# A degenerating PDE system for phase transitions and damage: global existence of weak solutions

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Variational Models and Methods for Evolution

Levico Terme (Italy), September 10-12, 2012

joint work with Riccarda Rossi (University of Brescia)

Supported by the FP7-IDEAS-ERC-StG Grant "EntroPhase" #256872

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- The potential future perspectives: to apply the *entropic formulation* to damage phenomena

# Mathematical problem arising from Thermomechanics

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**Damage** phenomena:

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**Damage** phenomena:

- **aim:** deal with diffuse interface models in thermoviscoelasticity accounting for
  - the absolute temperature $\theta$
  - the evolution of the displacement variables $\mathbf{u}$
  - the damage parameter $\chi$

  where the internal energy balance display nonlinear dissipation and the momentum equation contains $\chi$-dependent elliptic operators, which may degenerate at the *pure phases*

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\begin{aligned}
&c(\theta)\theta_t + \chi_t\theta - \rho\theta \operatorname{div} \mathbf{u}_t - \operatorname{div}(k(\theta)\nabla\theta)) = g + \chi|\varepsilon(\mathbf{u}_t)|^2 + |\chi_t|^2 \\
&\mathbf{u}_{tt} - \operatorname{div}(\chi\varepsilon(\mathbf{u}_t) + \chi\varepsilon(\mathbf{u}) - \rho\theta\mathbf{1}) = \mathbf{f} \\
&\chi_t + \partial I_{(-\infty,0]}(\chi_t) + A_s\chi + W'(\chi) \ni -\frac{|\varepsilon(\mathbf{u})|^2}{2} + \theta
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2. a *generalization of the principle of virtual powers* inspired by:
   - 2.1. the notion of *energetic solution* - A. Mielke and co-authors ([Bouchitté, Mielke, Roubíček, ZAMP. Angew. Math. Phys. (2009) and [Mielke, Roubíček, Zeman, Comput. Methods Appl. Mech. Engrg. (2010)]) for rate-independent processes for damage phenomena and
   - 2.2. a notion of *weak solution* introduced by [Heinemann, Kraus, WIAS preprint 1569 and WIAS preprint 1520, to appear on Adv. Math. Sci. Appl. (2010)] for non-degenerating isothermal diffuse interface models for phase separation and damage

## Entropic formulation: a phase transitions model

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We consider there a model for solid-liquid phase transitions associated to a nonlinear PDE system

$$\theta_t + \chi_t\theta - \Delta\theta = |\chi_t|^2$$
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**$\Longrightarrow$ a new notion of solution is needed**

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Finally, couple these relations to a suitable phase dynamics.

## The entropy production

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$$\int_0^T \int_\Omega s_t \varphi - \int_0^T \int_\Omega \frac{\mathbf{q}}{\theta} \cdot \nabla\varphi = \int_0^T \int_\Omega r\varphi \quad \forall \varphi \in \mathcal{D}(\overline{Q}_T),$$

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$r$ represents the entropy production rate. Then, in order to comply with the Clausius-Duhem inequality, we assume:

(i) $r$ is a nonnegative measure on $[0,T] \times \overline{\Omega} =: \overline{Q}_T$;

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Taking $\mathbf{q} = -\nabla \theta$, $s = \log \theta + \chi$, we get

$$\int_0^T \int_\Omega \Big( (\log \theta + \chi)\, \partial_t \varphi - \nabla \log \theta \cdot \nabla \varphi \Big) dxdt$$
$$\leq \int_0^T \int_\Omega \frac{1}{\theta} \left( -|\chi_t|^2 - \nabla \log \theta \cdot \nabla \theta \right) \varphi dxdt$$

for every test function $\varphi \in \mathcal{D}(\overline{Q}_T)$, $\varphi \geq 0$

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for every test function $\varphi \in \mathcal{D}(\overline{Q}_T)$, $\varphi \geq 0$

$\Rightarrow$ the total entropy is controlled by dissipation.

