Nonlocal diffuse-interface models for binary viscous incompressible fluids

Sergio Frigeri[1]

[1]Dipartimento di Matematica “F. Enriques”
Università degli Studi di Milano
www.mat.unimi.it/users/frigeri

MathProSpeM2012
INdAM, Roma, April 16-19 2012

Supported by the FP7-IDEAS-ERC-StG Grant “EntroPhase”

Local Cahn-Hilliard-Navier-Stokes systems

In $\Omega \times (0, \infty)$, $\Omega \subset \mathbb{R}^d$, $d = 2, 3$

$$
\begin{aligned}
& u_t + (u \cdot \nabla) u - \operatorname{div}(\nu(\varphi) D u) + \nabla \pi = \mu \nabla \varphi + h \\
& \operatorname{div}(u) = 0 \\
& \varphi_t + u \cdot \nabla \varphi = \operatorname{div}\,(k(\varphi) \nabla \mu) \\
& \mu = -\epsilon \Delta \varphi + \epsilon^{-1} F'(\varphi)
\end{aligned}
$$

μ chemical potential, first variation of the free energy

$$
E(\varphi) = \int_\Omega \left(\frac{\epsilon}{2} |\nabla \varphi|^2 + \frac{1}{\epsilon} F(\varphi) \right) dx
$$

Rigorous derivation by Gurtin, Polignone and Viñals '96

Local Cahn-Hilliard-Navier-Stokes systems

- F double-well potential
  - Regular, e.g.

$$F(s) = (1 - s^2)^2, \qquad \forall s \in \mathbb{R}$$

  - Singular, e.g.

$$F(s) = \frac{\theta}{2}((1+s)\log(1+s) + (1-s)\log(1-s)) - \frac{\theta_c}{2}s^2$$

  for all $s \in (-1, 1)$, with $\theta < \theta_c$
- Mathematical results by V.N. Starovoitov ('97), F. Boyer ('99), Abels '09, Abels and Feireisl '08 (existence of weak and strong solutions, uniqueness and regularity) and by Abels '09, Gal & Grasselli '09 , Zhao, Wu & Huang '09 (convergence to single equilibria), Abels '09, Gal & Grasselli '09, '10 and '11 (attractors).

Nonlocal Cahn-Hilliard-Navier-Stokes systems

- **Nonlocal free energy** (van der Waals)

$$\mathcal{E}(\varphi) = \frac{1}{4}\int_\Omega\int_\Omega J(x-y)(\varphi(x)-\varphi(y))^2 dxdy + \eta\int_\Omega F(\varphi(x))dx$$

  where $J : \mathbb{R}^d \to \mathbb{R}$ s.t. $J(x) = J(-x)$
  Local free energy is an approximation of the nonlocal one
- **Nonlocal chemical potential**

$$\mu = a\varphi - J * \varphi + \eta F'(\varphi)$$

  where

$$(J * \varphi)(x) := \int_\Omega J(x-y)\varphi(y)dy \quad a(x) := \int_\Omega J(x-y)dy$$

Nonlocal Cahn-Hilliard-Navier-Stokes systems

Consider in $\Omega \times (0, \infty)$ ($\Omega \subset \mathbb{R}^d$ bounded, $d = 2, 3$)

$$
\begin{aligned}
&\varphi_t + u \cdot \nabla \varphi = \Delta \mu \\
&\mu = a\varphi - J * \varphi + F'(\varphi) \\
&u_t - \mathrm{div}(\nu(\varphi) Du) + (u \cdot \nabla) u + \nabla \pi = \mu \nabla \varphi + h \\
&\mathrm{div}(u) = 0
\end{aligned}
$$

subject to

$$
\begin{aligned}
&\frac{\partial \mu}{\partial n} = 0 \quad u = 0 \qquad \text{on} \quad \partial\Omega \times (0, \infty) \\
&u(0) = u_0 \quad \varphi(0) = \varphi_0 \qquad \text{in} \quad \Omega
\end{aligned}
$$

Some literature on nonlocal models

- Cahn-Hilliard equation: Giacomin & Lebowitz '97 and '98; Chen & Fife '00; Gajewski '02; Gajewski & Zacharias '03; Han '04; Bates & Han '05 ; Colli, Krejčí, Rocca & Sprekels '07; Londen & Petzeltová '11
- Navier-Stokes-Korteweg systems (liquid-vapour phase transitions): Rohde '05
- several other contributions on nonlocal Allen-Cahn equations and phase-field systems (notably by Bates et al. and Sprekels et al.)

