Optimal control of multifrequency induction hardening

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joint work with
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- The process of Induction Hardening
  - classical induction hardening
  - multifrequency induction hardening
  - aims: simulation and optimization of the process

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  - the induction heating (Maxwell equations)

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  - ◇ stability estimates
  - ◇ first-order optimality conditions

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  - ◇ first-order optimality conditions
- Comment on some numerics ([Hömberg, Petzold, WIAS])

Hardening process

- In most structural components in mechanical engineering, the surface is particularly stressed. Therefore, the aim of surface hardening is to increase the hardness of the boundary layers of a workpiece by rapid heating and subsequent quenching
- This heat treatment leads to a change in microstructure, which produces the desired hardening effect. Typical examples of application are gear-wheels

Hardening process

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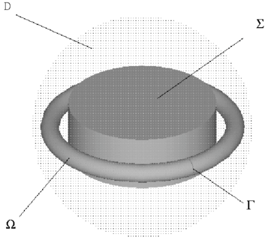
- This heat treatment leads to a change in microstructure, which produces the desired hardening effect. Typical examples of application are gear-wheels
- The mode of operation in induction hardening facilities relies on the transformer principle. A given current density in the induction coil (inductor) Ω induces eddy currents inside the workpiece Σ

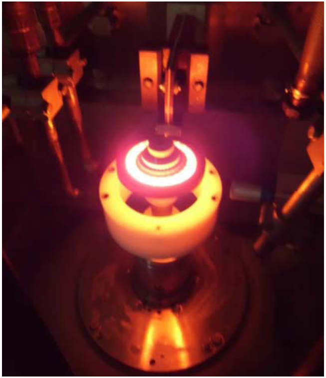


Advantages and Disadvantages

- Because of the Joule effect these eddy currents lead to an increase in temperature in the boundary layers of the workpiece
- Then the current is switched off and the workpiece is quenched by spray-water cooling

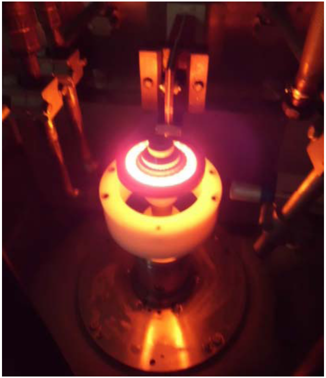
Advantages and Disadvantages

- Since the magnitude of the eddy currents decreases with growing distance from the workpiece surface because of the frequency dependent **skin-effect**, induction heating is suitable for surface hardening if the current frequency is big enough
- After heating, the workpiece is quenched by spray-water cooling and another phase transition leads to the desired hardening effect in the boundary layers of the workpiece



Advantages and Disadvantages

- There is a growing demand in industry for a more **precise process control**:
  - for weight reduction, especially in automotive industry, leading to components made of **thinner and thinner steel sheets**. The surface hardening of these sheets is a very delicate task, since one must be careful not to harden the complete sheet, which would lead to undesirable fatigue effects
  - for tendency to use high quality steels with only a **small carbon content**. Since the hardenability of a steel is directly related to its carbon content, already from a metallurgical point of view, the treatment of these steels is extremely difficult



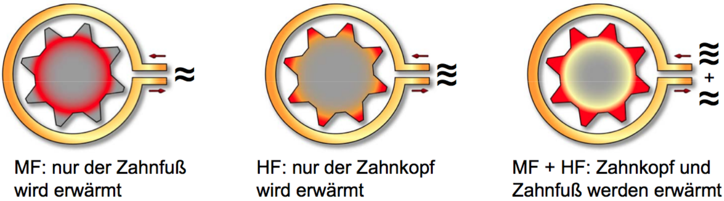
Advantage: Very fast and energy-efficient process

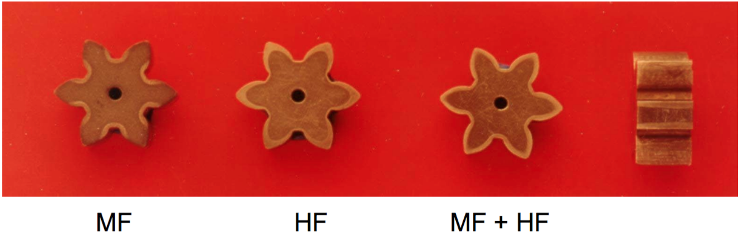
Drawback: Difficult to generate desired close to contour hardening profile for complex work pieces such as gears

MF+HF Actually simultaneous supply of medium and high frequences **(MF+HF)** gives the best result

Multi-frequency induction hardening

- Simultaneous supply of medium - and high frequency power on one induction coil
- Close to contour hardening profile for gears and other complex-shaped parts

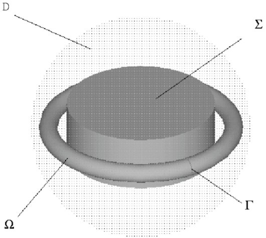




Only the bases heated	Only the tips heated	Both bases and tips heated

The model

- The reason why one can change the hardness of steel by thermal treatment lies in the occurring **phase transitions**
- In the case of surface hardening, owing to high cooling rates, most of the austenite is transformed to martensite by a diffusionless phase transition leading to the desired increase of hardness



