A weak formulation for a dynamic process in delamination with unilateral constraints

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WIAS

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Introduction: setting

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The variable $u : \Omega \to \mathbb{R}^d$ represents the displacement. The variable $z : \Gamma \to [0,1]$ represents the status of the adhesive.

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The variable $\sigma : \Omega \to \mathbb{R}^{\mathrm{d}\times\mathrm{d}}$ is the stress of the body. The constitutive equation for σ is

$$\sigma = \mathbb{C}^0 \mathrm{e(u)} + \mu \mathbb{C}^1 \mathrm{e}(\dot{\mathrm{u}}),$$

where $\mathrm{e(u)} := \frac{1}{2}(\nabla \mathrm{u} + \nabla \mathrm{u}^{\mathrm{T}})$, $\mathbb{C}^0$ is the elastic tensor and $\mathbb{C}^1$ is the elastic tensor for viscosity, $\mu > 0$ is the viscosity of the material. We suppose $\mathbb{C}^{\mathrm{i}}$ positive definite and constant on Ω (homogeneous material).

The general problem

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If $\mathrm{f} : \Omega \to \mathbb{R}^{\mathrm{d}}$, $\mathrm{g} : \partial_{\mathrm{N}}\Omega \to \mathbb{R}^{\mathrm{d}}$ represents the external forces, and $\mathrm{w} : \partial_{\mathrm{D}}\Omega \to \mathbb{R}^{\mathrm{d}}$ a boundary datum, then the law of dynamic reads

$$\rho\ddot{\mathrm{u}} - \mathrm{Div}\ \sigma = \mathrm{f}$$

where ρ is the constant density of the material, coupled with the Neumann condition $\sigma\nu = \mathrm{g}$ on $\partial_{\mathrm{N}}\Omega$, and the Dirichlet condition $\mathrm{u} = \mathrm{w}$ on $\partial_{\mathrm{D}}\Omega$.

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The relation with the variable z arises in the condition

$$\sigma\nu = -\mathbb{K}[\mathrm{u}]\mathrm{z} \quad \text{on } \Gamma$$

where $\mathbb{K}$ is the (constant, positive definite) elastic tensor of the adhesive, and $[\mathrm{u}] := \mathrm{u}^2 - \mathrm{u}^1$.
Let us introduce the delamination potential

$$\frac{1}{2}\int_\Gamma \mathbb{K}[\mathrm{u}] \cdot [\mathrm{u}]\mathrm{z}.$$

The internal variable

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Moreover there is a delamination threshold $\alpha \in L^\infty(\Gamma)$, with $\alpha > c > 0$, such that

$$\frac{1}{2}\mathbb{K}[u] \cdot [u] < \alpha \quad \Rightarrow \quad \dot{z} = 0 \tag{2a}$$

$$\tag{2b}$$

and

$$\dot{z}(\frac{1}{2}\mathbb{K}[u] \cdot [u] - \alpha) = 0, \tag{2c}$$

$$\frac{1}{2}\mathbb{K}[u] \cdot [u] - \alpha \leq 0, \tag{2d}$$

holding on the set $\{z > 0\} \subset \Gamma$.

Introduction of the constraint

Physically, the condition $[\mathrm{u}] \cdot \nu$ is the normal jump of the displacement on Γ. Thus if is is positive, means that the bodies are separating, while a negative value must be avoided, since it means that interpenetration of matter is taking place.

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The presence of (3) will provide an instantaneous normal reaction at Γ as soon as $[\mathrm{u}] \cdot \nu = 0$. Such reaction must have fixed sign too!
So we introduce the reaction term ξ in the equation for σ, i.e.,

$$-\sigma(\mathrm{t})\nu = \mathbb{K}[\mathrm{u}(\mathrm{t})]\mathrm{z}(\mathrm{t}) + \xi\nu \quad \text{on } \Gamma, \tag{4}$$

coupled with (3) and the condition

$$[\mathrm{u}] \cdot \nu > 0 \quad \Rightarrow \quad \xi = 0, \tag{5}$$

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This can be equivalently said writting

$$\xi \in \partial \mathrm{I}_{[0,+\infty)}([\mathrm{u}] \cdot \nu), \tag{7}$$

with $\partial \mathrm{I}_{[0,+\infty)}$ denotes the subdifferential of the characteristic function of the interval $[0, +\infty)$.

Simplifications (WLOG) and Generalizations

We treat the homogeneous Dirichlet datum case $\mathrm{u} = 0$ on $\partial_{\mathrm{D}}\Omega$. Moreover we assume all the elasticity tensors being the identity matrix, i.e., $\mathbb{C}^1 = \mathbb{C}^2 = \mathbb{K} = \mathrm{Id}$, and the constant $\rho = \mu = 1$. Finally we replace the symmetric gradient $\mathrm{e}(\mathrm{u})$ by the usual one $\nabla \mathrm{u}$ (wlog thanks to Korn).

