# A weak formulation for a dynamic process in delamination with unilateral constraint

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## Introduction: setting

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The variable $u : \Omega \to \mathbb{R}^d$ represents the displacement. The variable $z : \Gamma \to [0, 1]$ represents the status of the adhesive.

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The variable $\sigma : \Omega \to \mathbb{R}^{d \times d}$ is the stress of the body. The constitutive equation for $\sigma$ is

$$\sigma = \mathbb{C}^0 e(u) + \mu \mathbb{C}^1 e(\dot{u}),$$

where $e(u) := \frac{1}{2}(\nabla u + \nabla u^T)$, $\mathbb{C}^0$ is the elasticity tensor and $\mathbb{C}^1$ is the elasticity tensor for viscosity, $\mu > 0$ is the viscosity of the material. We suppose $\mathbb{C}^i$ positive definite and constant on $\Omega$ (homogeneous material).

# The general problem

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If $f : \Omega \to \mathbb{R}^d$, $g : \partial_N\Omega \to \mathbb{R}^d$ represents the external forces, and $w : \partial_D\Omega \to \mathbb{R}^d$ a boundary datum, then the law of dynamic reads

$$\rho\ddot{u} - \mathrm{Div}\ \sigma = f$$

where $\rho$ is the constant density of the material, coupled with the Neumann condition $\sigma\nu = g$ on $\partial_N\Omega$, and the Dirichlet condition $u = w$ on $\partial_D\Omega$.

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The relation with the variable $z$ arises in the condition

$$\sigma\nu = -\mathbb{K}[u]z \quad \text{on } \Gamma$$

where $\mathbb{K}$ is the (constant, positive definite) elasticity tensor of the adhesive, and $[u] := u^2 - u^1$.
Let us introduce the delamination potential

$$\frac{1}{2}\int_\Gamma \mathbb{K}[u] \cdot [u]z.$$

## The internal variable

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Moreover there is a delamination threshold $\alpha \in L^{\infty}(\Gamma)$, with $\alpha > c > 0$, such that

$$\frac{1}{2}\mathbb{K}[u] \cdot [u] < \alpha \quad \Rightarrow \quad \dot{z} = 0 \tag{2a}$$

$$\dot{z}(\frac{1}{2}\mathbb{K}[u] \cdot [u] - \alpha) = 0, \tag{2b}$$

and

$$\frac{1}{2}\mathbb{K}[u] \cdot [u] - \alpha \leq 0, \tag{2c}$$

holding on the set $\{z > 0\} \subset \Gamma$.

# Introduction of the constraint

Physically, the quantity $[u] \cdot \nu$ represents the normal jump of the displacement on $\Gamma$. Thus if it is positive, means that the bodies are separating, while a negative value must be avoided, since it means that interpenetration of matter is taking place.

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The presence of (3) will provide an instantaneous normal reaction at $\Gamma$ as soon as $[u] \cdot \nu = 0$. Such reaction must have fixed sign too!
So we introduce the reaction term $\xi$ in the equation for $\sigma$, i.e.,

$$-\sigma(t)\nu = \mathbb{K}[u(t)]z(t) + \xi\nu \quad \text{on } \Gamma, \tag{4}$$

coupled with (3) and the condition

$$[u] \cdot \nu > 0 \quad \Rightarrow \quad \xi = 0, \tag{5}$$

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This can be equivalently said writting

$$\xi \in \partial I_{[0,+\infty)}([u] \cdot \nu), \tag{7}$$

with $\partial I_{[0,+\infty)}$ denotes the subdifferential of the characteristic function of the interval $[0, +\infty)$.

# Simplifications (WLOG) and Generalizations

We treat the homogeneous Dirichlet datum case $u = 0$ on $\partial_D\Omega$. Moreover we assume all the elasticity tensors being the identity matrix, i.e., $\mathbb{C}^1 = \mathbb{C}^2 = \mathbb{K} = Id$, and the constant $\rho = \mu = 1$. Finally we replace the symmetric gradient $e(u)$ by the usual one $\nabla u$ (wlog thanks to Korn).