# The energy conservation and phase relation

The total energy has to be preserved. Hence

$$E(t) = E(0) \text{ for a.e. } t \in [0, T],$$

where

$$E \equiv \int_\Omega \left( \theta + W(\chi) + \frac{|\nabla \chi|^2}{2} \right) dx\,.$$

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Finally, the phase dynamics results as

$$\chi_t - \Delta\chi + W'(\chi) = \theta - \theta_c \quad \text{a.e. in } \Omega \times (0, T),$$

where $W$ is a double well or double obstacle potential: $W = \widehat{\beta} + \widehat{\gamma}$ where

$\widehat{\beta} : \mathbb{R} \to [0, +\infty]$ is proper, lower semi-continuous, convex function

$\widehat{\gamma} \in C^2(\mathbb{R})$, $\widehat{\gamma}' \in C^{0,1}(\mathbb{R})$ : $\widehat{\gamma}''(r) \geq -K$ for all $r \in \mathbb{R}$, $W(r) \geq c_w r^2$ for all $r \in \mathrm{dom}(\widehat{\beta})$

Examples: $\widehat{\beta}(r) = r\ln(r) + (1-r)\ln(1-r)$ or $\widehat{\beta}(r) = I_{[0,1]}(r)$.

## The existence theorem [E. Feireisl, H. Petzeltová, E.R., M2AS (2009)]

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$$\theta\in L^\infty(0,T;L^1(\Omega))\cap L^s(Q_T),\quad \theta(x,t)>0\quad \text{a. e. in } Q_T$$
$$\log(\theta)\in L^\infty(0,T;L^1(\Omega))\cap L^2(0,T;H^1(\Omega))\cap W^{1,1}(0,T;W^{-2,3/2}(\Omega))$$
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and *the total energy conservation*

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- However, in this case and similarly in many other situations, to prove that the solution has this extra regularity is **out of reach**
- **It can be suitable also in different applications** such as the ones related to phase transitions in viscoelastic materials, SMA, liquid crystal flows, etc.

## The generalized principle of virtual powers: damage phenomena

# The generalized principle of virtual powers in damage phenomena

The scope: The analysis of the initial boundary-value problem for the following PDE system:

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- $\theta$ is the absolute temperature of the system
- $\mathbf{u}$ the vector of *small displacements*
- $\chi$ is the damage parameter, assessing the soundness of the material in *damage* (for the completely *damaged* $\chi = 0$ and the *undamaged* state $\chi = 1$, respectively, while $0 < \chi < 1$: *partial damage*)

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$\Longrightarrow$ concentrate first on **degeneracy**

## The aim: deal with the possible degeneracy in the momentum equation

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$\Longrightarrow$ We shall approximate the system with a non-degenerating one, where we replace the momentum equation with

$$\mathbf{u}_{tt} - \mathrm{div}((\chi + \delta)\varepsilon(\mathbf{u}_t) + (\chi + \delta)\varepsilon(\mathbf{u})) = \mathbf{f} \qquad \text{for } \delta > 0$$

It seems to us that *both* the coefficients need to be truncated when taking the degenerate limit in the momentum equation. Indeed, on the one hand the truncation in front of $\varepsilon(\mathbf{u}_t)$ allows us to deal with the *main part* of the elliptic operator. On the other hand, in order to pass to the limit in the quadratic term on the right-hand side of $\chi$-eq., we will also need to truncate the coefficient of $\varepsilon(\mathbf{u})$.

**Cf.** [M. Frémond, Phase Change in Mechanics, Lecture Notes of UMI, Springer-Verlag, 2012]

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**The free-energy $\mathcal{F}$:**

$$\mathcal{F} = \int_\Omega \Big( f(\theta) + \chi \frac{|\varepsilon(\mathbf{u})|^2}{2} + \frac{a_s(\chi,\chi)}{2} + W(\chi) - \theta\chi \Big)\, \mathrm{d}x$$

- $f$ is a concave function
- $a_s(z_1, z_2) := \displaystyle\int_\Omega \int_\Omega \frac{(\nabla z_1(x) - \nabla z_1(y)) \cdot (\nabla z_2(x) - \nabla z_2(y))}{|x-y|^{d+2(s-1)}}\, \mathrm{d}x\, \mathrm{d}y$ is the bilinear form associated to the fractional $s-$Laplacian $A_s$
- $s > d/2$: we need the embedding of $H^s(\Omega)$ into $C^0(\overline{\Omega})$
- $W = \widehat{\beta} + \widehat{\gamma}$, $\widehat{\gamma} \in C^2(\mathbb{R})$, $\widehat{\beta}$ proper, convex, l.s.c., $\overline{\mathrm{dom}(\widehat{\beta})} = [0,1]$
- we could include the thermal expansion term $\rho\theta \mathrm{tr}(\varepsilon(\mathbf{u}))$ (neglect it in this presentation)

