MACH2019
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of degradation and restoration in Cultural Heritage
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Recent results on Additive Manufacturing Graded-material Design based on Phase-field and Topology Optimization

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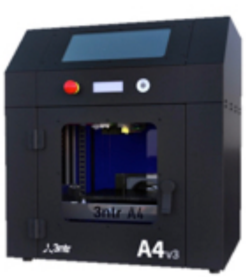
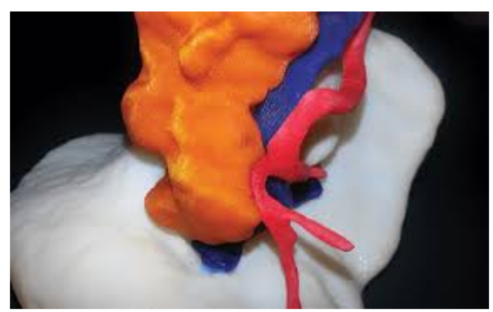


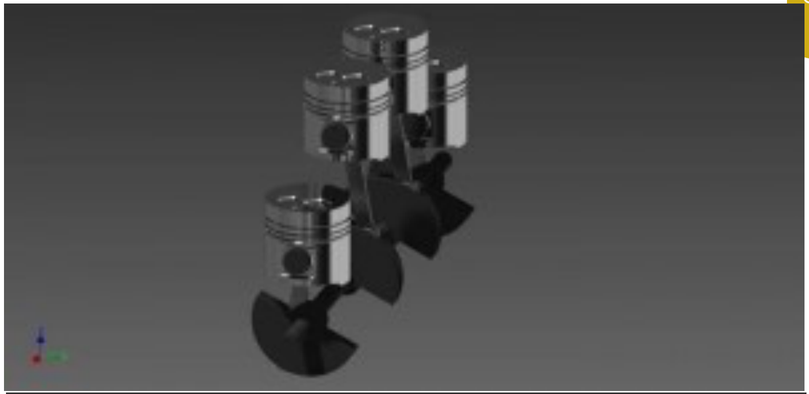



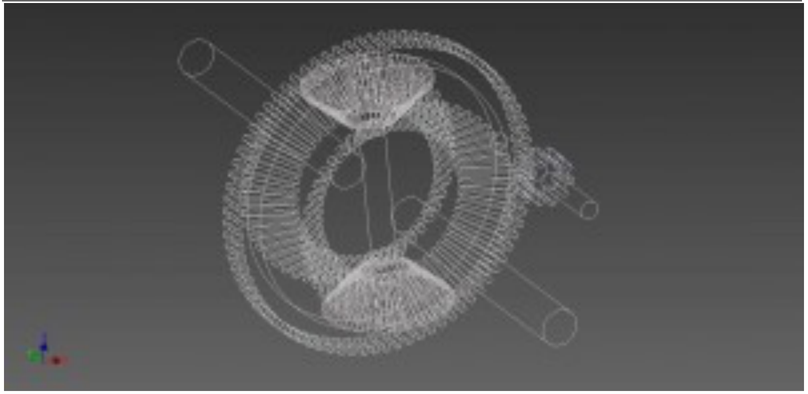

Additive Manufacturing

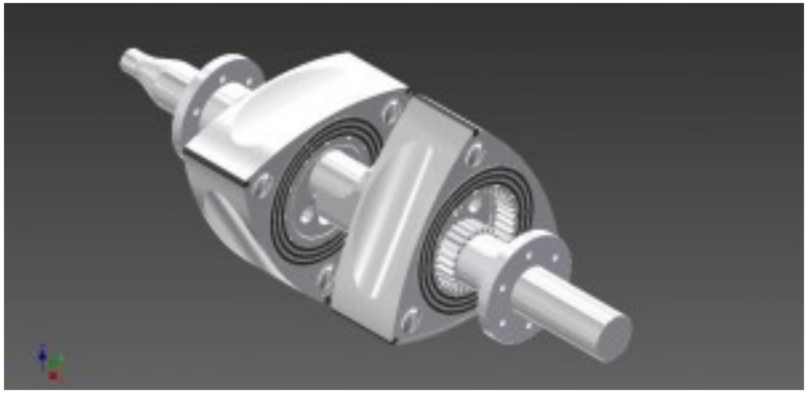
- ✓ AM is **deeply changing** paradigms in design and industrial production in comparison with more traditional technologies, like casting, stamping, and milling.

- ✓ Based on the fact that components or complete structures are constructed through sequences of **material layers deposition** and/or curing

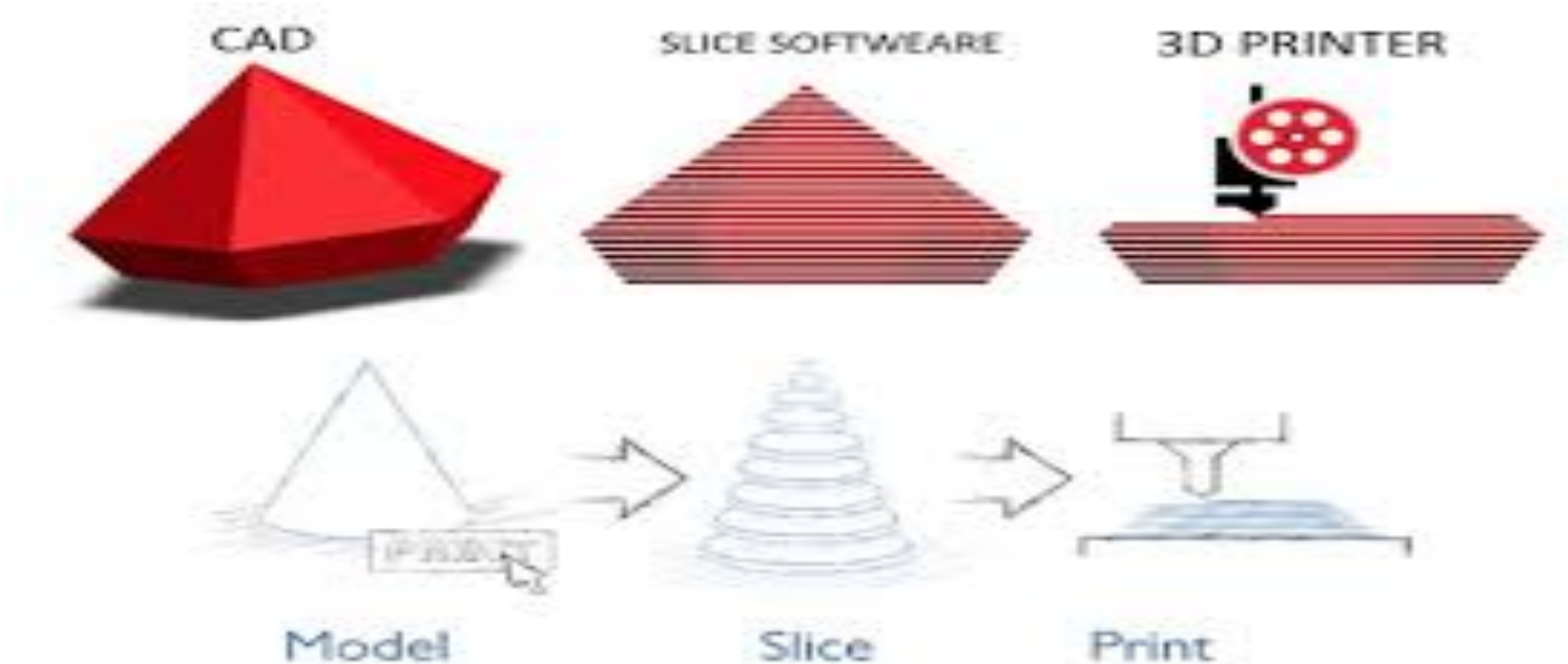
- ✓ Through deposition of fused material (FDM technology) or by melting/sintering of powders (SLS and SLM technologies).