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Let us rewrite all the equations

$$\ddot{u} - \Delta u - \Delta \dot{u} = f \quad \text{on } \Omega, \tag{8a}$$

$$-(\nabla u + \nabla \dot{u})\nu = [u]z + \xi\nu \quad \text{on } \Gamma, \tag{8b}$$

$$\frac{1}{2}|[u]|^2 < \alpha \quad \Rightarrow \quad \dot{z} = 0, \tag{8c}$$

and

$$\dot{z}(\frac{1}{2}|[u]|^2 - \alpha) = 0, \tag{8d}$$

$$\frac{1}{2}|[u]|^2 - \alpha \leq 0, \tag{8e}$$

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holding on the set $\{z > 0\} \subset \Gamma$.
On the other side, in orther to get a general constraint, we replace the function $I_{[0,+\infty)}$ by $j : \mathbb{R} \to [0, +\infty]$, being a convex and lower semicontinuous function such that $j(0) = \min j = 0$. Then the constraint reads

$$\xi \in \partial j([u] \cdot \nu). \tag{8f}$$

# Generalized constraint

We define

$$\mathcal{J}(v) := \int_0^T \int_\Gamma j(v)\,dx\,dt \quad v \in L^2([0,T] \times \Gamma). \tag{9}$$

The subdifferential of $\mathcal{J}$ on $L^2([0,T] \times \Gamma)$ is defined as the multivalued operator

$$\partial\mathcal{J} : L^2([0,T] \times \Gamma) \rightrightarrows L^2([0,T] \times \Gamma),$$

as follows: for $v, u \in L^2([0,T] \times \Gamma)$, we have

$$v \in \partial\mathcal{J}(u) \Leftrightarrow \mathcal{J}(w) - \mathcal{J}(u) \geq ((v, w-u)) \quad \forall w \in L^2([0,T] \times \Gamma). \tag{10}$$

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Setting $\beta := \partial j$, it is easy to see that $v \in \partial\mathcal{J}(u)$ if and only if

$$v(t, x) \in \beta(u(t, x)) \quad \text{for a.e. } (t, x) \in [0, T] \times \Gamma. \tag{11}$$

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We call this the pointwise interpretation of $\partial\mathcal{J}$, so we still denote it by $\beta := \partial\mathcal{J}$.

# Relaxation of the constraint

We consider the restriction of $\mathcal{J}$ to the space $\mathcal{H} \subset L^2([0, T] \times \Gamma)$, and we consider the subdifferential with respect to this new topology. This is the multivalued operator

$$\partial_{\mathcal{H}} \mathcal{J} : \mathcal{H} \rightrightarrows \mathcal{H}',$$

defined as follows: for $u \in \mathcal{H}$ and $\xi \in \mathcal{H}'$, we have

$$\xi \in \partial_{\mathcal{H}} \mathcal{J}(u) \quad \Leftrightarrow \quad \mathcal{J}(w) - \mathcal{J}(u) \geq \langle\langle \xi, w - u \rangle\rangle \quad \forall w \in \mathcal{H}, \tag{12}$$

where $\langle\langle \cdot, \cdot \rangle\rangle$ is the duality pairing between $\mathcal{H}$ and $\mathcal{H}'$.

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Note that the pointwise interpretation

$$\xi(t, x) \in \beta(u(t, x)) \quad \text{for a.e. } (t, x) \in [0, T] \times \Gamma,$$

does no longer make sense!