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Let us rewrite all the equations

$$\ddot{u} - \Delta u - \Delta \dot{u} = f \quad \text{on } \Omega, \tag{8a}$$

$$-(\nabla u + \nabla \dot{u})\nu = [u]z + \xi\nu \quad \text{on } \Gamma, \tag{8b}$$

$$\frac{1}{2}|[u]|^2 < \alpha \quad \Rightarrow \quad \dot{z} = 0, \tag{8c}$$

and

$$\dot{z}(\frac{1}{2}|[u]|^2 - \alpha) = 0, \tag{8d}$$

$$\frac{1}{2}|[u]|^2 - \alpha \leq 0, \tag{8e}$$

holding on the set $\{z > 0\} \subset \Gamma$.

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$$\frac{1}{2}|[\mathrm{u}]|^2 - \alpha \leq 0, \tag{8e}$$

holding on the set $\{\mathrm{z} > 0\} \subset \Gamma$.
On the other side, in orther to get a general constraint, we replace the function $I_{[0,+\infty)}$ by $\mathrm{j} : \mathbb{R} \to [0, +\infty]$, being a convex and lower semicontinuous function such that $\mathrm{j}(0) = \min \mathrm{j} = 0$. Then the constraint reads

$$\xi \in \partial \mathrm{j}([\mathrm{u}] \cdot \nu). \tag{8f}$$

Generalized constraint

We define

$$\mathcal{J}(\mathrm{v}) := \int_0^{\mathrm{T}} \int_{\Gamma} \mathrm{j}(\mathrm{v})\mathrm{dxdt} \quad \mathrm{v} \in \mathrm{L}^2([0, \mathrm{T}] \times \Gamma). \tag{9}$$

The subdifferential of $\mathcal{J}$ on $\mathrm{L}^2([0, \mathrm{T}] \times \Gamma)$ is defined as the multivalued operator

$$\partial\mathcal{J} : \mathrm{L}^2([0, \mathrm{T}] \times \Gamma) \rightrightarrows \mathrm{L}^2([0, \mathrm{T}] \times \Gamma),$$

as follows: for $\mathrm{v}, \mathrm{u} \in \mathrm{L}^2([0, \mathrm{T}] \times \Gamma)$, we have

$$\mathrm{v} \in \partial\mathcal{J}(\mathrm{u}) \Leftrightarrow \mathcal{J}(\mathrm{w}) - \mathcal{J}(\mathrm{u}) \geq (\!(\mathrm{v}, \mathrm{w} - \mathrm{u})\!) \quad \forall \mathrm{w} \in \mathrm{L}^2([0, \mathrm{T}] \times \Gamma). \tag{10}$$

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Setting $\beta := \partial \mathrm{j}$, it is easy to see that $\mathrm{v} \in \partial \mathcal{J}(\mathrm{u})$ if and only if

$$\mathrm{v}(\mathrm{t}, \mathrm{x}) \in \beta(\mathrm{u}(\mathrm{t}, \mathrm{x})) \quad \text{for a.e. } (\mathrm{t}, \mathrm{x}) \in [0, \mathrm{T}] \times \Gamma. \tag{11}$$

Generalized constraint

We define

$$\mathcal{J}(v) := \int_0^T \int_\Gamma j(v)\,dx\,dt \quad v \in L^2([0,T] \times \Gamma). \tag{9}$$

The subdifferential of $\mathcal{J}$ on $L^2([0,T] \times \Gamma)$ is defined as the multivalued operator

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as follows: for $v, u \in L^2([0,T] \times \Gamma)$, we have

$$v \in \partial\mathcal{J}(u) \Leftrightarrow \mathcal{J}(w) - \mathcal{J}(u) \geq (\!(v, w-u)\!) \quad \forall w \in L^2([0,T] \times \Gamma). \tag{10}$$

Setting $\beta := \partial j$, it is easy to see that $v \in \partial\mathcal{J}(u)$ if and only if

$$v(t,x) \in \beta(u(t,x)) \quad \text{for a.e. } (t,x) \in [0,T] \times \Gamma. \tag{11}$$

We call this the pointwise interpretation of $\partial\mathcal{J}$, so we still denote it by $\beta := \partial\mathcal{J}$.