$\exists$ weak sols (smooth potentials)

(A1) $J \in W^{1,1}(\mathbb{R}^d)$ s.t. $a(x) = \int_\Omega J(x-y)dy \geq 0$

(A2) $F \in C^2(\mathbb{R})$ and $\exists c_0 > 0$ s.t.

$$F''(s) + a(x) \geq c_0, \quad \forall s \in \mathbb{R}, \quad \text{a.e. } x \in \Omega$$

(A3) $\exists c_1 > 0$, $c_2 > 0$ and $q > 0$ s.t.

$$F''(s) + a(x) \geq c_1|s|^{2q} - c_2, \quad \forall s \in \mathbb{R}, \quad \text{a.e. } x \in \Omega$$

(A4) $\exists c_3 > 0$, $c_4 \geq 0$ and $p \in (1,2]$ s.t.

$$|F'(s)|^p \leq c_3|F(s)| + c_4, \quad \forall s \in \mathbb{R}$$

(A5) $h \in L^2_{loc}(\mathbb{R}^+; V'_{div})$ $\qquad \mathbb{R}^+ := [0, \infty)$

$\exists$ weak sols (smooth potentials)

Theorem (Colli, F. & Grasselli '11)

Assume (A1)–(A5). Then, if $u_0 \in H_{div}$, $\varphi_0 \in H$ with $F(\varphi_0) \in L^1(\Omega)$, for every $T > 0$ $\exists$ a weak sol $[u, \varphi]$ on $[0, T]$ corresponding to u_0 and φ_0 s.t.

$$u \in L^\infty(0, T; H_{div}) \cap L^2(0, T; V_{div})$$

$$\varphi \in L^\infty(0, T; L^{2+2q}(\Omega)) \cap L^2(0, T; V)$$

$$u_t \in L^{4/d}(0, T; V'_{div})$$

$$\varphi_t \in L^2(0, T; V') \quad \textit{if} \quad d = 2 \quad \textit{or} \quad d = 3 \textit{ and } q \geq 1/2$$

$$\mu \in L^2(0, T; V)$$

$\exists$ weak sols (smooth potentials)

Theorem (Colli, F. & Grasselli '11)

s.t. the energy inequality

$$\mathcal{E}(u(t),\varphi(t)) + \int_s^t (\nu\|\nabla u(\tau)\|^2 + \|\nabla\mu(\tau)\|^2)d\tau$$
$$\leq \mathcal{E}(u(s),\varphi(s)) + \int_s^t \langle h, u(\tau)\rangle d\tau$$

holds for all $t \geq s$ and for a.a. $s \in (0,\infty)$, including $s = 0$
We have set

$$\mathcal{E}(u(t),\varphi(t)) = \frac{1}{2}\|u(t)\|^2$$
$$+ \frac{1}{4}\int_\Omega\int_\Omega J(x-y)(\varphi(x,t)-\varphi(y,t))^2 dxdy + \int_\Omega F(\varphi(t))$$

$\exists$ weak sols (smooth potentials)

Main difficulty of the problem

- The nonlocal term implies that φ is not as regular as for the standard (local) CHNS system

$$\varphi \in L^2(H^1) \ \ (nonlocal), \ \text{instead of} \ \varphi \in L^\infty(H^1) \ \ (local)$$

Consequence

- Uniqueness of weak sols in 2D and regularity results (higher order estimates in 2D and 3D) are open issues

Asymptotic behavior in 2D (smooth potentials)

Corollary (Colli, F. & Grasselli '11)

Assume (A1)–(A5) and $d = 2$. *Then*

- *every weak sol* $z := [u, \varphi]$ *satisfies the energy identity*

$$\frac{d}{dt}\mathcal{E}(z) + \nu\|\nabla u\|^2 + \|\nabla\mu\|^2 = \langle h, u\rangle \qquad \forall t \geq 0$$