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**Theorem**

*There exists a bounded Borel measure $\mathcal{T}$ such that $\langle\langle \xi, \varphi \rangle\rangle = \int_0^T \int_\Gamma \varphi d\mathcal{T}$ for all $\varphi \in \mathcal{H} \cap C_0([0, T] \times \Gamma)$. Moreover if $\mathcal{T} = \mathcal{T}_a + \mathcal{T}_s$ then*

$$\mathcal{T}_a u \in L^1([0, T] \times \Gamma), \tag{13}$$

$$\mathcal{T}_a(t, x) \in \beta(u(t, x)) \text{ for a.e. } (t, x) \in [0, T] \times \Gamma, \tag{14}$$

$$\langle\langle \xi, u \rangle\rangle - \int_0^T \int_\Gamma \mathcal{T}_a u \, dxdt = \sup \left\{ \int_0^T \int_\Gamma z \, d\mathcal{T}_s, \ z \in C([0, T] \times \Gamma), \ |z| \leq 1 \right\}. \tag{15}$$

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It can be proved that, in the case that $j = I_{[0,+\infty)}$, denoting by $\mathcal{T}_s = \rho|\mathcal{T}_s|$,

$$\rho \in \partial j(u) \ \ |\mathcal{T}_s| - \text{a.e. in } [0,T] \times \Gamma. \tag{16}$$

This means that $\mathcal{T}_s$ is supported on the set where $u = 0$ and that here it holds $\rho = -1$.

# References

These results are adaptations of those contained in


- H. Brézis, *Intégrales convexes dans les espaces de Sobolev*, Israel J. Math., **13** (1972), 9–23.
- M. Grun-Rehomme, *Caractérisation du sous-différentiel d'intégrandes convexes dans les espaces de Sobolev* (French), J. Math. Pures Appl. (9), **56** (1977), 149–156.

# Approximation of $\beta_w$

We introduce $j^\epsilon$ the Moreau-Yosida regularization of $j$, and define the operator $\mathcal{J}^\epsilon$ on $L^2([0,T]\times\Gamma)$ as

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As for $j$, we set

$$\beta^\epsilon := \partial j^\epsilon,$$

the Yosida approximation of $\beta$. Recall that $\beta^\epsilon$ is globally $\epsilon^{-1}$-Lipschitz continuous.

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**Lemma**

*$\beta^\epsilon$ is a monotone operator on $\mathcal{H}$ into $\mathcal{H}'$. Moreover for $u \in \mathcal{H}$ then $\beta^\epsilon(u) \in \mathcal{H}'$ belongs to the subdifferential of $\mathcal{J}^\epsilon$ (as an operator on $\mathcal{H}$).*

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# Approximation of $\beta_w$

Following the theory of

- H. Attouch, “Variational Convergence for Functions and Operators”, Pitman, London, 1984.

we can then prove that the monotone operators $\beta^\epsilon$ tends to the maximal monotone operator $\beta_w$ in the sense of graph, i.e.,

$$\forall [x, y] \in \beta_w \quad \exists [x^\epsilon, y^\epsilon] \in \beta^\epsilon \quad \text{such that} \quad [x^\epsilon, y^\epsilon] \to [x, y],$$

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**Lemma**

*Let the monotone operator $\eta_n$ tends to the maximal monotone operator $\eta$ in the sense of graph (operators on $\mathcal{H}$ into $\mathcal{H}'$). Let $u_n \rightharpoonup u$ weakly in $\mathcal{H}$, $\xi_n \rightharpoonup \xi$ weakly in $\mathcal{H}'$, and assume $\xi_n \in \eta_n(u_n)$. If*

$$\limsup \langle\langle \xi_n, u_n \rangle\rangle \leq \langle\langle \xi, u \rangle\rangle,$$

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These are the ingredients we need!

# Energetic formulation of the evolution

We define a weak form of solution to problem (8).

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**Definition**

Let $u_0, v_0 \in V$, $z_0 \in \mathcal{Z}$, and $f \in L^2([0,T], V')$. Then $(u, z, \eta)$ is an energetic solution to (8) if

$$u \in H^1([0,T], V) \cap W^{1,\infty}([0,T], L^2(\Omega)), \tag{18a}$$

$$\dot{u} \in H^1([0,T], H^{-1}(\tilde{\Omega})) \cap BV(0,T; \tilde{H}^{-2}(\Omega)), \tag{18b}$$

$$z \in L^\infty([0,T], \mathcal{Z}) \cap BV(0,T; L^1(\Gamma)), \tag{18c}$$

$$\xi \in \mathcal{H}', \tag{18d}$$

is such that $u(0) = u_0$, $\dot{u}(0) = v_0$, $z(0) = z_0$, and satisfies conditions (a), (a'), (b), (c), and (d) below.