Relaxation of the constraint

We consider the restriction of $\mathcal{J}$ to the space $\mathcal{H} \subset \mathrm{L}^2([0,\mathrm{T}] \times \Gamma)$, and we consider the subdifferential with respect to this new topology. This is the multivalued operator

$$\partial_{\mathcal{H}} \mathcal{J} : \mathcal{H} \rightrightarrows \mathcal{H}',$$

defined as follows: for $\mathrm{u} \in \mathcal{H}$ and $\xi \in \mathcal{H}'$, we have

$$\xi \in \partial_{\mathcal{H}} \mathcal{J}(\mathrm{u}) \quad \Leftrightarrow \quad \mathcal{J}(\mathrm{w}) - \mathcal{J}(\mathrm{u}) \geq \langle\!\langle \xi, \mathrm{w} - \mathrm{u} \rangle\!\rangle \quad \forall \mathrm{w} \in \mathcal{H}, \tag{12}$$

where $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ is the duality pairing between $\mathcal{H}$ and $\mathcal{H}'$.

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Note that the pointwise interpretation

$$\xi(\mathrm{t}, \mathrm{x}) \in \beta(\mathrm{u}(\mathrm{t}, \mathrm{x})) \quad \text{for a.e. } (\mathrm{t}, \mathrm{x}) \in [0, \mathrm{T}] \times \Gamma,$$

does no longer make sense!

Properties of the weak constraint

However if $\xi \in \beta_{\mathrm{w}}(\mathrm{u})$ the following can be said:

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Theorem

There exists a bounded Borel measure $\mathcal{T}$ such that $\langle\langle \xi, \varphi \rangle\rangle = \int_0^{\mathrm{T}} \int_{\Gamma} \varphi \mathrm{d}\mathcal{T}$ for all $\varphi \in \mathcal{H} \cap \mathrm{C}_0([0,\mathrm{T}] \times \Gamma)$. Moreover if $\mathcal{T} = \mathcal{T}_{\mathrm{a}} + \mathcal{T}_{\mathrm{s}}$ then

$$\mathcal{T}_{\mathrm{a}}\mathrm{u} \in \mathrm{L}^1([0,\mathrm{T}] \times \Gamma), \tag{13}$$

$$\mathcal{T}_{\mathrm{a}}(\mathrm{t},\mathrm{x}) \in \beta(\mathrm{u}(\mathrm{t},\mathrm{x})) \text{ for a.e. } (\mathrm{t},\mathrm{x}) \in [0,\mathrm{T}] \times \Gamma, \tag{14}$$

$$\langle\langle \xi, \mathrm{u} \rangle\rangle - \int_0^{\mathrm{T}} \int_{\Gamma} \mathcal{T}_{\mathrm{a}}\mathrm{u}\, \mathrm{dxdt} = \sup \Big\{ \int_0^{\mathrm{T}} \int_{\Gamma} \mathrm{z}\, \mathrm{d}\mathcal{T}_{\mathrm{s}},\ \mathrm{z} \in \mathrm{C}([0,\mathrm{T}] \times \Gamma),\ |\mathrm{z}| \leq 1 \Big\}. \tag{15}$$

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It can be proved that, in the case that $\mathrm{j} = \mathrm{I}_{[0,+\infty)}$, denoting by $\mathcal{T}_{\mathrm{s}} = \rho|\mathcal{T}_{\mathrm{s}}|$,

$$\rho \in \partial \mathrm{j}(\mathrm{u}) \quad |\mathcal{T}_{\mathrm{s}}| - \text{a.e. in } [0,\mathrm{T}] \times \Gamma. \tag{16}$$

This means that $\mathcal{T}_{\mathrm{s}}$ is supported on the set where $\mathrm{u} = 0$ and that here it holds $\rho = -1$.

References

These results are adaptations of those contained in

- H. Brézis, Intégrales convexes dans les espaces de Sobolev, Israel J. Math., 13 (1972), 9–23.
- M. Grun-Rehomme, Caractérisation du sous-différentiel d'intégrandes convexes dans les espaces de Sobolev (French), J. Math. Pures Appl. (9), 56 (1977), 149–156.

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Approximation of β_{w}

We introduce j^ϵ the Moreau-Yosida regularization of j, and define the operator $\mathcal{J}^\epsilon$ on $L^2([0,T]\times\Gamma)$ as

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Recall that j^ϵ is globally ϵ^{-1}-Lipschitz continuous. As for j, we set

$$\beta^\epsilon := \partial \mathrm{j}^\epsilon,$$

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Lemma

β^ϵ is a monotone operator on $\mathcal{H}$ into $\mathcal{H}'$. Moreover for $\mathrm{u}\in\mathcal{H}$ then $\beta^\epsilon(\mathrm{u})\in\mathcal{H}'$ belongs to the subdifferential of $\mathcal{J}^\epsilon$ (as an operator on $\mathcal{H}$).

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Following the theory of

- H. Attouch, "Variational Convergence for Functions and Operators", Pitman, London, 1984.

we can then prove that the monotone operators β^ϵ tends to the maximal monotone operator β_{w} in the sense of graph, i.e.,

$$\forall [\mathrm{x}, \mathrm{y}] \in \beta_{\mathrm{w}} \quad \exists [\mathrm{x}^\epsilon, \mathrm{y}^\epsilon] \in \beta^\epsilon \quad \text{such that} \quad [\mathrm{x}^\epsilon, \mathrm{y}^\epsilon] \to [\mathrm{x}, \mathrm{y}],$$

where the convergence is intended with respect to the strong topology of $\mathcal{H} \times \mathcal{H}'$.