- *if* $h \in L^2_{tb}(\mathbb{R}^+; V'_{div})$, *i.e.*

$$\|h\|^2_{L^2_{tb}(\mathbb{R}^+; V'_{div})} := \sup_{t\geq 0}\int_t^{t+1} \|h(\tau)\|^2_{V'_{div}}\, d\tau < \infty$$

then every weak sol z *satisfies the dissipative estimate*

$$\mathcal{E}(z(t)) \leq \mathcal{E}(z_0)e^{-kt} + F(m)|\Omega| + K \qquad \forall t \geq 0$$

$m = \bar{\varphi}_0$ *and* $k, K \geq 0$ *are independent of* $z_0 := [u_0, \varphi_0]$

Asymptotic behavior in 2D (smooth potentials)

Existence of the global attractor (autonomous case)
For $m \geq 0$ given, set

$$\mathcal{X}_m = H_{div} \times \mathcal{Y}_m$$

$$\mathcal{Y}_m = \{\varphi \in H \ : \ F(\varphi) \in L^1(\Omega), \ |\bar{\varphi}| \leq m\}$$

Let $\mathcal{G}$ be the set of all weak sols corresponding to all initial data $z_0 = [u_0, \varphi_0] \in \mathcal{X}_m$.

Theorem (F. & Grasselli '11)

Suppose (A1)–(A4) hold and $h \in V'_{div}$. Then $\mathcal{G}$ is a generalized semiflow on $\mathcal{X}_m$ which possesses the global attractor $\mathcal{A}_m$

Asymptotic behavior in 3D (smooth potentials)

Theorem (F. & Grasselli '11)

Assume (A1)–(A4), $h \in L^2_{tb}(\mathbb{R}^+, V'_{div})$ and $d = 3$. Then the above dissipative estimate still holds for all $t > 0$ for all weak sols satisfying the energy inequality between s and t for a.a. $s \in (0, \infty)$, including $s = 0$, and for all $t \geq s$

Trajectory attractor approach (Chepyzhov & Vishik)

- phase space is a space of trajectories $\mathcal{K}^+_\Sigma$ on which the translation semigroup $\{T(t)\}$ acts
- the attraction of the trajectory attractor $\mathcal{A}_\Sigma$ is w.r.t. a suitable weak topology Θ^+_{loc} for the family of bounded (in a suitable norm or metric) subsets of $\mathcal{K}^+_\Sigma$

Trajectory attractor in 3D (smooth potentials)

Introduce the space

$$\mathcal{F}^+_{loc} = \Big\{[v,\psi] \in L^\infty_{loc}(\mathbb{R}^+; H_{div} \times L^{2+2q}(\Omega)) \cap L^2_{loc}(\mathbb{R}^+; V_{div} \times V) \ :$$
$$v_t \in L^{4/3}_{loc}(\mathbb{R}^+; V'_{div}),\ \psi_t \in L^2_{loc}(\mathbb{R}^+; V'),\ |\overline{\psi}(t)| \le m\Big\}$$

endowed with the topology Θ^+_{loc} of local weak convergence

Definition

For every $h \in L^2_{loc}(\mathbb{R}^+; V'_{div})$ the trajectory space $\mathcal{K}^+_h$ is the set of all weak sols $z = [v,\psi]$ (with external force h) in $\mathcal{F}^+_{loc}$ satisfying the energy inequality for all $t \ge s$ and for a.a. $s \in (0,\infty)$

Let $\mathcal{F}^+_b$ be a subspace of $\mathcal{F}^+_{loc}$ endowed with a norm used to define bounded subsets of $\mathcal{K}^+_\Sigma := \cup_{h\in\Sigma}\mathcal{K}^+_h$. We have $\mathcal{K}^+_\Sigma \subset \mathcal{F}^+_b$

Trajectory attractor in 3D (smooth potentials)

Set

$$\Sigma = \mathcal{H}_+(h_0) := \left[\{T(t)h_0, t \geq 0\}\right]_{L^2_{loc,w}(\mathbb{R}^+;H_{div})}$$

h_0 translation bounded in $L^2_{loc}(\mathbb{R}^+; H_{div})$

Theorem (F. & Grasselli '11)