The model

Hence a mathematical model for induction surface hardening has to account for

- the **electromagnetic effects** that lead to the surface heating
- the **thermomechanical effects**
- the **phase transitions** that are caused by the enormous changes in temperature during the heat treatment

The phase transitions

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z_0	initial composition
z_1	austenite
z_2	ferrite
z_3	pearlite
z_4	martensite

$\sum z_i = 1$

- At room temperature, in coil workpiece general, steel is a mixture of ferrite, pearlite, and martensite
- Upon heating, these phases are transformed to austenite
- Then, during cooling austenite is transformed back to a mixture of ferrite, pearlite, and martensite – we do not consider this part of the process – less interesting

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The actual phase distribution at the end of the heat treatment depends on the cooling strategy. In the case of surface hardening, owing to high cooling rates most of the austenite is transformed to martensite by a diffusionless phase transition leading to the desired increase of hardness.

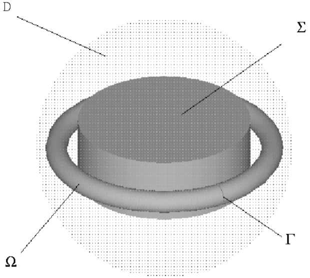
The evolution of the phase fraction of austenite $z = z_1$

- According to [Leblond and Deveaux (1984)], the formation of austenite cannot be described by the additivity rule, since for fixed temperature within the transformation range, one can get an equilibrium volume fraction of austenite less than one

- Therefore, they propose to use the rate law

$$
\begin{aligned}
z(0) &= 0 \quad \text{in } \Sigma \\
z_t(t) &= \frac{1}{\tau(\theta)} \max\{(z_{eq}(\theta) - z(t)), 0\} \\
&= \frac{1}{\tau(\theta)} (z_{eq}(\theta) - z(t))^+ \quad \text{in } Q := \Sigma \times (0, T)
\end{aligned}
$$

$z_{eq} \in [0,1]$ is an equilibrium fraction of austenite.



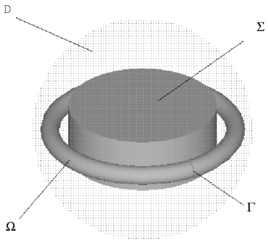
The magnetic field

In the eddy current problems we get the Maxwell's equations in $D \times (0, T)$

$$\operatorname{curl} H = J, \quad \operatorname{curl} E = -B_t, \quad \operatorname{div} B = 0$$

where

- E is the electric field
- B the magnetic induction
- H the magnetic field
- J the spatial current density
- and we have neglected the electric displacement in the first relation ($|\frac{\partial D}{\partial t}| << |J|$)



The magnetic field

Assume the Ohm's law and a linear relation between the magnetic induction and the magnetic field

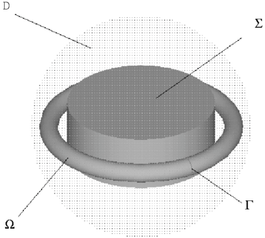
$$J = \sigma E\,, \qquad B = \mu H$$

where the electrical conductivity σ and the magnetic permeability μ (sufficiently regular and bounded for below and above) may depend both on the spatial variables and also on the phase parameter z

$$\sigma(x,z) = \begin{cases} 0, & x \in D \setminus (\Omega \cup \Sigma) \\ \sigma_w(z), & x \in \Sigma \\ \sigma_i, & x \in \Omega \end{cases}$$

and

$$\mu(x,z) = \begin{cases} \mu_0, & x \in D \setminus (\Omega \cup \Sigma) \\ \mu_w(z), & x \in \Sigma \\ \mu_i, & x \in \Omega \end{cases}$$



The magnetic vector potential

Since $\operatorname{div} B = 0$, we can introduce the magnetic vector potential A such that

$$B = \operatorname{curl} A \quad \text{in } D$$

and, since A is not uniquely defined, we impose the Coulomb gauge

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Using $\operatorname{curl} E + B_t = 0$ and $B = \operatorname{curl} A$, we define the scalar potential ϕ by

$$E + A_t = -\nabla\phi \quad \text{in } D \times (0, T)$$

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$$E + A_t = -\nabla\phi \quad \text{in } D \times (0, T)$$

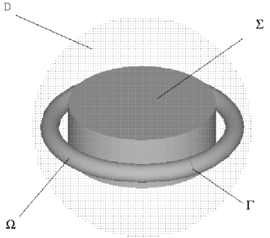
and we get the total current density

$$J = \sigma E = \underbrace{-\sigma A_t}_{J_{eddy}} \quad \underbrace{-\sigma\nabla\phi}_{J_{source}} \quad \text{in } D \times (0, T)$$

Since $J_{source} = 0$ in $D \setminus \Omega$, we get

$$\underbrace{\sigma A_t + \operatorname{curl} \underbrace{\left(\frac{1}{\mu} \operatorname{curl} A\right)}_{H}}_{J} + \sigma\chi_\Omega\nabla\phi = 0 \text{ in } D \times (0, T) \text{ and } -\operatorname{div}(\sigma\nabla\phi) = 0 \text{ in } \Omega$$