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where the convergence is intended with respect to the strong topology of $\mathcal{H} \times \mathcal{H}'$. The following holds:

Lemma

Let the monotone operator η_{n} tends to the maximal monotone operator η in the sense of graph (operators on $\mathcal{H}$ into $\mathcal{H}'$). Let $\mathrm{u}_{\mathrm{n}} \rightharpoonup \mathrm{u}$ weakly in $\mathcal{H}$, $\xi_{\mathrm{n}} \rightharpoonup \xi$ weakly in $\mathcal{H}'$, and assume $\xi_{\mathrm{n}} \in \eta_{\mathrm{n}}(\mathrm{u}_{\mathrm{n}})$. If

$$\limsup \langle\langle \xi_{\mathrm{n}}, \mathrm{u}_{\mathrm{n}} \rangle\rangle \leq \langle\langle \xi, \mathrm{u} \rangle\rangle,$$

then $\xi \in \eta(\mathrm{u})$.

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we can then prove that the monotone operators β^ϵ tends to the maximal monotone operator $\beta_{\rm w}$ in the sense of graph, i.e.,

$$\forall[{\rm x},{\rm y}] \in \beta_{\rm w} \quad \exists[{\rm x}^\epsilon,{\rm y}^\epsilon] \in \beta^\epsilon \quad \text{such that} \quad [{\rm x}^\epsilon,{\rm y}^\epsilon] \to [{\rm x},{\rm y}],$$

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Lemma

Let the monotone operator $\eta_{\rm n}$ tends to the maximal monotone operator η in the sense of graph (operators on $\mathcal{H}$ into $\mathcal{H}'$). Let ${\rm u_n} \rightharpoonup {\rm u}$ weakly in $\mathcal{H}$, $\xi_{\rm n} \rightharpoonup \xi$ weakly in $\mathcal{H}'$, and assume $\xi_{\rm n} \in \eta_{\rm n}({\rm u_n})$. If

$$\limsup \langle\!\langle \xi_{\rm n}, {\rm u_n} \rangle\!\rangle \leq \langle\!\langle \xi, {\rm u} \rangle\!\rangle,$$

then $\xi \in \eta({\rm u})$.

These are the ingredients we need!

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Definition

Let $\mathrm{u}_0, \mathrm{v}_0 \in \mathrm{V}$, $\mathrm{z}_0 \in \mathcal{Z}$, and $\mathrm{f} \in \mathrm{L}^2([0,\mathrm{T}],\mathrm{V}')$. Then $(\mathrm{u},\mathrm{z},\eta)$ is an energetic solution to (8) if

$$\mathrm{u} \in \mathrm{H}^1([0,\mathrm{T}],\mathrm{V}) \cap \mathrm{W}^{1,\infty}([0,\mathrm{T}],\mathrm{L}^2(\Omega)), \tag{18a}$$

$$\dot{\mathrm{u}} \in \mathrm{H}^1([0,\mathrm{T}],\mathrm{H}^{-1}(\tilde{\Omega})) \cap \mathrm{BV}(0,\mathrm{T};\tilde{\mathrm{H}}^{-2}(\Omega)), \tag{18b}$$

$$\mathrm{z} \in \mathrm{L}^\infty([0,\mathrm{T}],\mathcal{Z}) \cap \mathrm{BV}(0,\mathrm{T};\mathrm{L}^1(\Gamma)), \tag{18c}$$

$$\eta \in \mathcal{H}', \tag{18d}$$

is such that $\mathrm{u}(0) = \mathrm{u}_0$, $\dot{\mathrm{u}}(0) = \mathrm{v}_0$, $\mathrm{z}(0) = \mathrm{z}_0$, and satisfies conditions (a), (a'), (b), (c), and (d) below.

Energetic formulation of the evolution

We define a weak form of solution to problem (8).

Definition

Let $u_0, v_0 \in V$, $z_0 \in \mathcal{Z}$, and $f \in L^2([0,T], V')$. Then (u, z, η) is an energetic solution to (8) if

$$u \in H^1([0,T], V) \cap W^{1,\infty}([0,T], L^2(\Omega)), \tag{18a}$$

$$\dot{u} \in H^1([0,T], H^{-1}(\tilde{\Omega})) \cap BV(0,T; \tilde{H}^{-2}(\Omega)), \tag{18b}$$

$$z \in L^\infty([0,T], \mathcal{Z}) \cap BV(0,T; L^1(\Gamma)), \tag{18c}$$

$$\eta \in \mathcal{H}', \tag{18d}$$

is such that $u(0) = u_0$, $\dot{u}(0) = v_0$, $z(0) = z_0$, and satisfies conditions (a), (a'), (b), (c), and (d) below.