Let (A1)–(A4) hold. In addition, suppose that (A4) holds with $p \in (1, 3/2]$ *and that* $2q + 2 = p'$*. If*

$$h_0 \in L^2_{tb}(\mathbb{R}^+; H_{div})$$

then $\{T(t)\}$ *acting on* $\mathcal{K}^+_{\mathcal{H}_+(h_0)}$ *possesses the uniform (with respect to* $h \in \mathcal{H}_+(h_0)$*) trajectory attractor* $\mathcal{A}_{\mathcal{H}_+(h_0)}$*. This set is strictly invariant, bounded in* $\mathcal{F}^+_b$ *and compact in* Θ^+_{loc}*. In addition, if* F *has growth* ≤ 6*, then* $\mathcal{K}^+_{\mathcal{H}_+(h_0)}$ *is closed in* Θ^+_{loc}*, and* $\mathcal{A}_{\mathcal{H}_+(h_0)} \subset \mathcal{K}^+_{\mathcal{H}_+(h_0)}$

∃ weak sols (singular potentials)

$$F = F_1 + F_2 \quad F_1 \in C^{2+2q}(-1,1)\ \ q \in \mathbb{N} \quad F_2 \in C^2([-1,1])$$

(P1) $F_1^{(2+2q)}(s) \geq c_1 > 0$ near $s = \pm 1$

(P2) For each $k = 0, 1, \cdots, 2+2q$ and each $j = 0, 1, \cdots, q$

$$F_1^{(k)}(s) \geq 0 \qquad \text{near } s = 1$$

$$F_1^{(2j+2)}(s) \geq 0 \qquad F_1^{(2j+1)}(s) \leq 0 \qquad \text{near } s = -1$$

(P3) $F_1^{(2+2q)}$ non-decreasing (increasing) near $s = 1$ ($s = -1$)

(P4) $\exists\ \alpha, \beta \in \mathbb{R}$ with $\alpha + \beta > -\min_{[-1,1]} F_2''$ s.t.

$$F_1''(s) \geq \alpha \qquad \forall s \in (-1,1), \quad a(x) \geq \beta \qquad \text{a.e. } x \in \Omega$$

(P5) $\lim_{s \to \pm 1} F_1'(s) = \pm\infty$

Remark

(P1)-(P5) satisfied for the logarithmic double-well potential F for any $q \in \mathbb{N}$. In particular, setting

$$F_1(s) = \frac{\theta}{2}((1+s)\log(1+s)+(1-s)\log(1-s)) \quad F_2(s) = -\frac{\theta_c}{2}s^2$$

then (P4) satisfied iff $\beta > \theta_c - \theta$

Remark

- *(P1), (P2)* $\Rightarrow F_\epsilon(s) \geq c_q|s|^{2+2q} - d_q$
- *(P2), (P4)* $\Rightarrow F''_\epsilon(s) + a(x) \geq c_0 > 0$

for ϵ small enough, where F_ϵ is a regular approximation of F with $(2+2q)$-growth

$\exists$ weak sols (singular potentials)

Theorem (F. & Grasselli '12)

Assume (A1), (A5) and that (P1)–(P5) hold for some fixed positive integer q. Let $u_0 \in H_{div}$, $\varphi_0 \in L^\infty(\Omega)$ *with* $F(\varphi_0) \in L^1(\Omega)$. *In addition, assume that* $|\overline{\varphi_0}| < 1$. *Then, for every* $T > 0$ $\exists$ *a weak sol* $z := [u, \varphi]$ *on* $[0, T]$ *corresponding to* $[u_0, \varphi_0]$ *such that* $\overline{\varphi}(t) = \overline{\varphi_0}$ *for all* $t \in [0, T]$ *and*

$$\varphi \in L^\infty(Q), \quad |\varphi(x,t)| < 1 \text{ a.e } (x,t) \in Q := \Omega \times (0,T)$$
$$\varphi \in L^\infty(0, T; L^{2+2q}(\Omega))$$

Furthermore, the energy inequality holds between s and t, for all $t \geq s$ *and for a.a.* $s \in (0, \infty)$, *including* $s = 0$. *If* $d = 2$, *every weak sol z satisfies the energy identity*

Main steps of the proof

- Approximate problem with potential F_ϵ
- Uniform (w.r.t. ϵ) estimates for the approximate sol $z_\epsilon = [u_\epsilon, \varphi_\epsilon]$
- Use $|\overline{\varphi_0}| < 1$ to control the averages $\{\overline{\mu_\epsilon}\}$
- Pass to the limit $z_\epsilon \to z$
- Use (P5) to show that $|\varphi| < 1$ in $\Omega \times (0, T)$ and hence that $z = [u, \varphi]$ is indeed a sol