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(a) for all $\varphi \in \mathcal{V}$,

$$\begin{aligned} &-(\!(\dot{u}, \dot{\varphi})\!) + (\dot{u}(T), \varphi(T)) + (\!(\nabla \dot{u}, \nabla \varphi)\!) + (\!(\nabla u, \nabla \varphi)\!) + \langle\!\langle \xi, [\varphi] \cdot \nu \rangle\!\rangle \\ &= (u_1, \varphi(0)) + \langle\!\langle f, \varphi \rangle\!\rangle - (\!(z[u], [\varphi])\!)^\Gamma. \end{aligned} \tag{19}$$

# Energetic formulation of the evolution

(a') for all $t \in [0, T]$ there exists $\xi_t \in \mathcal{H}'$ such that also the local version of (19) holds

$$\begin{aligned} &- (\!(\dot u, \dot\varphi)\!)_t + (\dot u(t), \varphi(t)) + (\!(\nabla \dot u, \nabla\varphi)\!)_t + (\!(\nabla u, \nabla\varphi)\!)_t + \langle\!\langle \xi_t, [\varphi]\cdot\nu\rangle\!\rangle_t \\ &= (u_1, \varphi(0)) + \langle\!\langle f, \varphi\rangle\!\rangle_t - (\!(z[u], [\varphi])\!)^{\Gamma}_t, \end{aligned} \tag{20}$$

for all $\varphi \in \mathcal{V}_t$. Moreover $\xi_t$ satisfies the property that, for all $\varphi \in \mathcal{H}_t$ with $\varphi(t) = 0$, we have

$$\langle\!\langle \xi_t, \varphi\rangle\!\rangle_t = \langle\!\langle \xi, \tilde\varphi\rangle\!\rangle, \tag{21}$$

where $\tilde\varphi$ is the extension to $\mathcal{H}$ of $\varphi \in \mathcal{H}_{t,0}$ such that $\varphi(s) = 0$ for $s \in [t, T]$.

(b) We have

$$\xi \in \beta_w([u]\cdot\nu), \tag{22}$$

and for all $t \in [0, T]$ it also holds that

$$\xi_t \in \beta_{w,t}([u\llcorner_{[0,t]}]\cdot\nu).$$

(c) for almost every $x \in \Gamma$ the function $t \mapsto z(t,x)$ is nonincreasing and

$$\text{either} \quad \frac{1}{2}|[u(t,x)]|^2 \leq \alpha(x) \quad \text{or} \quad z(t,x) = 0 \quad \text{for a.e. } x \in \Gamma \tag{23}$$

for all $t \in [0, T]$.

(c') for all times $t_1$ and $t_2$ with $0 \leq t_1 < t_2 \leq T$ it holds

$$\int_\Gamma z(t_2)(\frac{1}{2}|[u(t_2)]|^2 - \alpha)dx - \int_\Gamma z(t_1)(\frac{1}{2}|[u(t_1)]|^2 - \alpha)dx \\ - \int_{t_1}^{t_2} \int_\Gamma z[u] \cdot [\dot{u}] dx dt = 0. \tag{24}$$

(d) for all $t \in [0, T]$ the following energy inequality holds

$$\frac{1}{2}\|\dot{u}(t)\|_H^2 + \int_\Gamma j([u(t)] \cdot \nu)dx + \frac{1}{2}\int_\Gamma z(t)|[u](t)|^2 dx + \frac{1}{2}\|\nabla u(t)\|^2 \\ + \int_0^T \|\nabla \dot{u}\|^2 dt - (\alpha, z(t))_\Gamma + (\alpha, z_0)_\Gamma \leq \\ \frac{1}{2}\|v_0\|_H^2 + \int_\Gamma j([u_0] \cdot \nu)dx + \frac{1}{2}\int_\Gamma z_0|[u_0]|^2 dx + \frac{1}{2}\|\nabla u_0\|^2 + \langle\langle f, \dot{u} \rangle\rangle_t. \tag{25}$$

# Energetic formulation of the approximate evolution

In order to prove the existence of an energetic solution we introduce the following approximate evolutions.