(a) for all $\varphi \in \mathcal{V}$,

$$\begin{aligned} &- (\!(\dot{u}, \dot{\varphi})\!) + (\dot{u}(T), \varphi(T)) + (\!(\nabla \dot{u}, \nabla \varphi)\!) + (\!(\nabla u, \nabla \varphi)\!) + \langle\!\langle \eta, [\varphi] \cdot \nu \rangle\!\rangle \\ &= (u_1, \varphi(0)) + \langle\!\langle f, \varphi \rangle\!\rangle - (\!(z[u], [\varphi])\!)^\Gamma. \end{aligned} \tag{19}$$

Energetic formulation of the evolution

(a') for all $t \in [0, T]$ there exists $\eta_t \in \mathcal{H}'$ such that also the local version of (19) holds

$$- ((\dot{u}, \dot{\varphi}))_t + (\dot{u}(t), \varphi(t)) + ((\nabla \dot{u}, \nabla \varphi))_t + ((\nabla u, \nabla \varphi))_t + \langle\langle \eta_t, [\varphi] \cdot \nu \rangle\rangle_t$$
$$= (u_1, \varphi(0)) + \langle\langle f, \varphi \rangle\rangle_t - ((z[u], [\varphi]))_t^{\Gamma}, \tag{20}$$

for all $\varphi \in \mathcal{V}_t$. Moreover η_t satisfies the property that, for all $\varphi \in \mathcal{H}_t$ with $\varphi(t) = 0$, we have

$$\langle\langle \eta_t, \varphi \rangle\rangle_t = \langle\langle \eta, \tilde{\varphi} \rangle\rangle, \tag{21}$$

where $\tilde{\varphi}$ is the extension to $\mathcal{H}$ of $\varphi \in \mathcal{H}_{t,0}$ such that $\varphi(s) = 0$ for $s \in [t, T]$.

(b) We have

$$\eta \in \beta_w([u] \cdot \nu), \tag{22}$$

and for all $t \in [0, T]$ it also holds that

$$\eta_t \in \beta_{w,t}([u\llcorner_{[0,t]}] \cdot \nu).$$

Energetic formulation of the evolution

(c) for almost every $x \in \Gamma$ the function $t \mapsto z(t,x)$ is nonincreasing and

$$\text{either} \quad \frac{1}{2}|[u(t,x)]|^2 \leq \alpha(x) \quad \text{or} \quad z(t,x) = 0 \quad \text{for a.e. } x \in \Gamma \tag{23}$$

for all $t \in [0,T]$.

(d) the following energy inequality holds

$$\begin{aligned} &\frac{1}{2}\|\dot{u}(t)\|_H^2 + \int_\Gamma j([u(t)]\cdot\nu)dx + \frac{1}{2}\int_\Gamma z(t)|[u](t)|^2 dx + \frac{1}{2}\|\nabla u(t)\|^2 \\ &+ \int_0^T \|\nabla \dot{u}\|^2 dt - (\alpha, z(t))_\Gamma + (\alpha, z_0)_\Gamma \leq \\ &\frac{1}{2}\|v_0\|_H^2 + \int_\Gamma j([u_0]\cdot\nu)dx + \frac{1}{2}\int_\Gamma z_0|[u_0]|^2 dx + \frac{1}{2}\|\nabla u_0\|^2 + \langle\langle f, \dot{u}\rangle\rangle_t, \end{aligned} \tag{24}$$

for all $t \in [0,T]$.

Energetic formulation of the approximate evolution

In order to prove the existence of an energetic solution we introduce the following approximate evolutions.

Energetic formulation of the approximate evolution

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Definition

Let $\epsilon \in (0,1)$, $u_0, v_0 \in V$, $z_0 \in \mathcal{Z}$, and $f \in L^2([0,T],V')$. Then (u^ϵ, z^ϵ) is an ϵ-approximation of the energetic solution (4) if

$$u^\epsilon \in H^1([0,T],V) \cap W^{1,\infty}([0,T],L^2(\Omega)), \tag{25a}$$

$$\dot{u}^\epsilon \in H^1([0,T],V'), \tag{25b}$$

$$z^\epsilon \in L^\infty([0,T],\mathcal{Z}) \cap BV(0,T;L^1(\Gamma)), \tag{25c}$$

is such that $u^\epsilon(0) = u_0$, $\dot{u}^\epsilon(0) = v_0$, $z^\epsilon(0) = z_0$, and satisfies conditions (a^ϵ), (b^ϵ), and (c^ϵ) below.