Asymptotic behavior in 2D (singular potentials)

- $\mathcal{G}$ set of all weak sols corresponding to all initial data $z_0 = [u_0, \varphi_0] \in \mathcal{X}_{m_0} := H_{div} \times \mathcal{Y}_{m_0}$, $m_0 \in (0,1)$

$$\mathcal{Y}_{m_0} := \{\varphi \in L^\infty(\Omega) : |\varphi| < 1, \ \ F(\varphi) \in L^1(\Omega), |\overline{\varphi}| \leq m_0\}$$

Theorem (F. & Grasselli '12)

Let $d = 2$ and suppose that (A1), (P1)–(P5) hold and that $h \in V'_{div}$. Then $\mathcal{G}$ is a generalized semiflow on $\mathcal{X}_{m_0}$. If F is bounded in $(-1, 1)$, then $\mathcal{G}$ possesses the global attractor

Trajectory attractor in 3D (singular potentials)

For $m_0 \in (0,1)$

$$\mathcal{F}^+_{loc} = \Big\{ [v,\psi] \in L^\infty_{loc}(\mathbb{R}^+; H_{div} \times L^{2+2q}(\Omega)) \cap L^2_{loc}(\mathbb{R}^+; V_{div} \times V) \ :$$
$$v_t \in L^{4/3}_{loc}(\mathbb{R}^+; V'_{div}),\ \psi_t \in L^2_{loc}(\mathbb{R}^+; V'),\ |\overline{\psi}(t)| \le m_0$$
$$\psi \in L^\infty(Q_M),\ |\psi| < 1 \text{ in } Q_M,\ \forall M > 0 \Big\}, \quad Q_M := \Omega \times (0,M)$$

with the topology Θ^+_{loc} of local weak convergence, and

$$\mathcal{F}^+_{b} = \Big\{ [v,\psi] \in L^\infty(\mathbb{R}^+; H_{div} \times L^{2+2q}(\Omega)) \cap L^2_{tb}(\mathbb{R}^+; V_{div} \times V) \ :$$
$$v_t \in L^{4/3}_{tb}(\mathbb{R}^+; V'_{div}),\ \psi_t \in L^2_{tb}(\mathbb{R}^+; V'), |\overline{\psi}(t)| \le m_0,$$
$$\psi \in L^\infty(Q_\infty),\ |\psi| < 1 \text{ a.e. in } Q_\infty,\ F(\psi) \in L^\infty(\mathbb{R}^+; L^1(\Omega)) \Big\}$$

metric subspace used to define bounded subsets of the space of trajectories $\mathcal{K}^+_{\mathcal{H}_+(h_0)}$

Trajectory attractor in 3D (singular potentials)

Theorem (F. & Grasselli '12)

Let $d = 3$ and assume that (A1), (P1)-(P5) hold and $h_0 \in L^2_{tb}(\mathbb{R}^+; H_{div})$. Then, the translation semigroup $\{T(t)\}$ acting on $\mathcal{K}^+_{\mathcal{H}_+(h_0)}$ possesses the uniform (w.r.t. $h \in \mathcal{K}^+_{\mathcal{H}_+(h_0)}$) trajectory attractor $\mathcal{A}_{\mathcal{H}_+(h_0)}$. This set is strictly invariant, bounded in $\mathcal{F}^+_b$ and compact in Θ^+_{loc}. In addition, if F is bounded on $(-1, 1)$, then $\mathcal{K}^+_{\mathcal{H}_+(h_0)}$ is closed in Θ^+_{loc}, and $\mathcal{A}_{\mathcal{H}_+(h_0)} \subset \mathcal{K}^+_{\mathcal{H}_+(h_0)}$

Some developments and open issues

In progress

- nonlocal CHNS system with degenerate mobility

$$m(\varphi) = 1 - \varphi^2$$

- nonlocal Ladyzhenskaya-Cahn-Hilliard models
- robustness of the trajectory attractor (w.r.t. the approximating the potential)
- strong trajectory attractor in 2D

Open issues

- uniqueness for $N = 2$ and existence of strong sols
- connectedness of $\mathcal{A}_m$
- unmatched densities
- compressible models