q)$, $\nabla\boldsymbol{u}\in L^{18/5}(Q)$, $\nabla\varphi\in L^{10/3}(Q)$) and comparison in the mod. mom. bal. eq. yield the bounds (not uniform in δ)

$$\big(\widetilde{\rho}(\varphi)\boldsymbol{u}\big)_t \in L^{30/29}(0,T;D(A^{3/2})'), \qquad \widetilde{\rho}(\varphi)\boldsymbol{u}\in L^{15/7}(0,T;W^{1,15/7}(\Omega)^3)$$

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- By Aubin-Lions we deduce strong convergence for $\widetilde{\rho}(\varphi)\boldsymbol{u}$ in $L^{15/7}(Q)$, strong convergence for φ in $L^2(0,T;H^{2-}(\Omega))\Longrightarrow$ strong convergence for $\boldsymbol{u}$ in $L^{6-}(Q)$. These allow to pass to the limit in the weak formulation of the mod. mom. bal. eq.

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Passing to the limit as $\epsilon \to 0$ we then find that $[\boldsymbol{u}, \varphi]$ solves Problem $\mathbf{P}_\delta$ given by

$$(\rho\boldsymbol{u})_t + \mathrm{div}(\rho\boldsymbol{u} \otimes \boldsymbol{u}) - \nu\Delta\boldsymbol{u} + \delta A^3\boldsymbol{u} + \nabla\pi + \mathrm{div}(\boldsymbol{u} \otimes \widetilde{\boldsymbol{J}}) = \mu\nabla\varphi$$

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$$+ \int_0^t \left\|\sqrt{m(\varphi_\delta)}\nabla\mu_\delta\right\|^2 d\tau \leq \int_\Omega \frac{1}{2}\rho(\varphi_{0\delta})\boldsymbol{u}_0^2 + E(\varphi_{0\delta}) + \frac{\delta}{2}\|\nabla\varphi_{0\delta}\|^2$$

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$$+ \int_0^t \left\|\sqrt{m(\varphi_\delta)}\nabla\mu_\delta\right\|^2 d\tau \leq \int_\Omega \frac{1}{2}\rho(\varphi_{0\delta})\boldsymbol{u}_0^2 + E(\varphi_{0\delta}) + \frac{\delta}{2}\|\nabla\varphi_{0\delta}\|^2$$

- weaker estimates for $[\boldsymbol{u}_\delta, \varphi_\delta]$ (uniform in δ). Then, $[\boldsymbol{u}_\delta, \varphi_\delta] \to [\boldsymbol{u}, \varphi]$ and by compactness arguments these estimates are enough to pass to the limit in Problem $\mathbf{P}_\delta$ as $\delta \to 0$

Step III: pass to the limit as $\delta \to 0$ in Problem $\mathbf{P}_\delta$

- $\forall \delta > 0$ Problem $\mathbf{P}_\delta$ admits a weak sol $[\boldsymbol{u}_\delta, \varphi_\delta]$ satisfying

$$\int_\Omega \frac{1}{2}\rho(\varphi_\delta)\boldsymbol{u}_\delta^2 + E(\varphi_\delta) + \frac{\delta}{2}\|\nabla\varphi_\delta\|^2 + \nu\int_0^t \|\nabla \boldsymbol{u}_\delta\|^2 d\tau + \delta\int_0^t \|A^{3/2}\boldsymbol{u}_\delta\|^2 d\tau$$
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Passing to the limit as $\delta \to 0$ we then find that $[\boldsymbol{u}, \varphi]$ solves the original Problem **P**

$$(\rho\boldsymbol{u})_t + \mathrm{div}(\rho\boldsymbol{u}\otimes\boldsymbol{u}) - \nu\Delta\boldsymbol{u} + \nabla\pi + \mathrm{div}(\boldsymbol{u}\otimes\widetilde{\boldsymbol{J}}) = \mu\nabla\varphi$$
$$\mathrm{div}(\boldsymbol{u}) = 0$$
$$\varphi_t + \boldsymbol{u}\cdot\nabla\varphi = \mathrm{div}(m(\varphi)\nabla\mu)$$
$$\widetilde{\boldsymbol{J}} := -\beta m(\varphi)\nabla\mu, \qquad \beta = (\widetilde{\rho}_2 - \widetilde{\rho}_1)/2$$

+ B.C and I.C.788

The singular potential is not regularized and the original problem is approximated with a one-parameter family of Problems $\widetilde{\mathbf{P}}_\delta$ given by

$$
\begin{aligned}
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&\mathrm{div}(\boldsymbol{u}) = 0\\
&\varphi_t + \boldsymbol{u}\cdot\nabla\varphi = \mathrm{div}(m(\varphi)\nabla\mu)\\
&\mu = a\varphi - J*\varphi + F'(\varphi) - \delta\Delta\varphi\\
&\widetilde{\boldsymbol{J}} := -\beta m(\varphi)\nabla\mu\\
&\boldsymbol{u} = 0, \qquad \frac{\partial\mu}{\partial\boldsymbol{n}} = \frac{\partial\varphi}{\partial\boldsymbol{n}} = 0, \qquad \text{on } \Gamma\\
&\boldsymbol{u}(0) = \boldsymbol{u}_0, \qquad \varphi(0) = \varphi_{0\delta}
\end{aligned}
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- use a time-discretization argument (as in Abels, Depner and Garcke) to prove that $\widetilde{\mathbf{P}}_\delta$ has a sol
- pass to the limit as $\delta \to 0$ in Problem $\widetilde{\mathbf{P}}_\delta$ as in the previous Step III

- compressible models
- **non-isothermal model(s)**
  (Eleuteri, Rocca & Schimperna, Discrete Contin. Dyn. Syst. '15 for the local CHNS)
- multicomponent models