Additive Manufacturing



- To create the object from 3D model, the corresponding STL file must be imported in a "**slicing**" software.
- Slicing software generate the 3D printer machine code, which contains the necessary instructions to make the object.
- Finally, the object can be subjected to post-processing operations, to remove any support structure and improve mechanical and chromatic features.

AM at large scales

- ✓ AM was used to design small objects, prototypes and in turn is now gradually being used for large-scale achievements, such as building houses or bridges or building restoration:

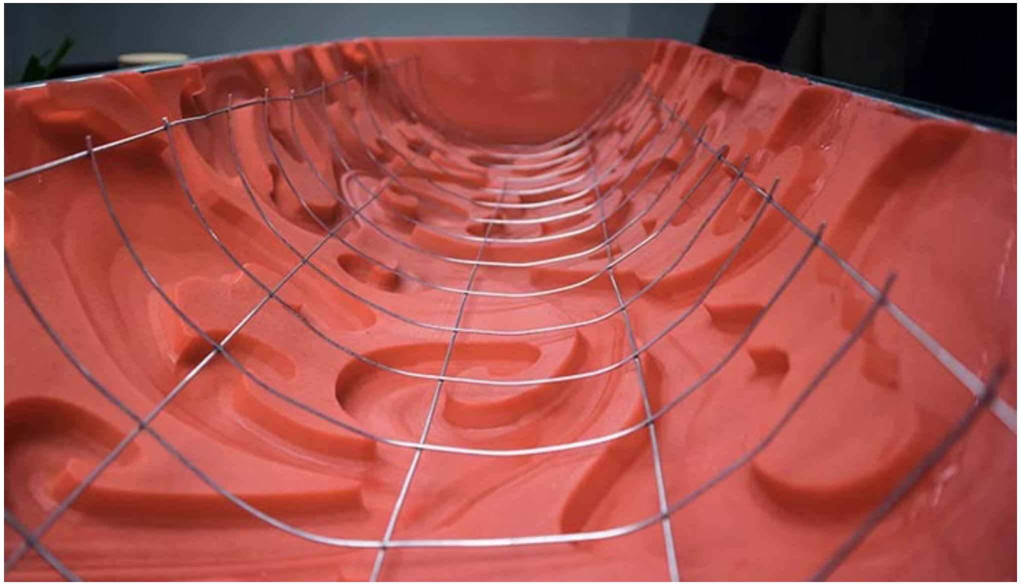
The first 3D printed pedestrian bridge in the world opened to the public on December 14 in Madrid

Additive manufacturing at the service of architecture in New York

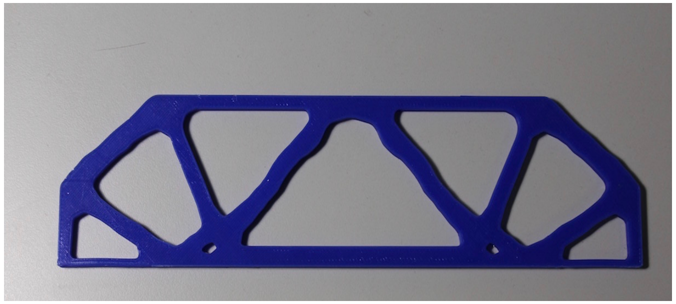
Restoration of buildings via 3D printing

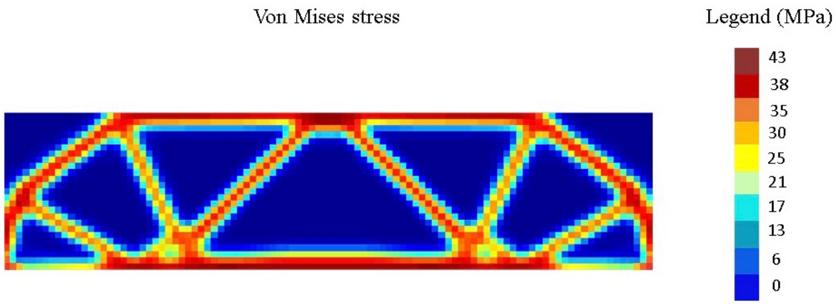
- ✓ A New York-based architecture firm EDG has developed a new method of digital sculpture, called “Modern Ornamental”, used to restore a building in New York City.
- ✓ This method relies on 3D printing to make complex molds to produce concrete structures.
- ✓ A cheaper, faster process that would better maintain a building.

Rather than creating solid 3D printed parts, which can often be expensive and time-consuming, the team created different molds to be filled with standard concrete.



Our main interest





- We focus on the problem of **structural optimization:** to find the best way to distribute a material in order to minimize an objective functional.

- The **shape of the domain is a-priori unknown**, while known quantities are the applied loads as well as regions where we want to have holes or material.

- Our main interest is to find regions which should be filled by material in order to optimize some properties of the sample.

- As the boundaries of these regions are unknown this is a free boundary problem we use here the **phase-field method**.

- This topic is of particular interest in the industrial field: used to predict and maximize the performance of a structure at the design stage.