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**Definition**

Let $\epsilon \in (0,1)$, $u_0, v_0 \in V$, $z_0 \in \mathcal{Z}$, and $f \in L^2([0,T], V')$. Then $(u^\epsilon, z^\epsilon)$ is an $\epsilon$-approximation of the energetic solution (4) if

$$u^\epsilon \in H^1([0,T], V) \cap W^{1,\infty}([0,T], L^2(\Omega)), \tag{26a}$$

$$\dot{u}^\epsilon \in H^1([0,T], V'), \tag{26b}$$

$$z^\epsilon \in L^\infty([0,T], \mathcal{Z}) \cap BV(0,T; L^1(\Gamma)), \tag{26c}$$

is such that $u^\epsilon(0) = u_0$, $\dot{u}^\epsilon(0) = v_0$, $z^\epsilon(0) = z_0$, and satisfies conditions (a$^\epsilon$), (b$^\epsilon$), and (c$^\epsilon$) below.

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(a$^\epsilon$) for every time $t \in [0,T]$, it holds

$$\begin{aligned} &-((\dot{u}^\epsilon, \dot{\varphi}))_t + (\dot{u}^\epsilon(t), \varphi(t)) + ((\nabla \dot{u}^\epsilon, \nabla \varphi))_t + ((\nabla u^\epsilon, \nabla \varphi))_t + \langle\langle \beta^\epsilon([u^\epsilon]\cdot\nu), [\varphi]\cdot\nu \rangle\rangle_t \\ &= (u_1, \varphi(0)) + \langle\langle f, \varphi \rangle\rangle_t - ((z^\epsilon [u^\epsilon], [\varphi]))_t^\Gamma, \end{aligned} \tag{27}$$

for all $\varphi \in \mathcal{V}$.

# Energetic formulation of the approximate evolution

($b^\epsilon$) for almost every $x \in \Gamma$ the function $t \mapsto z^\epsilon(t,x)$ is nonincreasing and

$$\text{either} \quad \frac{1}{2}|[u^\epsilon(t,x)]|^2 \leq \alpha(x) \quad \text{or} \quad z^\epsilon(t,x) = 0 \quad \text{for a.e. } x \in \Gamma \tag{28}$$

for all $t \in [0, T]$.

($c^\epsilon$) the following energy balance holds

$$\frac{1}{2}\|\dot{u}^\epsilon(t)\|_H^2 + \int_\Gamma j^\epsilon([u^\epsilon(t)] \cdot \nu)dx + \frac{1}{2}\int_\Gamma z^\epsilon(t)|[u^\epsilon](t)|^2 dx + \frac{1}{2}\|\nabla u^\epsilon(t)\|^2$$

$$+ \int_0^T \|\nabla \dot{u}^\epsilon\|^2 dt - (\alpha, z^\epsilon(t))_\Gamma + (\alpha, z_0)_\Gamma =$$

$$\frac{1}{2}\|v_0\|_H^2 + \int_\Gamma j^\epsilon([u_0] \cdot \nu)dx + \frac{1}{2}\int_\Gamma z_0|[u_0]|^2 dx + \frac{1}{2}\|\nabla u_0\|^2 + \langle\langle f, \dot{u}^\epsilon \rangle\rangle_t, \tag{29}$$

for all $t \in [0, T]$.

# Existence of evolutions

For all $\epsilon \in (0,1)$, existence of an approximate solution can be obtained by time discretization and by an implicit Euler scheme. This is standard and we can adapt, for instance, results by

- T. Roubicek, *Adhesive contact of visco-elastic bodies and defect measures arising by vanishing viscosity*. SIAM J. Math. Anal., **45** (2013), 101-126,

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We want to pass to the limit as $\epsilon \to 0$. The following Theorem holds true.