Boundary conditions



- Introduction of boundary conditions on ∂D
  (1) Perfect electric conductor $E \times n = 0$
  (2) Perfect magnetic conductor $H \times n = 0$

(1) leads to

$$A \times n = 0 \quad \text{(tangential component vanish on } \partial D\text{)}$$

(2) leads to

$$\mu^{-1} \operatorname{curl} A \times n = 0$$

- Inside the inductor (a closed tube) we fix a section Γ and model the current density which is generated by the hardening machine by an interface condition on Γ

$$\sigma \nabla \phi \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega$$
$$[\sigma \nabla \phi \cdot \tilde{n}] = 0 \quad \text{and } [\phi] = U_0 \quad \text{on } \Gamma$$

where n, $\mathbf{n}$, and $\tilde{n}$ denote normal unit vectors to ∂D, $\partial\Omega$, and Γ, resp.

Eliminating the scalar potential

- For a given coil geometry (here a torus with rectangular cross-section), the source current density $J_{source} := \sigma \nabla \phi$ can be precomputed analytically
- From $\mathrm{div}(\sigma \nabla \phi) = 0$ one obtains in cylindrical coordinates

$$\phi = C_1 \varphi \quad \text{and consequently } J_{source} = \sigma C_1 \, (0, 1/r, 0)^T_{(r,\varphi,z)}$$

where $C_1 = U_0/(2\pi)$ for a given voltage

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- For a given source current in the coil C_1 is computed from

$$\int_\Gamma J_{source}\cdot\tilde{n}\,\mathrm{da} = I_{\mathrm{coil}}$$

- In cartesian coordinates one obtains for a given source current

$$J_{source} = \frac{I_{\mathrm{coil}}}{\log(r_A/r_I)h}\begin{pmatrix}-y/(x^2+y^2)\\ x/(x^2+y^2)\\ 0\end{pmatrix}$$

- It can be used as control for optimization $J_{source} = u(t)J_0$ where J_0 is the spatial current density prescribed in the induction coil Ω

$$J_0(x) = \begin{cases} J_i(x), & x\in\Omega\\ 0, & x\in D\setminus\Omega\end{cases}$$

and $u = u(t)$ **denotes a time-dependent control on** $[0, T]$

The energy balance

Assuming to have as constant density $\rho = 1$ (for simplicity), the internal energy balance results as

$$e_t + \operatorname{div} q = JE = \sigma|A_t + \nabla\phi|^2 = \sigma|A_t|^2 \quad \text{in } \Sigma \times (0, T)$$

where

- e denotes the internal energy of the system
- q the heat flux, which, accordingly to the standard Fourier law is assumed as follows

$$q = -\kappa\nabla\theta, \quad \kappa > 0$$

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Boundary conditions: neglect the possible radiative heat transfer between the inductor and the workpiece assuming

$$\kappa\frac{\partial\theta}{\partial\nu} + \eta\theta = g \quad \text{in } \partial\Sigma \times (0, T)$$

where ν denotes the outward unit normal vector to $\partial\Sigma$, η stands for an heat transfer coefficient and g is a given boundary source

The thermodynamical consistency

From the Helmholtz relation $e = \psi + \theta s$, where $\psi = \psi(\theta, z)$ denotes the free energy of the system, we have that the Clausius-Duhem inequality

$$
\begin{aligned}
\theta\left(s_t + \operatorname{div}\left(\frac{q}{\theta}\right)\right) &= e_t - \psi_\theta\theta_t - \psi_z z_t - \theta_t s - \frac{q}{\theta}\nabla\theta + \operatorname{div} q \\
&= e_t + \operatorname{div} q - (\psi_\theta + s)\theta_t - \psi_z z_t + \frac{\kappa|\nabla\theta|^2}{\theta} \\
&= \sigma|A_t|^2 - (\psi_\theta + s)\theta_t - \psi_z z_t + \frac{\kappa|\nabla\theta|^2}{\theta} \\
&\geq 0
\end{aligned}
$$

is satisfied e.g. if we assume the standard relations $\psi_\theta + s = 0$ and $L\tau(\theta)z_t = -\psi_z$, and hence $\psi_z = -L(z_{eq}(\theta) - z)^+$, for some positive constant $L > 0$.

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$$e_t = c_v\theta_t + (\psi_z + \theta s_z)z_t = c_v\theta_t + f(\theta, z)z_t$$

where we have denoted for simplicity by

$$f(\theta, z) = \psi_z - \theta(\psi_z)_\theta = -L(z_{eq}(\theta) - z)^+ + L\theta z'_{eq}(\theta)H(z_{eq}(\theta) - z)$$

H=Heaviside function.