Outline

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- Additive Manufacturing: single-material Design
- Additive Manufacturing: graded-material Design
- Phase-field method for the topology optimization problem
- Optimality conditions
- Numerical results

Phase-field and Topology optimization

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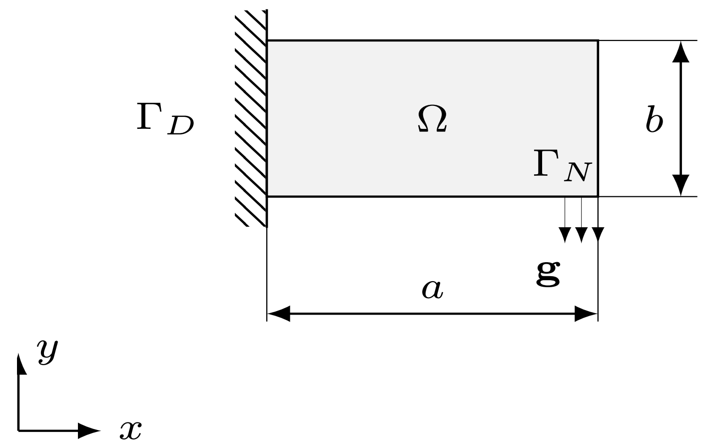
- Consider a domain, $\Omega \subset R^n$, with $n = 3$ or 2
- Denote by **u** the displacement and $\varepsilon(\mathbf{u})$ the symmetric strain
- Introduce a **scalar phase field variable** $\Phi \in [0,1]$ describing material presence
  - $\Phi \equiv 0$ corresponds to no material
  - $\Phi \equiv 1$ indicates material

$$
\begin{aligned}
-\mathrm{div}\,[\mathbb{C}\boldsymbol{\varepsilon}(\mathbf{u})] &= \mathbf{0} && \text{in } \Omega \\
\mathbf{u} &= \mathbf{0} && \text{on } \Gamma_D \\
[\mathbb{C}\boldsymbol{\varepsilon}(\mathbf{u})] &= \mathbf{g} && \text{on } \Gamma_N
\end{aligned}
$$

$$\mathbb{C}(\phi) = \mathbb{C}_A \phi^p + \mathbb{C}_B (1-\phi)^p$$



Single material: the functional

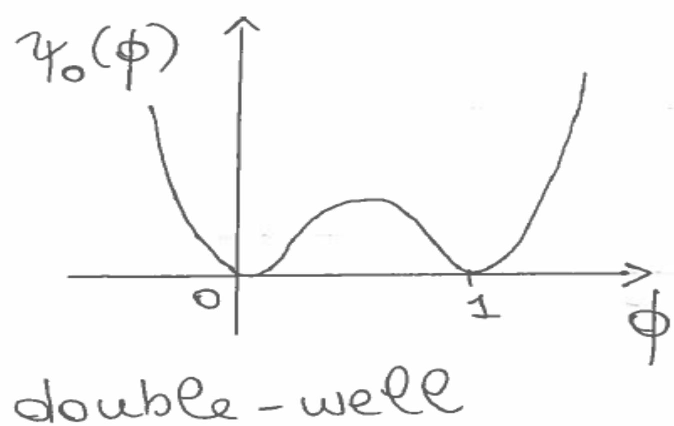
- **Optimized topology** aims at minimizing compliance, complemented with a measure of the perimeter which is here regularized by a Ginzburg-Landau type potential and subject to the volume constraint

$$\int_\Omega \phi \mathrm{d}\Omega = m \mid \Omega \mid$$

$$\mathcal{J}(\phi, \mathbf{u}(\phi)) = \int_{\Gamma_N} \mathbf{g} \cdot \mathbf{u}(\phi) \mathrm{d}\Gamma + \kappa \int_\Omega \left[\frac{\gamma}{2} \parallel \nabla \phi \parallel^2 + \frac{1}{\gamma} \psi_0(\phi) \right] \mathrm{d}\Omega$$

$$\psi_0(\phi) = (\phi - \phi^2)^2$$

ϒ represents the tickness of the interface between $\phi=0$ and $\phi=1$



Single Material: optimization problem

Problem $(\mathcal{P})$:

$$\min_{\phi} \quad \mathcal{J}(\phi, \mathbf{u}(\phi))$$

such that the following constraints are satisfied:

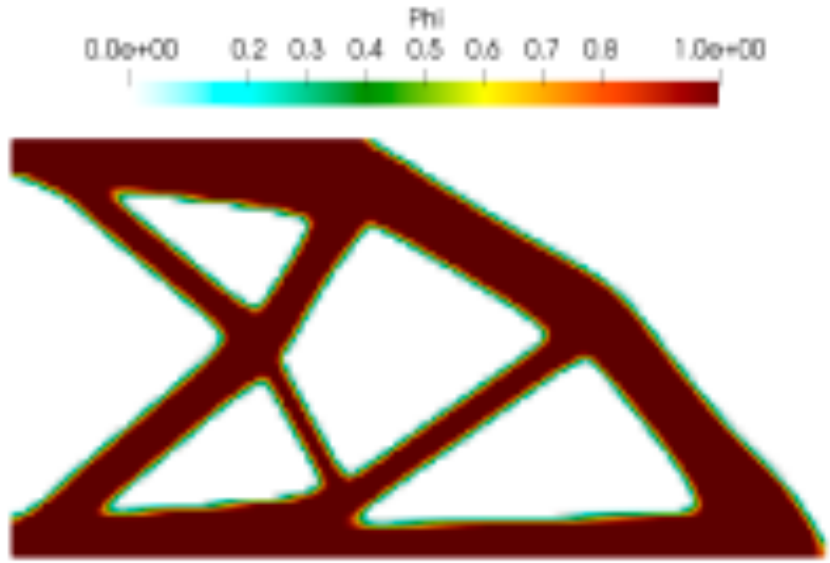
$$\int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{u}) : \mathbb{C}(\phi)\boldsymbol{\varepsilon}(\mathbf{v}) \mathrm{d}\Omega = \int_{\Gamma_N} \mathbf{g} \cdot \mathbf{v} \mathrm{d}\Gamma.$$

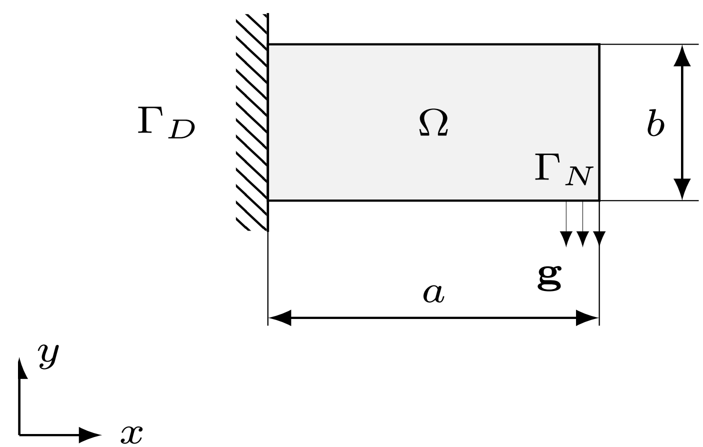
$$\mathcal{M}(\phi) = \int_{\Omega} \phi \mathrm{d}\Omega - m \mid \Omega \mid = 0,$$

with $\phi \in H^1(\Omega)$ satisfying the constraint:

$$0 \leq \phi \leq 1 \qquad \text{a.e. in } \Omega.$$

Single material: cantilever optimized beam





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The Graded Material Design

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- ✓ A single material is gradually distributed through the body.
- ✓ The result is a structure with with graded stiffness values alternating regions of soft material with other with stiffer material
- ✓ AM technologies allow us to grade density of a body almost in a continuous way varying the amount of distributed material point by point during the printing process.