Energetic formulation of the approximate evolution

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$$z^\epsilon \in L^\infty([0,T], \mathcal{Z}) \cap BV(0,T; L^1(\Gamma)), \tag{25c}$$

is such that $u^\epsilon(0) = u_0$, $\dot{u}^\epsilon(0) = v_0$, $z^\epsilon(0) = z_0$, and satisfies conditions (a^ϵ), (b^ϵ), and (c^ϵ) below.

(a^ϵ) for every time $t \in [0,T]$, it holds

$$\begin{aligned} &-(\!(\dot{u}^\epsilon, \dot{\varphi})\!)_t + (\dot{u}^\epsilon(t), \varphi(t)) + (\!(\nabla \dot{u}^\epsilon, \nabla \varphi)\!)_t + (\!(\nabla u^\epsilon, \nabla \varphi)\!)_t + \langle\!\langle \beta^\epsilon([u^\epsilon]\cdot\nu), [\varphi]\cdot\nu \rangle\!\rangle_t \\ &= (u_1, \varphi(0)) + \langle\!\langle f, \varphi \rangle\!\rangle_t - (\!(z^\epsilon [u^\epsilon], [\varphi])\!)_t^\Gamma, \end{aligned} \tag{26}$$

for all $\varphi \in \mathcal{V}$.

Energetic formulation of the approximate evolution

(b^ϵ) for almost every $x \in \Gamma$ the function $t \mapsto z^\epsilon(t,x)$ is nonincreasing and

$$\text{either} \quad \frac{1}{2}|[u^\epsilon(t,x)]|^2 \le \alpha(x) \quad \text{or} \quad z^\epsilon(t,x) = 0 \quad \text{for a.e. } x \in \Gamma \tag{27}$$

for all $t \in [0,T]$.

(c^ϵ) the following energy balance holds

$$\begin{aligned}
&\frac{1}{2}\|\dot{u}^\epsilon(t)\|_H^2 + \int_\Gamma j^\epsilon([u^\epsilon(t)]\cdot\nu)dx + \frac{1}{2}\int_\Gamma z^\epsilon(t)|[u^\epsilon](t)|^2 dx + \frac{1}{2}\|\nabla u^\epsilon(t)\|^2 \\
&+ \int_0^T \|\nabla \dot{u}^\epsilon\|^2 dt - (\alpha, z^\epsilon(t))_\Gamma + (\alpha, z_0)_\Gamma = \\
&\frac{1}{2}\|v_0\|_H^2 + \int_\Gamma j^\epsilon([u_0]\cdot\nu)dx + \frac{1}{2}\int_\Gamma z_0|[u_0]|^2 dx + \frac{1}{2}\|\nabla u_0\|^2 + \langle\langle f, \dot{u}^\epsilon\rangle\rangle_t,
\end{aligned} \tag{28}$$

for all $t \in [0,T]$.

Existence of evolutions

For all $\epsilon \in (0,1)$, existence of an approximate solution can be obtained by time discretization and by an implicit Euler scheme. This is standard and we can adapt, for instance, results by

- T. Roubicek, Adhesive contact of visco-elastic bodies and defect measures arising by vanishing viscosity. SIAM J. Math. Anal., 45 (2013), 101-126,

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We want to pass to the limit as $\epsilon \to 0$. The following Theorem holds true.

Theorem

Let $(\mathrm{u}^\epsilon, \mathrm{z}^\epsilon)$ be approximate solutions. Then there exists $(\mathrm{u}, \mathrm{z}, \eta)$ energetic solution as in Definition 4 such that, up to a subsequence,

$$\mathrm{u}^\epsilon \to \mathrm{u} \quad \text{strongly in } \mathrm{H}^1(0,\mathrm{T};\mathrm{L}^2(\Omega)) \text{ and weakly in } \mathrm{H}^1(0,\mathrm{T};\mathrm{V}), \tag{29a}$$

$$\dot{\mathrm{u}}^\epsilon \rightharpoonup \dot{\mathrm{u}} \quad \text{weakly in } \mathrm{H}^1(0,\mathrm{T};\mathrm{H}^{-1}(\tilde{\Omega})) \text{ and weakly* in } \mathrm{BV}(0,\mathrm{T};\tilde{\mathrm{H}}^{-2}(\Omega)), \tag{29b}$$

$$\mathrm{z}^\epsilon(\mathrm{t}) \rightharpoonup \mathrm{z}(\mathrm{t}) \quad \text{weakly* in } \mathrm{L}^\infty(\Gamma) \text{ for all } \mathrm{t} \in [0,\mathrm{T}], \tag{29c}$$

$$\beta_\epsilon([\mathrm{u}^\epsilon]\cdot\nu) \rightharpoonup \eta \quad \text{weakly in } \mathcal{H}' \text{ and in } \mathcal{V}'. \tag{29d}$$

Sketch of the proof

In order to get the convergences above we should prove suitable apriori estimates for $(\mathrm{u}^\epsilon, \mathrm{z}^\epsilon, \beta_\epsilon([\mathrm{u}^\epsilon] \cdot \nu))$.