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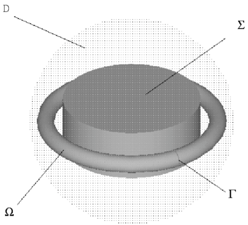
H=Heaviside function. The internal energy balance results

$$c_v\theta_t + \operatorname{div} q = \sigma|A_t|^2 - f(\theta, z)z_t \quad \text{in } \Sigma \times (0, T)$$

The PDE system

- Model consists of vector-potential formulation of Maxwell's equation, heat equation and rate law for phase fractions
- Source term u can be used as control for optimization



$$\sigma A_t + \operatorname{curl} \frac{1}{\mu} \operatorname{curl} A = J_0 u \qquad \text{in } D \times (0, T)$$

$$\theta_t - \operatorname{div} \kappa \nabla \theta = \sigma |A_t|^2 + \mathcal{F}(\theta, z) z_t \qquad \text{in } \Sigma \times (0, T)$$

$$z_t = \frac{1}{\tau(\theta)} \left(z_{\text{eq}}(\theta) - z\right)^+ \qquad \text{in } \Sigma \times (0, T)$$

where

- $J_0 = \left(-y/(x^2 + y^2), x/(x^2 + y^2), 0\right)^T$
- $\mathcal{F}(\theta, z) = L(z_{eq}(\theta) - \theta z'_{eq}(\theta) - z)$

Analysis and control of Joule heating models – a few references

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Resistance heating

- heat source $h = \sigma|\nabla\varphi|^2 \quad \longrightarrow$ thermistor problem
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Induction heating – time domain

- heat source $h = \sigma(\theta)|\nabla\varphi + A_t|^2$
- Bossavit, Rodrigues (1994); Hömberg, Sokolowski (2003): optimal shape design; Hömberg (2004): including mechanical effects $\longrightarrow$ the L^1-regularity of the r.h.s in the internal energy balance $\longrightarrow$ weak solutions via Boccardo-Gallouët estimates

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Induction heating – frequency domain

- Bachinger, Langer, Schöberl: *Numerical analysis of nonlinear multiharmonic eddy current problems*. Numer. Math. **100** (2005)
- Druet, Klein, Sprekels, Tröltzsch, Yousept: *Optimal control of 3D state-constrained induction heating problems with nonlocal radiation effects* SICON **49** (2011)
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- Well-posedness of state system

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- Well-posedness of state system
- Optimality conditions

Solution space for vector potential

$$\mathbb{X} = \left\{ v \in \mathbb{L}^2(D) \,\middle|\, \mathrm{curl}\, v \in \mathbb{L}^2(D)\,,\ \mathrm{div}\, v = 0\,, n \times v\Big|_{\partial D} = 0 \right\}$$

Assume $\partial D \in C^{1,1}$. Then $\mathbb{X}$, equipped with the norm

$$\|v\|_{\mathbb{X}} = \|\,\mathrm{curl}\, v\|_{\mathbb{L}^2(D)}\,,$$

is a closed subspace of $\mathbb{H}^1(D)$

Assumptions

(i) $\sigma(x,z),\ \mu(x,z):\ \overline{D}\times[0,1]\to\mathbb{R}$ are continuous and Lipschitz continuous (w.r.t. z for almost all $x\in D$) function s.t.

$$\underline{\sigma}\le\sigma(x,z)\le\overline{\sigma}\quad\text{in }\overline{D}\times[0,1]$$

$$\underline{\mu}\le\mu(x,z)\le\overline{\mu}\quad\text{in }\overline{D}\times[0,1]\,;$$

(ii) $u\in H^1(0,T)$;

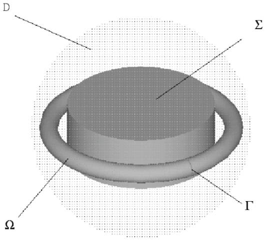
(iii) $J_0\ :\ D\to\mathbb{R}^3$ is an $\mathbb{L}^2_{curl}(D)$-function;

(iv) $\tau,\ z_{eq}\in C^2(\mathbb{R})$ and

$$\tau_*\le\tau(\theta)\le\tau^*,\ 0\le z_{eq}(\theta)\le 1$$
$$\|\tau\|_{C^2(\mathbb{R})},\ \|z_{eq}\|_{C^2(\mathbb{R})}\le M$$
$$|-z_{eq}(\theta)+\theta z'_{eq}(\theta)|,|\theta z''_{eq}(\theta)|\le M\quad\text{for all }\theta\in\mathbb{R}\,;$$

(v) $g\in L^\infty(0,T;L^\infty(\partial\Sigma))$;