Modeling of Graded material

- Collaboration with F. Auricchio, E. Bonetti, M. Carraturo, D. Hömberg, A.Reali

- **IDEA:** Introduce a new grading **scalar phase field variable**

$$\chi \in [0,\phi]$$

Material stiffness can continuously vary from a **stiff material $\chi = \phi$** to a soft material $\chi = 0$ and let the material tensor be:

$$\mathbb{C}(\phi,\chi) = \mathbb{C}(\chi)\phi^p + \gamma_\phi^2 \mathbb{C}(\chi)(1-\phi)^p,$$

$$\mathbb{C}(\chi) = \mathbb{C}_A \chi + \mathbb{C}_B(\phi - \chi),$$

where the soft material $\mathbb{C}_B$ is defined as:

$$\mathbb{C}_B = \frac{1}{\beta}\mathbb{C}_A,$$

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Optimization: graded material

The new graded-material optimization functional is

$$\mathcal{J}^M(\phi,\chi,\mathbf{u}(\phi,\chi)) = \int_{\Gamma_N} \mathbf{g}\cdot\mathbf{u}(\phi,\chi)\mathrm{d}\Gamma +$$

$$\kappa_\phi \int_\Omega \left[\frac{\gamma_\phi}{2} \mid \nabla\phi \mid^2 + \frac{1}{\gamma_\phi}\psi_0(\phi)\right]\mathrm{d}\Omega + \kappa_\chi \int_\Omega \frac{\gamma_\chi}{2} \mid \nabla\chi \mid^2 \mathrm{d}\Omega$$

- Minimization problem is solved employing Allen-Cahn gradient flow, i.e. a steepest descent pseudo-time stepping method, with a time-step increment

- Alternate solution of gradient flow and of equilibrium problem

The new optimization problem

$$\min_{\phi,\chi} \quad \mathcal{J}^M(\phi,\chi,\mathbf{u}(\phi,\chi)),$$

$$\int_\Omega \boldsymbol{\varepsilon}(\mathbf{u}) : \mathbb{C}(\phi,\chi)\boldsymbol{\varepsilon}(\mathbf{v})\mathrm{d}\Omega = \int_{\Gamma_N} \mathbf{g}\cdot\mathbf{v}\mathrm{d}\Gamma,$$

$$\mathcal{M}(\phi) = \int_\Omega \phi\mathrm{d}\Omega - m \mid \Omega \mid = 0,$$

where $\phi,\chi \in H^1(\Omega)$, under the constraint

$0 \leq \phi \leq 1 \quad$ a.e. in Ω, $\quad \boldsymbol{\Phi_{ad}}$

and the additional constraint on χ:

$0 \leq \chi \leq \phi \quad$ a.e. in Ω. $\quad \boldsymbol{\Xi_{ad}}$

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Lagrangean/optimality conditions

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$$\mathcal{L}^M(\phi, \chi, \mathbf{u}, \lambda, \mathbf{p}) =$$

$$\mathcal{J}^M(\phi, \chi, \mathbf{u}) + \lambda \mathcal{M}(\phi) + \mathcal{S}^M(\phi, \chi, \mathbf{u}, \mathbf{p}),$$

$$\mathcal{S}^M(\phi, \chi, \mathbf{u}, \mathbf{p}) = \int_\Omega \varepsilon(\mathbf{u}) : \mathbb{C}(\phi, \chi)\varepsilon(\mathbf{p}) \mathrm{d}\Omega - \int_{\Gamma_N} \mathbf{g} \cdot \mathbf{p} \mathrm{d}\Gamma.$$

$$D_\phi \mathcal{L}^M(\bar{\phi}, \bar{\chi}, \bar{\mathbf{u}}, \bar{\lambda}, \bar{\mathbf{p}}) (\phi - \bar{\phi}) \geq 0 \quad \forall \phi \in \Phi_{ad}$$

and $\bar{\mathbf{p}} = \bar{\mathbf{u}}.$

$$D_\chi \mathcal{L}^M (\bar{\phi}, \bar{\chi}, \bar{\mathbf{u}}, \bar{\lambda}, \bar{\mathbf{p}}) (\chi - \bar{\chi}) \geq 0 \quad \forall \chi \in \Xi_{ad},$$

Allen-Cahn gradient flow method

The optimal control problem can be solved as in the single-material case by means of the Allen-Cahn gradient flow, leading to the following set of equations:

$$\frac{\gamma_\phi}{\tau}\int_\Omega(\phi_{n+1}-\phi_n)v_\phi \mathrm{d}\Omega+\kappa_\phi\gamma_\phi\int_\Omega\nabla\phi\cdot\nabla v_\phi \mathrm{d}\Omega+ \int_\Omega v_\phi\lambda \mathrm{d}\Omega-\int_\Omega v_\phi\frac{\partial\mathcal{E}^M(\phi_n,\chi_n,\mathbf{u}_n)}{\partial\phi}\mathrm{d}\Omega \frac{\kappa_\phi}{\gamma_\phi}\int_\Omega\frac{\partial\psi_0(\phi_n)}{\partial\phi}v_\phi \mathrm{d}\Omega=0, \quad (21)$$

$$\frac{\gamma_\chi}{\tau}\int_\Omega(\chi_{n+1}-\chi_n)v_\chi \mathrm{d}\Omega+\kappa_\chi\gamma_\chi\int_\Omega\nabla\chi\cdot\nabla v_\chi \mathrm{d}\Omega- \int_\Omega v_\chi\frac{\partial\mathcal{E}^M(\phi_n,\chi_n,\mathbf{u}_n)}{\partial\chi}\mathrm{d}\Omega=0, \quad (22)$$