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Some estimates are straightforward and follows by the energy balance (c^ϵ). These are

$$
\begin{aligned}
&\|\mathrm{u}^\epsilon\|_{\mathrm{H}^1(0,\mathrm{t};\mathrm{V})} \leq \mathrm{M},\\
&\int_\Gamma \mathrm{j}^\epsilon([\mathrm{u}^\epsilon(\mathrm{t})]\cdot\nu) \leq \mathrm{M} \text{ for all } \mathrm{t}\in[0,\mathrm{T}],\\
&\frac{1}{2}\int_\Gamma |[\mathrm{u}^\epsilon](\mathrm{t})|^2 \mathrm{z}^\epsilon(\mathrm{t}) \leq \mathrm{M} \text{ for all } \mathrm{t}\in[0,\mathrm{T}],\\
&\|\mathrm{z}^\epsilon\|_{\mathrm{L}^\infty(0,\mathrm{t};\mathcal{Z})} \leq \mathrm{M},\\
&\|\mathrm{z}^\epsilon\|_{\mathrm{BV}(0,\mathrm{T};\mathrm{L}^1(\Gamma))} \leq \mathrm{M}.
\end{aligned}
$$

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In order to get the convergences above we should prove suitable apriori estimates for $(u^\epsilon, z^\epsilon, \beta_\epsilon([u^\epsilon]\cdot\nu))$.
Some estimates are straightforward and follows by the energy balance (c^ϵ). These are

$$\|u^\epsilon\|_{H^1(0,t;V)} \le M,$$
$$\int_\Gamma j^\epsilon([u^\epsilon(t)]\cdot\nu) \le M \text{ for all } t \in [0,T],$$
$$\frac{1}{2}\int_\Gamma |[u^\epsilon](t)|^2 z^\epsilon(t) \le M \text{ for all } t \in [0,T],$$
$$\|z^\epsilon\|_{L^\infty(0,t;\mathcal{Z})} \le M,$$
$$\|z^\epsilon\|_{BV(0,T;L^1(\Gamma))} \le M.$$

Crucial is the following one:

Lemma

For all $\epsilon \in (0,1)$ it holds

$$\|\beta_\epsilon([u^\epsilon]\cdot\nu)\|_{L^1(0,T;L^1(\Gamma))} \le M. \tag{30}$$

Sketch of the proof

This can be obtained letting $\bar{\psi} \in \mathrm{H}^{\frac{1}{2}}(\Gamma; \mathbb{R}^{\mathrm{d}})$ such that $\bar{\psi} \cdot \nu = 1$ on the whole Γ and extending (harmonic) it to an element $\varphi \in \mathrm{V}$ which is 0 on Ω_2, and set $\Psi(\mathrm{t}, \mathrm{x}) := \varphi(\mathrm{x})$ for all $\mathrm{t} \in [0, \mathrm{T}]$. Testing the weak equation (a^{ϵ}) by $\mathrm{u} - \delta\Psi$, $\delta > 0$,

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$$(\dot{u}^\epsilon(T), u^\epsilon(T)) - (u_1^\epsilon, u_0^\epsilon) - \int_0^T \|\dot{u}^\epsilon\|_2^2 + \int_{\Omega_1} u^\epsilon(t), \delta\Psi dx - \int_{\Omega_1} u_0^\epsilon, \delta\Psi dx + \int_0^T \|\nabla u^\epsilon\|_2^2 dt$$

$$+ \frac{1}{2}\|\nabla u^\epsilon(T)\|_2^2 - \frac{1}{2}\|\nabla u_0^\epsilon\|_2^2 + (\!(\nabla u, \delta\nabla\psi\rangle)\!)^{\Omega_1} + (\nabla u^\epsilon(T), \delta\nabla\Psi)^{\Omega_1} - (\nabla u_0^\epsilon, \delta\nabla\Psi)^{\Omega_1}$$

$$+ \int_0^T \int_\Gamma \beta_\epsilon([u^\epsilon] \cdot \nu)([u^\epsilon] \cdot \nu - \delta) dx dt + (\!(z^\epsilon, |[u^\epsilon]|^2 - \delta u^\epsilon \cdot \nu)\!) = \langle\!\langle f, u^\epsilon \rangle\!\rangle + \langle\!\langle f, \delta\Psi \rangle\!\rangle^{\Omega_1}.$$

and then using the previous estimates and the fact that $|\beta_\epsilon(x)| \leq \delta^{-1}|\beta_\epsilon(x)(x-\delta)|$ for $\epsilon \in (0,1)$.