(vi) $A_0\in\mathbb{X}\cap\mathbb{H}^3(D)$, $\theta_0\in W^{2,5/3}(\Sigma)$



Weak formulation

Find a triple (A, θ, z) with the regularity properties

$$A \in H^2(0,T;\mathbb{L}^2(D)) \cap W^{1,\infty}(0,T;\mathbb{X}), \quad \operatorname{curl} A \in L^\infty(0,T;\mathbb{L}^6(D)) \tag{1}$$

$$\theta \in W^{1,5/3}(0,T;L^{5/3}(\Sigma)) \cap L^{5/3}(0,T;W^{2,5/3}(\Sigma)) \cap L^2(0,T;H^1(\Sigma)) \cap L^\infty(Q) \tag{2}$$

$$z \in W^{1,\infty}(0,T;W^{1,\infty}(\Sigma)), \quad 0 \le z < 1 \text{ a.e. in } Q \tag{3}$$

solving the following system

$$\int_{\Omega\cup\Sigma} \sigma(x,z) A_t \cdot v \, \mathrm{d}x + \int_D \frac{1}{\mu(x,z)} \operatorname{curl} A \cdot \operatorname{curl} v \, \mathrm{d}x = \int_\Omega J_0(x) u(t) \cdot v \, \mathrm{d}x \tag{4}$$

$$\text{for all } v \in \mathbb{X}, \text{ a.e. in } (0,T)$$

$$\theta_t - \Delta\theta = -\mathcal{F}(\theta,z) z_t + \sigma(x,z)|A_t|^2 \quad \text{a.e. in } Q \tag{5}$$

$$z_t = \frac{1}{\tau(\theta)} (z_{eq}(\theta) - z)^+ \quad \text{a.e. in } Q \tag{6}$$

$$\frac{\partial\theta}{\partial\nu} + \theta = g \quad \text{a.e. on } \partial\Sigma \times (0,T) \tag{7}$$

$$A(0) = A_0, \quad \text{a.e. in } D, \quad \theta(0) = \theta_0, \quad z(0) = 0 \quad \text{a.e. in } \Sigma \tag{8}$$

Well-posedness result

Theorem 1: There exists a unique solution to (4)-(8) satisfying the regularities (1)-(3) and the following estimate

$$\|A\|_{H^2(0,T;\mathbb{L}^2(D))\cap W^{1,\infty}(0,T;\mathbb{X})} + \|\operatorname{curl} A\|_{L^\infty(0,T;\mathbb{L}^6(D))}$$
$$+ \|\theta\|_{W^{1,5/3}(0,T;L^{5/3}(\Sigma))\cap L^{5/3}(0,T;W^{2,5/3}(\Sigma))\cap L^2(0,T;H^1(\Sigma))\cap L^\infty(Q)} + \|z\|_{W^{1,\infty}(0,T;W^{1,\infty}(\Sigma))} \leq S$$

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If we denote by (A_i, θ_i, z_i) $(i = 1, 2)$ two triples of solutions corresponding to data $(A_{0,i}, \theta_{0,i}, u_i)$, then, there exists a positive constant $C = C(S)$ such that the following stability estimate holds true

$$\begin{aligned}
&\|(A_1 - A_2)(t)\|^2_{\mathbb{L}^2(D)} + \|\operatorname{curl}(A_1 - A_2)\|^2_{\mathbb{L}^2(D\times(0,T))} \\
&\quad + \|\partial_t(A_1 - A_2)(t)\|^2_{\mathbb{L}^2(D)} + \|\operatorname{curl}(\partial_t(A_1 - A_2))\|^2_{\mathbb{L}^2(D\times(0,T))} \\
&\quad + \|(\theta_1 - \theta_2)(t)\|^2_{L^2(\Sigma)} + \|\theta_1 - \theta_2\|^2_{L^2(0,T;H^1(\Sigma))} \\
&\quad + \|(z_1 - z_2)(t)\|^2_{H^1(\Sigma)} + \|\partial_t(z_1 - z_2)\|^2_{L^2(0,T;H^1(\Sigma))} \\
&\le C \Big(\|A_{0,1} - A_{0,2}\|^2_{\mathbb{X}} + \|(\partial_t(A_1 - A_2))(0)\|^2_{\mathbb{L}^2(D)} + \|\theta_{0,1} - \theta_{0,2}\|^2_{L^2(\Sigma)} \\
&\quad + \|u_1 J_0 - u_2 J_0\|^2_{L^2(0,T)} + \|u_1' J_0 - u_2' J_0\|^2_{L^2(0,T)} \Big) \quad \text{for all } t \in [0, T]
\end{aligned}$$

Idea of the proof of Thm 1

- Local existence via Shauder fixed point argument exploiting the Lipschitz continuity of the map $\theta \mapsto z$ from $W^{1,p}(0, T_0; L^p(\Sigma))$ in itself

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  1. Exploit the regularity and boundedness of the solution z to the ODE getting

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  $$\int_{\Omega\cup\Sigma} \sigma(x,z) A_t \cdot v \,\mathrm{d}x + \int_D \frac{1}{\mu(x,z)} \operatorname{curl} A \cdot \operatorname{curl} v \,\mathrm{d}x = \int_\Omega J_0(x) u(t) \cdot v \,\mathrm{d}x$$
  and use Est. 1 and the assumptions on μ and σ, getting
  $$\|A\|_{H^1(0,T_0;\mathbb{X})\cap W^{1,\infty}(0,T_0;\mathbb{L}^2(D))} + \|A_t\|_{\mathbb{L}^{10/3}(D\times(0,T_0))} \le C$$