Under the volume constraint

$$\int_\Omega v_\lambda(\phi-m)\mathrm{d}\Omega=0.$$

$$m_\phi=m=\frac{1}{|\,\Omega\,|}\int_\Omega\phi \mathrm{d}\Omega.$$

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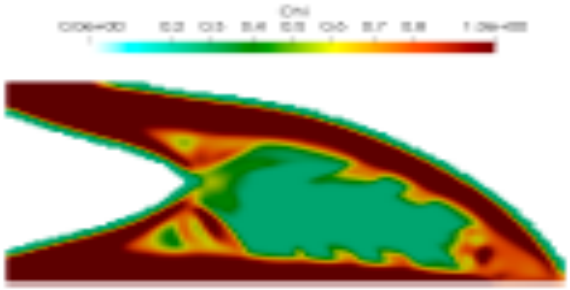
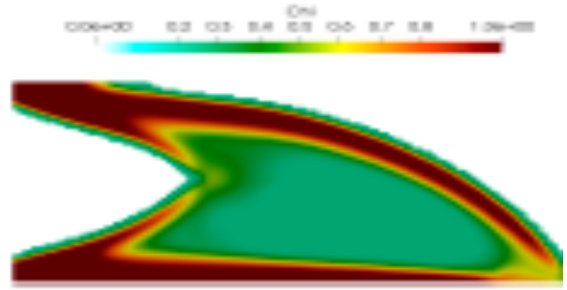
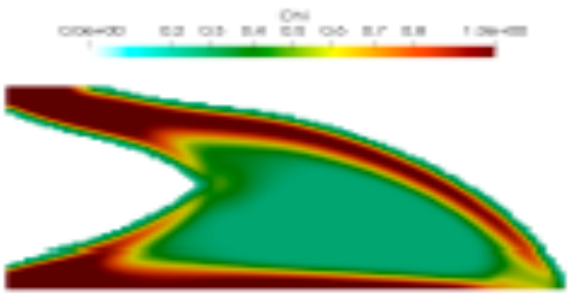
The cantilever beam



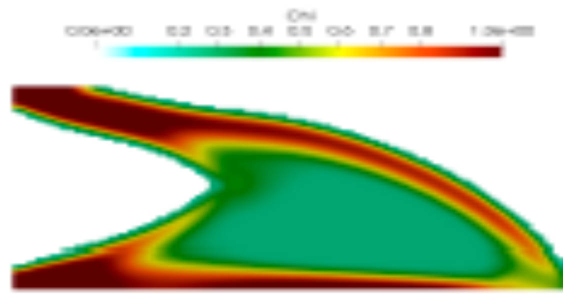
(a) $\gamma_\chi = 0.001$



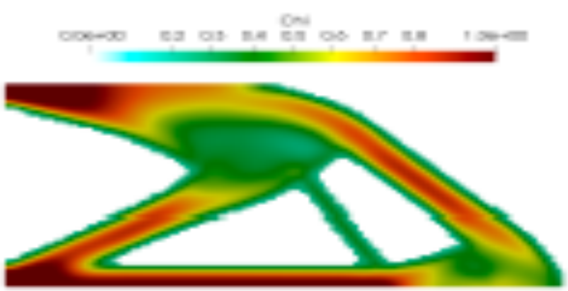
(b) $\gamma_\chi = 0.005$



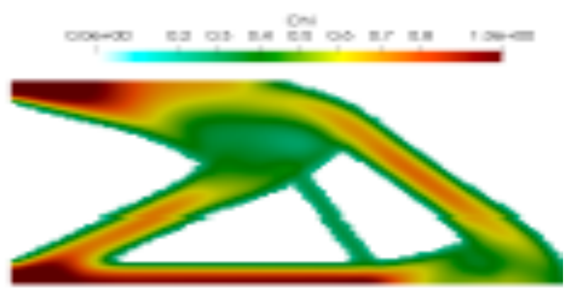
(c) $\gamma_\chi = 0.010$



(d) $\gamma_\chi = 0.020$



(e) $\gamma_\chi = 0.050$



(f) $\gamma_\chi = 0.100$

Simply supported beam: sensitivity w.r.t. the softening parameter

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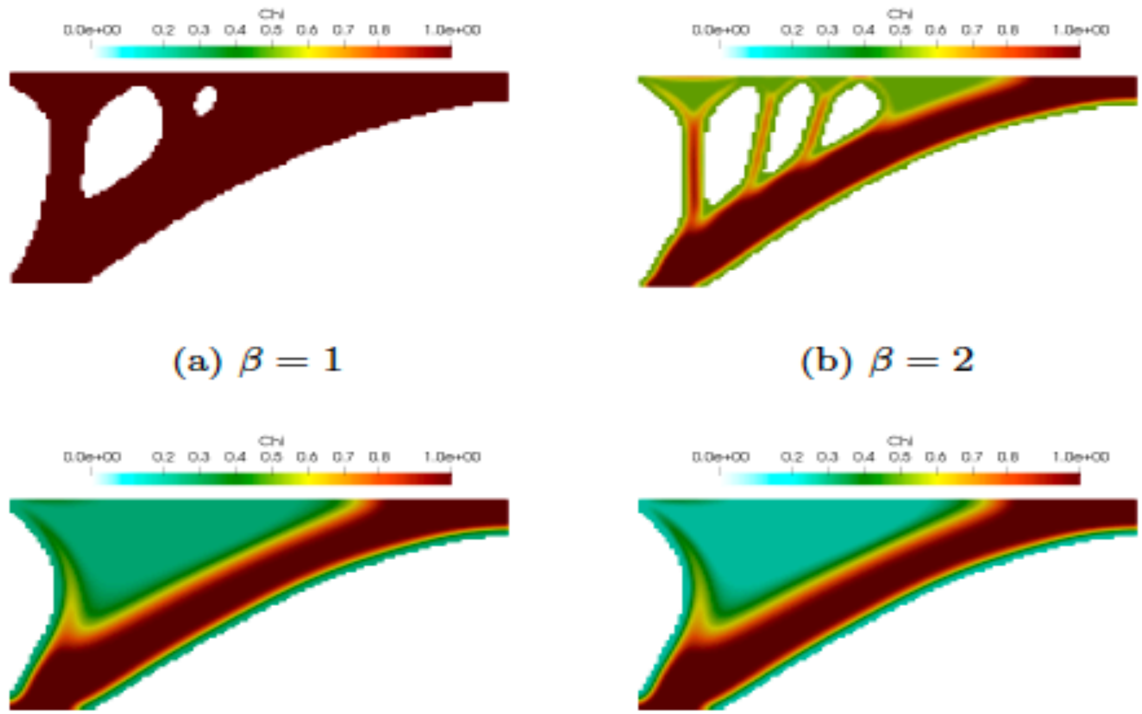