**Theorem**

*Let $(u^\epsilon, z^\epsilon)$ be approximate solutions. Then there exists $(u, z, \xi)$ energetic solution as in Definition 4 such that, up to a subsequence,*

$$u^\epsilon \to u \quad \text{strongly in } H^1(0,T;L^2(\Omega)) \text{ and weakly in } H^1(0,T;V), \tag{30a}$$

$$\dot{u}^\epsilon \rightharpoonup \dot{u} \quad \text{weakly in } H^1(0,T;H^{-1}(\tilde{\Omega})) \text{ and weakly* in } BV(0,T;\tilde{H}^{-2}(\Omega)), \tag{30b}$$

$$z^\epsilon(t) \rightharpoonup z(t) \quad \text{weakly* in } L^\infty(\Gamma) \text{ for all } t \in [0,T], \tag{30c}$$

$$\beta_\epsilon([u^\epsilon]\cdot\nu) \rightharpoonup \xi \quad \text{weakly in } \mathcal{H}' \text{ and in } \mathcal{V}'. \tag{30d}$$

# Sketch of the proof

In order to get the convergences above we should prove suitable apriori estimates for $(u^\epsilon, z^\epsilon, \beta_\epsilon([u^\epsilon] \cdot \nu))$.

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Some estimates are straightforward and follows by the energy balance ($c^\epsilon$). These are

$$
\begin{aligned}
&\|u^\epsilon\|_{H^1(0,t;V)} \leq M,\\
&\int_\Gamma j^\epsilon([u^\epsilon(t)]\cdot\nu) \leq M \text{ for all } t\in[0,T],\\
&\frac{1}{2}\int_\Gamma |[u^\epsilon](t)|^2 z^\epsilon(t) \leq M \text{ for all } t\in[0,T],\\
&\|z^\epsilon\|_{L^\infty(0,t;\mathcal{Z})} \leq M,\\
&\|z^\epsilon\|_{BV(0,T;L^1(\Gamma))} \leq M.
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$$\|z^\epsilon\|_{L^\infty(0,t;\mathcal{Z})} \leq M,$$
$$\|z^\epsilon\|_{BV(0,T;L^1(\Gamma))} \leq M.$$

Crucial is the following one:

**Lemma**

*For all $\epsilon \in (0,1)$ it holds*

$$\|\beta_\epsilon([u^\epsilon]\cdot\nu)\|_{L^1(0,T;L^1(\Gamma))} \leq M. \tag{31}$$

# Sketch of the proof

This can be obtained letting $\bar{\psi} \in H^{\frac{1}{2}}(\Gamma; \mathbb{R}^d)$ such that $\bar{\psi} \cdot \nu = 1$ on the whole $\Gamma$ and extending (harmonic) it to an element $\varphi \in V$ which is 0 on $\Omega_2$, and set $\Psi(t, x) := \varphi(x)$ for all $t \in [0, T]$. Testing the weak equation ($\mathrm{a}^\epsilon$) by $u - \delta\Psi$, $\delta > 0$,

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$$(\dot{u}^\epsilon(T), u^\epsilon(T)) - (u_1^\epsilon, u_0^\epsilon) - \int_0^T \|\dot{u}^\epsilon\|_2^2 + \int_{\Omega_1} u^\epsilon(t), \delta\Psi dx - \int_{\Omega_1} u_0^\epsilon, \delta\Psi dx + \int_0^T \|\nabla u^\epsilon\|_2^2 dt$$

$$+ \frac{1}{2}\|\nabla u^\epsilon(T)\|_2^2 - \frac{1}{2}\|\nabla u_0^\epsilon\|_2^2 + (\!(\nabla u, \delta\nabla\psi\rangle)\!)^{\Omega_1} + (\nabla u^\epsilon(T), \delta\nabla\Psi)^{\Omega_1} - (\nabla u_0^\epsilon, \delta\nabla\Psi)^{\Omega_1}$$

$$+ \int_0^T \int_\Gamma \beta_\epsilon([u^\epsilon]\cdot\nu)([u^\epsilon]\cdot\nu - \delta) dxdt + (\!(z^\epsilon, |[u^\epsilon]|^2 - \delta u^\epsilon \cdot \nu)\!) = \langle\!\langle f, u^\epsilon \rangle\!\rangle + \langle\!\langle f, \delta\Psi \rangle\!\rangle^{\Omega_1}.$$

and then using the previous estimates and the fact that $|\beta_\epsilon(x)| \leq \delta^{-1}|\beta_\epsilon(x)(x-\delta)|$ for $\epsilon \in (0,1)$.