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  3. Test
  $$\theta_t - \Delta\theta = -\mathcal{F}(\theta,z)z_t + \sigma(x,z)|A_t|^2$$
  by θ and using maximal regularity results for parabolic equations, we get
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  4. Differentiating (formally) the A-eq. and taking $v = A_{tt}$, arguing by comparison and using the results of [Kawanago, '93] in order to prove the L^∞-estimate for θ, we get the desired regularity of solutions and we can prolpongate it over the whole time interval $[0, T]$

Stability estimate

- Differentiate in time equation

$$\int_{\Omega\cup\Sigma} \sigma(x,z) A_t \cdot v \, \mathrm{d}x + \int_D \frac{1}{\mu(x,z)} \,\mathrm{curl}\, A \cdot \mathrm{curl}\, v \, \mathrm{d}x = \int_\Omega J_0(x) u(t) \cdot v \, \mathrm{d}x$$

  and take the difference of the two differentiated relations in two solutions (A_i, θ_i, z_i) and test the difference by $(A_1 - A_2)_t$

- Use the Lipschitz continuity properties of the solution map to the ODE in z

- Take the differences of the equations for θ and test by $\theta_1 - \theta_2$, exploit the regularity of A and sum up the two relations

Optimal control problem – Existence

- cost functional

$$\mathcal{J}(A,\theta,z;u) = \frac{\beta_1}{2}\int_0^T\int_\Sigma (\theta(x,t)-\theta_d(x,t))^2\,dxdt + \frac{\beta_2}{2}\int_\Sigma (z(x,T)-z_d)^2\,dx + \frac{\beta_3}{2}\|u\|^2_{H^1(0,T)}$$

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- control problem **(CP)**
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- **Theorem 2: (CP)** has a solution $u \in \mathcal{U}_{ad}$

Optimal control problem – Differentiability

- **Lipschitz-continuity** the control-to-state mapping

$$S \,:\, u \mapsto (A, \theta, z)$$

is Lipschitz continuous from $H^1(0, T)$ to $\mathcal{Z}$ by Theorem 1

- **Fréchet differentiability** the control-to-state mapping

$$S \,:\, u \mapsto (A, \theta, z)$$

is Fréchet differentiable from $H^1(0, T)$ to $\mathcal{Y}$: $\mathcal{Z} \subset \mathcal{Y}$

Optimal control problem – First-order necessary conditions of optimality

- **adjoint system**

$$
\begin{aligned}
-\sigma\alpha_t - \operatorname{curl}\left(\frac{1}{\mu}\operatorname{curl}\alpha\right) - \sigma'(x,z)z_t\alpha &= -2(\sigma A_t\vartheta)_t \\
-\vartheta_t - k\Delta\vartheta + f_\theta(\theta,z)\vartheta &= f_\theta\zeta + \beta_1(\theta-\theta_d) \\
-\zeta_t - f_z(\theta,z)\zeta + \sigma' A_t\cdot\alpha - \sigma'|A_t|^2\vartheta &= \frac{\mu'}{\mu^2}\operatorname{curl} A\cdot\operatorname{curl}\alpha - f_z\vartheta
\end{aligned}
$$

$$
\begin{aligned}
\alpha\times n &= 0 \quad \text{in } \partial D\times(0,T) \\
k\frac{\partial\vartheta}{\partial\nu} + \kappa\vartheta &= 0 \quad \text{in } \partial\Sigma\times(0,T) \\
\vartheta(T) &= 0\,,\ \zeta(T) = z(x,T) - z_d(x) \quad \text{in } \Sigma \\
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\end{aligned}
$$

- **variational inequality**

$$
\int_0^T \left(\beta_3\bar{u}(t) - \int_D \alpha(x,t)\cdot J_0(x,t)dx\right)(u-\bar{u})dt
$$

$$
+\int_0^T \beta_3 u'(t)(u'(t)-\bar{u}'(t))dt \geq 0 \qquad \text{for all } u\in\mathcal{U}_{ad}\subset H^1(0,T)
$$

Numerical results – Challenges [Hömberg, Petzold]

- **Multiple time scales**
  Magnetic vector potential and heat conductance live on different time scales (Averaging method)

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- **Nonlinear material data**
  Magnetic permeability μ depends on temperature *and* magnetic field H (Linearization)
- **3D**
  Time consuming simulation in 3D (Model reduction to tackle optimal control problem numerically)

Numerical realization – multiple time scales

- Time scale for Maxwell's equation governed by frequency of source current: $f \approx 10\ \mathrm{kHz} - 100\ \mathrm{kHz}$, consequetly $\tau \sim 10^{-5}\mathrm{s}$
- Time scale for heat equation governed by heat diffusion:

$$\tau \sim \frac{c_p \rho L^2}{k} \approx 1\ \mathrm{s}$$

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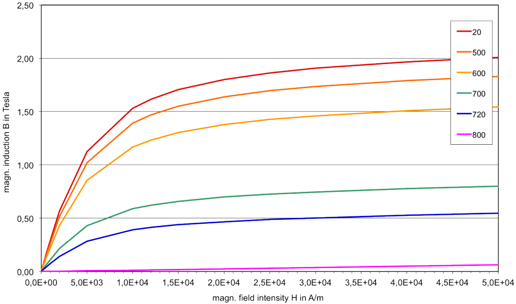
- *Alternating computation:*
  - Solve for A with fixed temperature *on fast time-scale*
  - Compute Joule heat by averaging electric energy $Q = \frac{1}{T}\int_0^T \sigma \left|\frac{\partial A}{\partial t}\right|^2 \mathrm{d}t$
  - Solve heat equation with fixed magnetic potential *on slow time-scale* (one time step)
  - Update A since σ and μ change with temperature