(a) $\beta = 1$ (b) $\beta = 2$

(c) $\beta = 3$ (d) $\beta = 4$

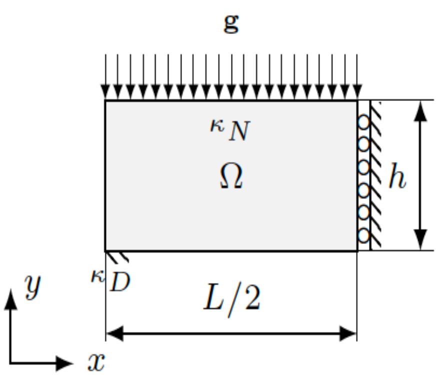
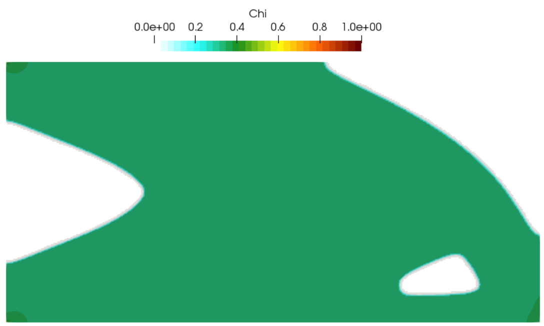
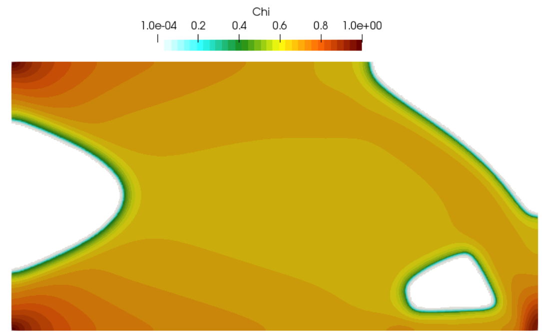


Fig. 6: Simply-supported beam: Sensitivity study of the softening factor β. Increasing the values of the softening factor, i.e., employing a softer material, the optimized structure does not present anymore the typical holes resulting from a single-material optimization 6a. Voids are now replaced by a region of soft material.

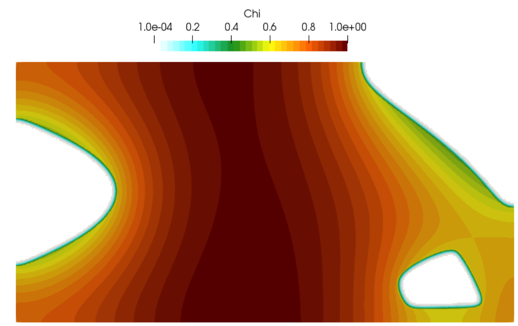
Optimization of a cantilever beam: sensitivity w.r.t. k_χ



k_χ=40



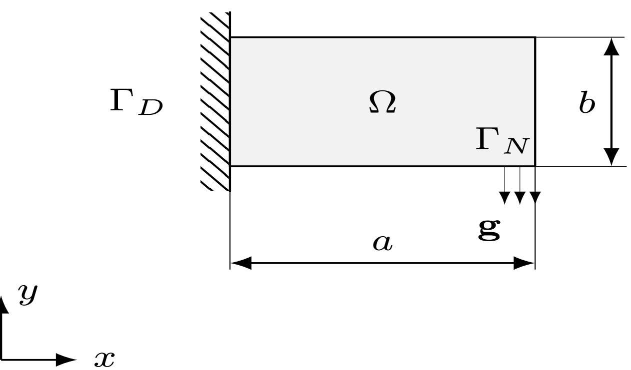
k_χ=4000



k_χ=400000

$$\mathcal{J}^M(\phi,\chi,\mathbf{u}(\phi,\chi)) = \int_{\Gamma_N} \mathbf{g}\cdot\mathbf{u}(\phi,\chi)\mathrm{d}\Gamma +$$

$$\kappa_\phi \int_\Omega \left[\frac{\gamma_\phi}{2} \mid \nabla\phi \mid^2 + \frac{1}{\gamma_\phi}\psi_0(\phi)\right]\mathrm{d}\Omega + \kappa_\chi \int_\Omega \frac{\gamma_\chi}{2} \mid \nabla\chi \mid^2 \mathrm{d}\Omega,$$



Compliance and material fraction index

Table 1: Cantilever beam: Sensitivity study of compliance and material fraction index m_χ for the parameter κ_2.

κ_2	compliance $\left[\frac{mm}{N}\right]$	m_χ	convergence
40	7325	0.241	NO
4000	4166	0.527	YES
400000	3762	0.673	YES
full dense material	3130	0.8	YES

$$m_\chi = \frac{1}{|\Omega|}\int_\Omega \chi d\Omega,$$

- ✓ The lowest compliance is achieved with the single stiffer material
- ✓ Using the graded material we can obtain FGM structure with relatively low compliance using considerably LESS MATERIAL