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$$+ \frac{1}{2}\|\nabla u^\epsilon(T)\|_2^2 - \frac{1}{2}\|\nabla u_0^\epsilon\|_2^2 + ((\nabla u, \delta\nabla\psi\rangle))^{\Omega_1} + (\nabla u^\epsilon(T), \delta\nabla\Psi)^{\Omega_1} - (\nabla u_0^\epsilon, \delta\nabla\Psi)^{\Omega_1}$$

$$+ \int_0^T \int_\Gamma \beta_\epsilon([u^\epsilon] \cdot \nu)([u^\epsilon] \cdot \nu - \delta) dx dt + ((z^\epsilon, |[u^\epsilon]|^2 - \delta u^\epsilon \cdot \nu)) = \langle\langle f, u^\epsilon \rangle\rangle + \langle\langle f, \delta\Psi \rangle\rangle^{\Omega_1}.$$

and then using the previous estimates and the fact that $|\beta_\epsilon(x)| \leq \delta^{-1}|\beta_\epsilon(x)(x - \delta)|$ for $\epsilon \in (0, 1)$.

The previous estimate implies

$$\|\dot{u}^\epsilon\|_{W^{1,1}(0,T;\tilde{H}^{-2}(\Omega))} \leq M, \tag{31}$$

for all $\epsilon \in (0, 1)$.

Sketch of the proof

Now we employ Helly selection principle combined with a generalized version of Aubin-Lions compactness principle entail the strong convergence

$$u^\epsilon \to u \quad \text{strongly in } H^1(0,T;L^2(\Omega)) \tag{32}$$

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Another key fact is the following:

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There holds

$$\|\beta_\epsilon([u^\epsilon]\cdot\nu)\|_{\mathcal{H}'} \leq M, \tag{34}$$

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Using the previous estimate we argue as before using an arbitrary function $\psi \in H^{\frac{1}{2}}(\Gamma;\mathbb{R}^d)$ and suitably extending it to V.
We have then found

$$\beta_\epsilon([u^\epsilon]\cdot\nu) \rightharpoonup \eta,$$

for some $\eta \in \mathcal{H}'$.

We are now ready to prove (b). To this aim we apply Lemma 3, and then we have to check that

$$\limsup_{\epsilon \to 0} \langle\!\langle \beta^\epsilon(\mathrm{u}^\epsilon), \mathrm{u}^\epsilon \cdot \nu \rangle\!\rangle \leq \langle\!\langle \eta, \mathrm{u} \cdot \nu \rangle\!\rangle. \tag{35}$$

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Thanks to the convergences above it is easily seen that (a) holds. Using (26), we write

$$\langle\langle \beta^\epsilon([u^\epsilon]\cdot\nu), u^\epsilon\rangle\rangle = \|\dot{u}^\epsilon\|^2_{L^2(0,T;L^2)} - (\dot{u}^\epsilon(T), u^\epsilon(T)) + (u_1, u_0) - \frac{1}{2}\|\nabla u^\epsilon(T)\|^2$$
$$+ \frac{1}{2}\|\nabla u_0\|^2 - \|\nabla u^\epsilon\|^2_{L^2(0,T;L^2)} - \int_0^T \int_\Gamma z^\epsilon(t)[u^\epsilon(t)]^2 dxdt + ((f, u^\epsilon)). \tag{36}$$

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Taking the lim sup we get

$$\le \|\dot u\|^2_{L^2(0,T;L^2)} - (\dot u(T), u(T)) + (u_1, u_0) - \frac{1}{2}\|\nabla u(T)\|^2 + \frac{1}{2}\|\nabla u_0\|^2 - \|\nabla u\|^2_{L^2(0,T;L^2)} + \int_0^T \int_\Gamma z(t)[u(t)]^2 dxdt + ((f, u)) = \langle\langle \eta, u \cdot \nu \rangle\rangle, \tag{37}$$

and the claim follows.

Some Remarks

We have seen that η and η_t coincides with Borel measures concentrated on the set where $[u] \cdot \nu = 0$. Thanks to the fact that

$$\dot{u} \in BV(0, T; \tilde{H}^{-2}(\Omega)),$$

it can be proved that $\eta_t \equiv \eta \llcorner_{[0,t] \times \Gamma}$ outside a set of countable many times.

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In dimension $d = 1$ energy balance holds!

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In some sense the presence of such jumps is the main difficulty to prove an enery balance (if it holds). Outside these points an energy balance holds true!!!

In dimension $d = 1$ energy balance holds!

The behaviour of the constraint is quite independent of the flow rule of the variable z. For instance, the same behaviour takes place when we add a viscosity term in the flow rule

$$\frac{1}{2}|[u]|^2 + \dot{z} - \alpha \leq 0$$

(S.-G. Schimperna).

THANK YOU FOR ATTENTION!

Some references:

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