Thermal and electrical conductivity, heat capacity and density

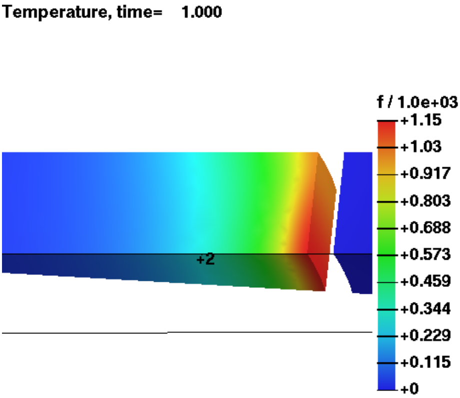
- Material data depend on temperature

  Electrical conductivity $\sigma(\theta)$
  Thermal conductivity $\kappa(\theta)$
  Density $\rho(\theta)$
  Specific heat capacity $c_p(\theta)$

- Nonlinear relation between magnetic induction B and magnetic field H:
  Magnetization curve $B = \mu(\theta, H)H$



Example 1, HF: temperature and growth of austenite



(Video: austenite.mp4)

Figure : temperature and austenite growth at high frequency

Example 1, HF vs. MF: Adaptive grid

- Source current in induction coil $I_0 = 5000\ \mathrm{A}$ at $f = 10\ \mathrm{kHz}$
- Heating time $1.0\ \mathrm{s}$
- Nonlinear data for $\sigma, c_p, \rho, \kappa, \mu_r$
- Adaptive grid with approx. 50000 DOF

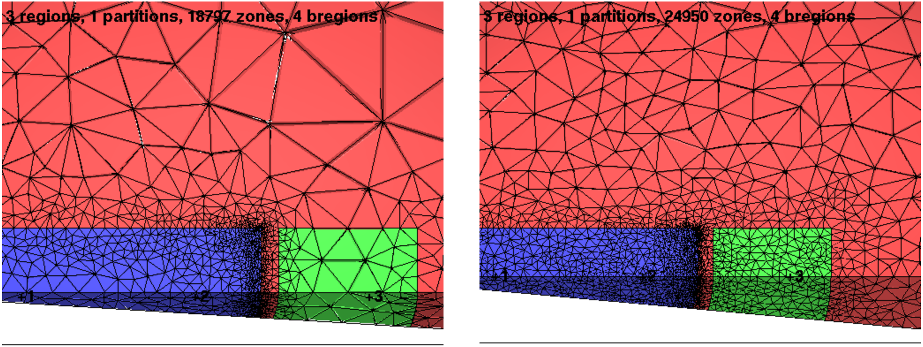


Figure : Comparison of the adaptive grid between HF (left) and MF (right)

Example 1, HF vs. MF: Temperature

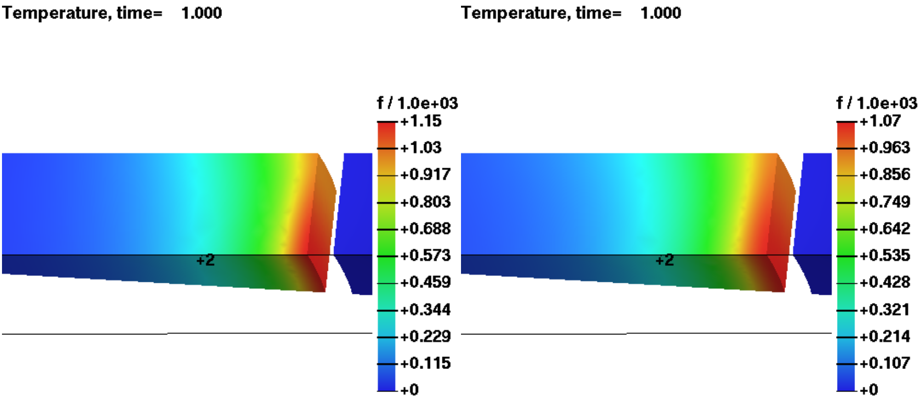
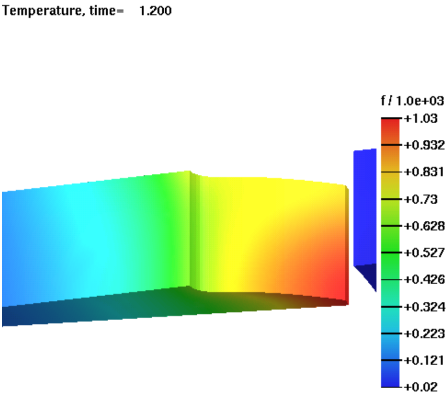