Printed cantilever beam: FDM 3D printer at the ProtoLab in Pavia

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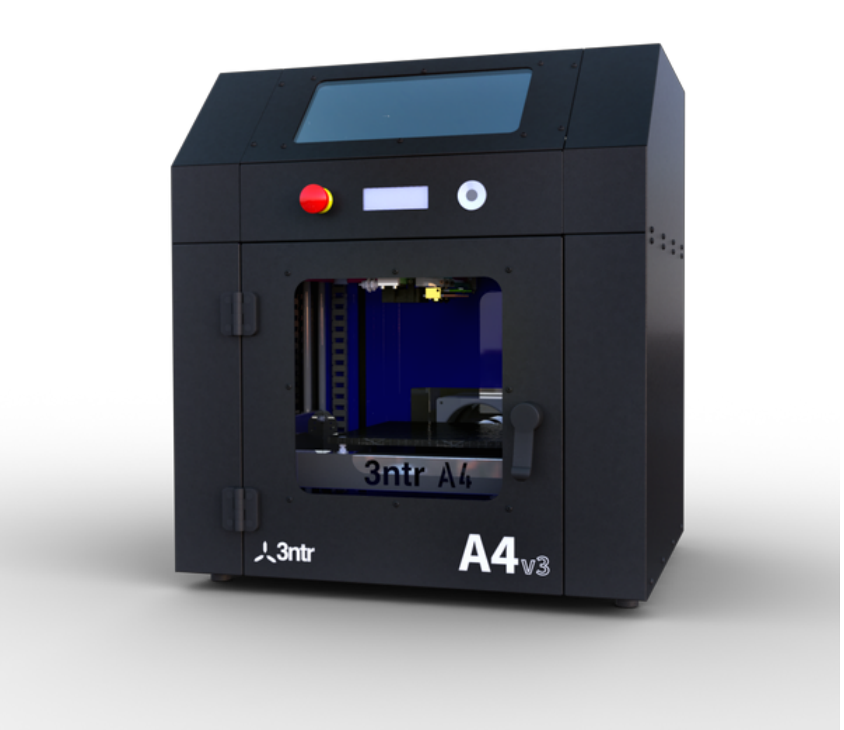
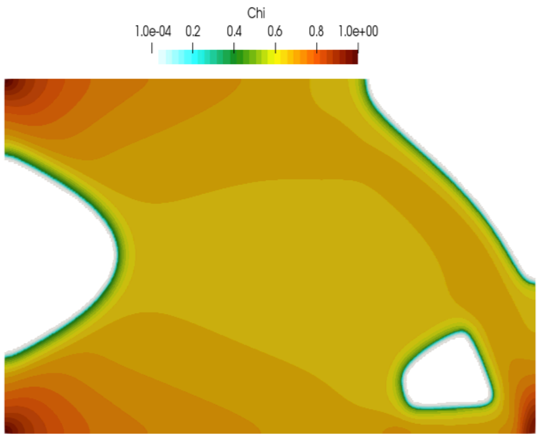
- ✓ This machine prints a filament of thermoplastic polymer which is first heated and then extruded through a printing nozzle
- ✓ Then it is deposited layer by layer until the desired object is obtained.



A 3D printing workflow for optimized FGM structures

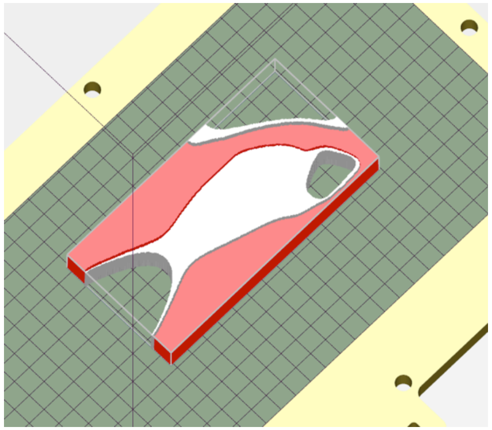


(a)

Extraction of STL files

(b)

Extrusion

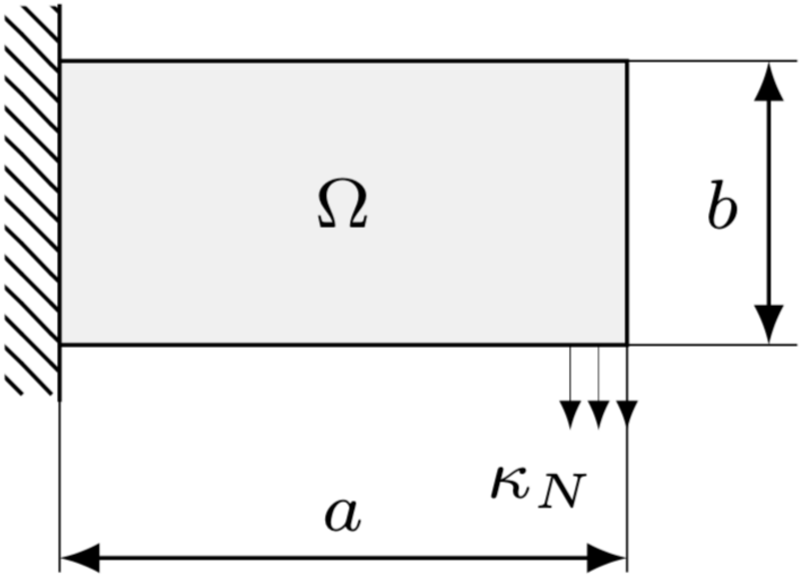


(c)

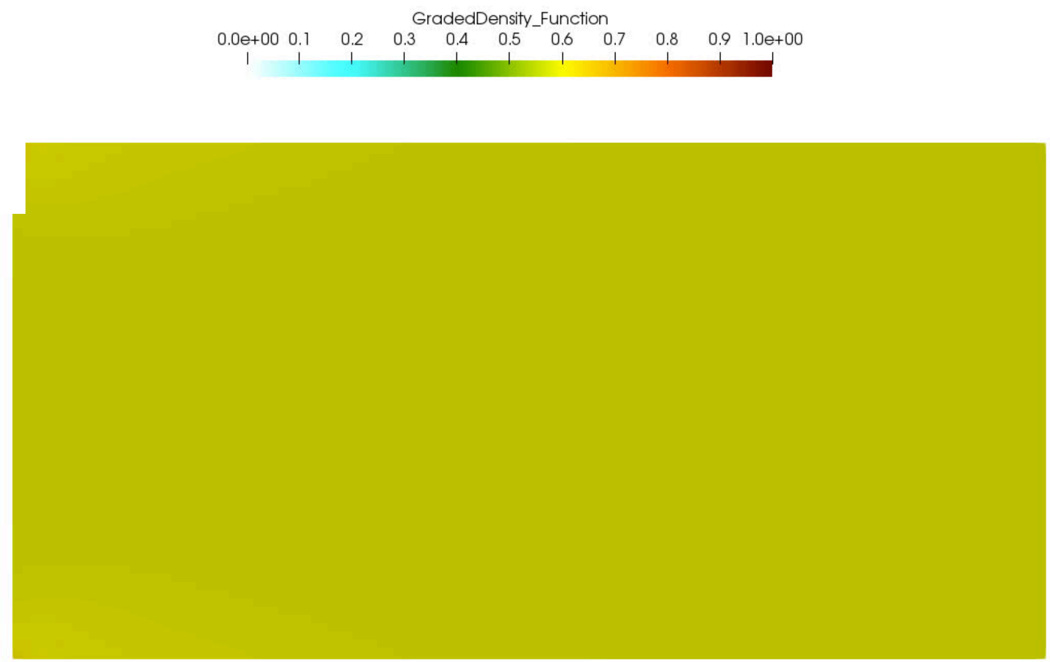
Printing

(d)

Optimization: graded material



Grey = no material
Color = material
Different color = different materials



OPEN PROBLEMS:

- **Introduce constraint on stress and displacement**
- **The stiffness is strongly influenced by the micro-structure of the partially filled regions**