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The previous estimate implies

$$\|\dot{u}^\epsilon\|_{W^{1,1}(0,T;\tilde{H}^{-2}(\Omega))} \le M, \tag{32}$$

for all $\epsilon \in (0,1)$.

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Now we employ Helly selection principle combined with a generalized version of Aubin-Lions compactness principle entail the strong convergence

$$u^\epsilon \to u \quad \text{strongly in } H^1(0, T; L^2(\Omega)) \tag{33}$$

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Another key fact is the following:

**Lemma**

*There holds*

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Using the previous estimate we argue as before using an arbitrary function $\psi \in H^{\frac{1}{2}}(\Gamma;\mathbb{R}^d)$ and suitably extending it to $V$.
We have then found

$$\beta_\epsilon([u^\epsilon]\cdot\nu) \rightharpoonup \xi,$$

for some $\xi \in \mathcal{H}'$.

# Sketch of the proof

We are now ready to prove (b). To this aim we apply Lemma 3, and then we have to check that

$$\limsup_{\epsilon \to 0} \langle\!\langle \beta^\epsilon(u^\epsilon), u^\epsilon \rangle\!\rangle \leq \langle\!\langle \xi, u \rangle\!\rangle. \tag{36}$$

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Thanks to the convergences above it is easily seen that (a) holds. Using (27), we write

$$\langle\langle \beta^\epsilon([u^\epsilon] \cdot \nu), u^\epsilon \rangle\rangle = \|\dot{u}^\epsilon\|^2_{L^2(0,T;L^2)} - (\dot{u}^\epsilon(T), u^\epsilon(T)) + (u_1, u_0) - \frac{1}{2}\|\nabla u^\epsilon(T)\|^2$$
$$+ \frac{1}{2}\|\nabla u_0\|^2 - \|\nabla u^\epsilon\|^2_{L^2(0,T;L^2)} - \int_0^T \int_\Gamma z^\epsilon(t)[u^\epsilon(t)]^2 dxdt + ((f, u^\epsilon)). \tag{37}$$

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Taking the lim sup we get

$$\leq \|\dot u\|^2_{L^2(0,T;L^2)} - (\dot u(T), u(T)) + (u_1, u_0) - \frac{1}{2}\|\nabla u(T)\|^2 + \frac{1}{2}\|\nabla u_0\|^2 - \|\nabla u\|^2_{L^2(0,T;L^2)}$$
$$+ \int_0^T \int_\Gamma z(t)[u(t)]^2 dxdt + ((f, u)) = \langle\!\langle \xi, u \rangle\!\rangle, \tag{38}$$

and the claim follows.

# Some Remarks

We have seen that $\xi$ and $\xi_t$ coincides with Borel measures concentrated on the set where $[u] \cdot \nu = 0$. Thanks to the fact that

$$\dot{u} \in BV(0, T; H^{-2}(\Omega)),$$

it can be proved that $\xi_t \equiv \xi \llcorner_{[0,t] \times \Gamma}$ outside a set of countable many times.

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In dimension $d = 1$ energy balance holds!

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In some sense the presence of such jumps is the main difficulty to prove an enery balance (if it holds). Outside these points an energy balance holds true!!!

In dimension $d = 1$ energy balance holds!
The behaviour of the constraint is quite independent of the flow rule of the variable $z$. For instance, the same behaviour takes place when we add a viscosity term in the flow rule

$$\frac{1}{2}|[u]|^2 + \dot{z} - \alpha \leq 0$$

(S.-G. Schimperna).

THANK YOU FOR ATTENTION!

Some references:


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