Figure : Comparison of the temperature profile after 1s between HF (left) and MF (right)

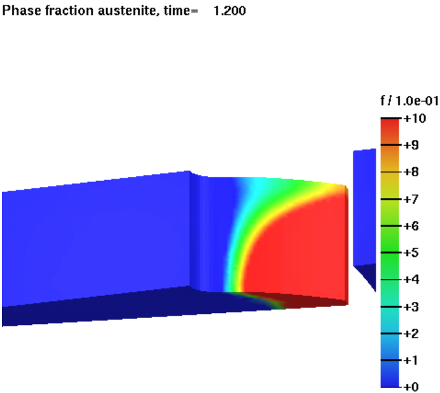
Example 2, HF: temperature



(Video: temperature.mp4)

Figure : temperature evolution at high frequency

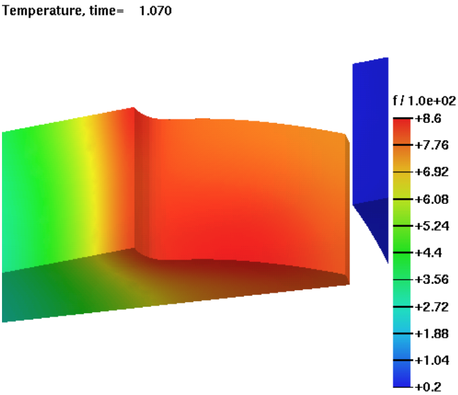
Example 2, HF: growth of austenite



(Video: austenite.mp4)

Figure : austenite evolution at high frequency

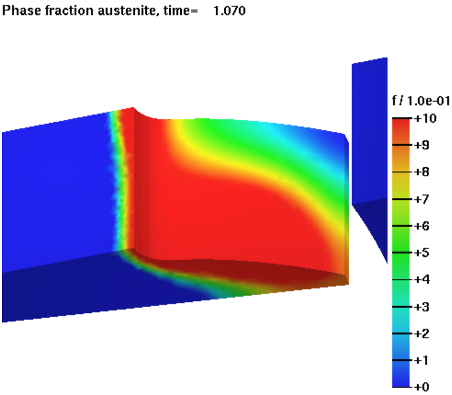
Example 2, MF: temperature



(Video: temperature.mp4)

Figure : temperature evolution at medium frequency

Example 2, MF: growth of austenite



(Video: austenite.mp4)

Figure : austenite evolution at medium frequency

Example 2: MF vs. HF

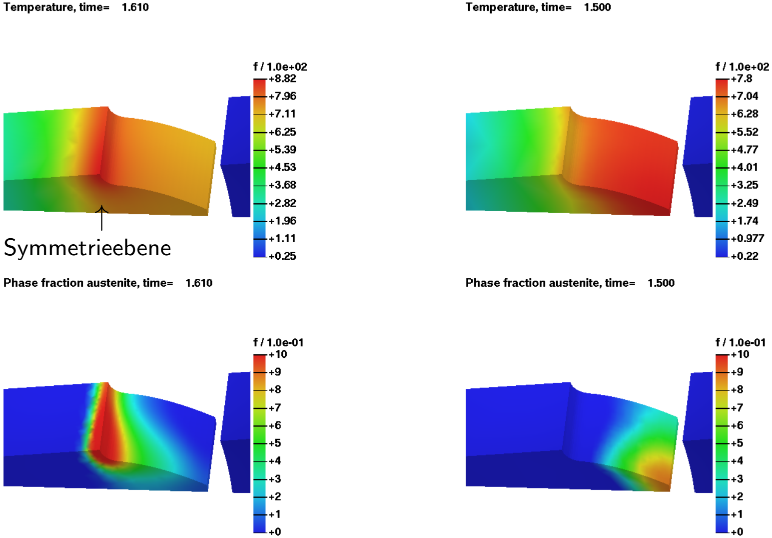


Figure : Temperature and austenite growth, $f = 10\ \mathrm{kHz}, 200\ \mathrm{kHz}$

Open problems

- The case of temperature-dependent electrical conductivity σ: in that case (cf. e.g. [Hömberg (2004)]) we get only existence of weak solutions $\longrightarrow$ parabolic equation for θ with L^1-r.h.s. because we cannot differentiate A-equation
- The case of magnetic permeability μ depending both on the temperature θ and on the magnetic field H
- Include Mechanical effects
- Numerical Optimal Control

Thanks for your attention!

cf. http://www.mat.unimi.it/users/rocca/

Avertisement:

- CIRM-ERC Workshop "DIMO-2013" Diffuse Interface Models, Levico Terme (Italy) September 10-13, 2013. Organized by Pierluigi Colli, E. R., Giulio Schimperna
- International School on "Recent advances in partial differential equations and applications", Milano (Italy) June 17 – 22, 2013. Coordinators: E. R. and Enrico Valdinoci
- Spring School on "Rate-independent evolutions and hysteresis modelling", Milano (Italy) May 27 – 31, 2013. Coordinators: Stefano Bosia, Michela Eleuteri, E. R. and Enrico Valdinoci.