The stress constraint

In the mathematical part of the paper we handle a more general functional including stress constraint:

$$(2.5)\quad \mathcal{J}(\mathbf{u},\boldsymbol{\sigma},\varphi,\chi) = \kappa_1 \int_\Omega \left(\frac{\mathcal{W}(\varphi)}{\gamma} + \gamma \frac{|\nabla\varphi|^2}{2} \right) \mathrm{d}x + \kappa_2 \int_\Omega \left(I_C(\varphi,\chi) + \frac{|\nabla\chi|^2}{2} \right) \mathrm{d}x + \kappa_3 \int_\Omega \varphi\,(\mathbf{f}\cdot\mathbf{u})\,\mathrm{d}x + \kappa_3 \int_{\Gamma_g} \mathbf{g}\cdot\mathbf{u}\,\mathrm{d}x + \kappa_5 \int_\Omega F(\boldsymbol{\sigma})\,\mathrm{d}x$$

over $(\mathbf{u},\boldsymbol{\sigma},\varphi,\chi) \in \mathcal{U}_{ad}$, and subject to the stress-strain state relation

$$(2.6)\quad -\operatorname{div}\boldsymbol{\sigma} = \varphi\,\mathbf{f} \quad \text{in } \Omega$$

$$(2.7)\quad \boldsymbol{\sigma}\cdot\mathbf{n} = \mathbf{g} \quad \text{on } \boldsymbol{\Gamma}_{\mathbf{g}}$$

$$(2.8)\quad \boldsymbol{\sigma} = \mathbb{K}(\varphi,\chi)\varepsilon(\mathbf{u}) \quad \text{in } \Omega$$

$$F(\boldsymbol{\sigma}) = \left(\Phi(\boldsymbol{\sigma})^2 - \Phi_{max}^2\right)_+$$

where $(\cdot)_+$ denotes the positive part function and we can choose, for example, $\Phi(\boldsymbol{\sigma}) = \sqrt{\frac{\sum_{i,j}(\lambda_i-\lambda_j)^2}{2}}$, being $\{\lambda_j\}$ the eigenvalues of the stress $\boldsymbol{\sigma}$

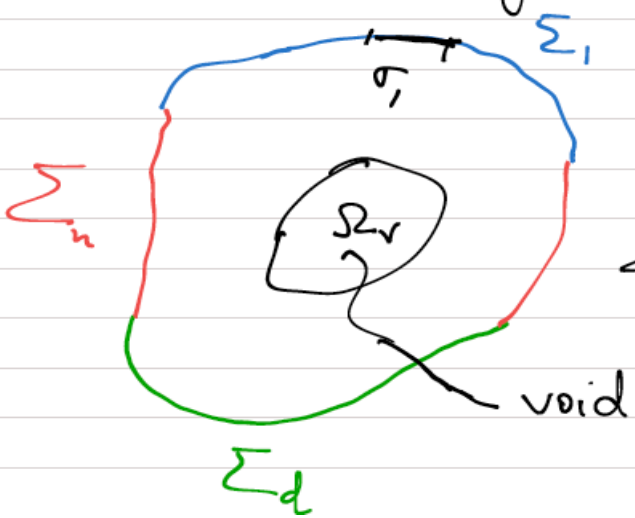
Open problem: include it in the simulations

A different approach: inverse problem

Joint with
F.Auricchio, E. Beretta,
C. Cavaterra, A. Reali

We start from the boundary value problem

$$
(1.1)\quad
\begin{cases}
\mathrm{div}(\mathbb{C}^M \hat{\nabla} u) = 0 & \text{in } \Omega_M \\
u = 0 & \text{on } \Sigma_d \\
(\mathbb{C}^M \hat{\nabla} u)\nu = 0 & \text{on } \Sigma_n \cup \Sigma \\
(\mathbb{C}^M \hat{\nabla} u)\nu = g & \text{on } \Sigma_1
\end{cases}
$$



where

$\Omega_M = \Omega \setminus \overline{\Omega_V}$

$\Sigma = \partial\Omega_V$

We know that there exists a unique variational solution

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Assume that we can measure the displacement $u|_{\sigma_1} = \bar{u}$ where $\sigma_1 \subset \Sigma_1$

goal

Minimize

$$F(u, \Omega_r) = k_1 \int_{\sigma_1} |u - \bar{u}|^2 ds + k_2 \, vol(\Omega_M)$$

$$+ k_3 \, Per(\Sigma) + k_4 \int_{\Sigma_1} g \cdot u$$

perimeter of Σ — compliance

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Let now for $\varphi \in H^1(\Omega, [0,1])$

$$\mathbb{C}_\varepsilon(\varphi) = \mathbb{C}^M \varphi + \varepsilon^2 \mathbb{C}^2 (1-\varphi)$$

(Setting $\mathbb{C}^M = \mathbb{C}^1$) and where $\mathbb{C}^2$ is an elast. tensor with same prop. as $\mathbb{C}^M$

Then,

$$\mathbb{C}_\varepsilon(1) = \mathbb{C}^M$$

$$\mathbb{C}_\varepsilon(0) = \varepsilon^2 \mathbb{C}^2$$

The new problem

Consider then $u_\varepsilon(\varphi) \in \overset{\cdot}{H}(\Omega)$ the unique weak solution to

$$
1) \begin{cases}
\operatorname{div}\left(\mathbb{C}_\varepsilon(\varphi) \hat{\nabla} u \right) = 0 & \text{in } \Omega \\
u = 0 & \text{on } \Sigma_d \\
\left(\mathbb{C}_\varepsilon(\varphi) \hat{\nabla} u \right) \nu = 0 & \text{on } \Sigma_n \\
\left(\mathbb{C}_\varepsilon(\varphi) \hat{\nabla} u \right) \nu = g & \text{on } \Sigma_1
\end{cases}
$$

The new minimization problem

$$\min_{\varphi \in H^1(\Omega, [0,1])} F_\varepsilon(u, \varphi) \qquad (u = u(\varphi))$$

where

$$F_\varepsilon = k_1 \int_{\sigma_1} |u - \bar{u}|^2 \, ds + k_2 \int_\Omega \varphi \, dx + k_3 \int_\Omega \varepsilon \frac{|\nabla \varphi|^2}{2} + \frac{\Psi(\varphi)}{\varepsilon} + k_4 \int_{\Sigma_1} g \cdot u$$

$$\Psi(\varphi) = \varphi(1 - \varphi)$$

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It is possible to get the optimality condition for F at fixed ε.

Open problems:

✓ Let ε go to zero: difficult problem

✓ Cavity reconstruction (work in progress with F. Auricchio and Alexander Viguerie)

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Thank you for the attention!

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