**Cf.** [M. Frémond, Phase Change in Mechanics, Lecture Notes of UMI, Springer-Verlag, 2012]

**The free-energy $\mathcal{F}$:**

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**The pseudo-potential $\mathcal{P}$:**

$$\mathcal{P} = \frac{k(\theta)}{2} |\nabla\theta|^2 + \frac{1}{2}|\chi_t|^2 + \chi \frac{|\varepsilon(\mathbf{u}_t)|^2}{2} + I_{(-\infty,0]}(\chi_t)$$

- $k$ the heat conductivity: coupled conditions with the specific heat $c(\theta) = f(\theta) - \theta f'(\theta)$
- $I_{(-\infty,0]}(\chi_t) = 0$ if $\chi_t \in (-\infty, 0]$, $I_{(-\infty,0]}(\chi_t) = +\infty$ otherwise

# The modelling

The momentum equation

$$\mathbf{u}_{tt} - \operatorname{div}\sigma = \mathbf{f} \quad \left(\sigma = \sigma^{nd} + \sigma^{d} = \frac{\partial \mathcal{F}}{\partial \varepsilon(\mathbf{u})} + \frac{\partial \mathcal{P}}{\partial \varepsilon(\mathbf{u}_t)}\right) \quad \text{becomes}$$

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$$B - \operatorname{div}\mathbf{H} = 0 \quad \left(B = \frac{\partial \mathcal{F}}{\partial \chi} + \frac{\partial \mathcal{P}}{\partial \chi_t}, \mathbf{H} = \frac{\partial \mathcal{F}}{\partial \nabla \chi}\right) \quad \text{becomes}$$

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The internal energy balance

$$e_t + \operatorname{div}\mathbf{q} = g + \sigma : \varepsilon(\mathbf{u}_t) + B\chi_t + \mathbf{H}\cdot\nabla\chi_t \quad \left(e = \mathcal{F} - \theta\frac{\partial \mathcal{F}}{\partial \theta}, \quad \mathbf{q} = \frac{\partial \mathcal{P}}{\partial \nabla \theta}\right)$$

becomes

$$\boxed{c(\theta)\theta_t + \chi_t\theta - \operatorname{div}(k(\theta)\nabla\theta) = g + |\chi_t|^2 + \chi|\varepsilon(\mathbf{u}_t)|^2}$$

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- For the analysis of the degenerate limit $\delta \searrow 0$ we have carefully adapted techniques from [Bouchitté, Mielke, Roubíček, ZAMP (2009)] and [Mielke, Roubíček, Zeman, Comput. Methods Appl. Mech. Engrg. (2011)] to the case of a *rate-dependent* equation for $\chi$, also coupled with the temperature equation.

## Energy vs Enthalpy

In order to deal with the low regularity of $\theta$, rewrite the internal energy equation

$$c(\theta)\theta_t + \chi_t\theta - \mathrm{div}(k(\theta)\nabla\theta) = g$$

as the enthalpy equation

$$w_t + \chi_t\Theta(w) - \mathrm{div}(K(w)\nabla w) = g \quad \text{where}$$

$$w = h(\theta) := \int_0^\theta c(s)\,\mathrm{d}s, \quad \Theta(w) := \begin{cases} h^{-1}(w) & \text{if } w \geq 0, \\ 0 & \text{if } w < 0, \end{cases} \quad K(w) := \frac{k(\Theta(w))}{c(\Theta(w))}$$

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We assume that

- $\mathrm{c} \in C^0([0,+\infty);[0,+\infty))$
- $\exists\, \sigma_1 \geq \sigma > \frac{2d}{d+2}$ : $\quad c_0(1+\theta)^{\sigma-1} \leq \mathrm{c}(\theta) \leq c_1(1+\theta)^{\sigma_1-1} \Longrightarrow h$ is strictly increasing
- the function $k : [0,+\infty) \to [0,+\infty)$ is continuous, and

$$\exists c_2,\, c_3 > 0 \ \ \forall\theta \in [0,+\infty) : \quad c_2\mathrm{c}(\theta) \leq k(\theta) \leq c_3(\mathrm{c}(\theta)+1)$$

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$\Longrightarrow \exists\, \bar{c} > 0 \ \ \forall\, w \in \mathbb{R} : \ c_2 \leq K(w) \leq \bar{c}$

$\Longrightarrow$ for every $s \in (1,\infty)\ \exists\, C_s > 0 \ \ \forall\, w \in L^1(\Omega) : \quad \|\Theta(w)\|_{L^s(\Omega)} \leq C_s(\|w\|_{L^{s/\sigma}(\Omega)}^{1/\sigma} + 1)$

# The approximating non-degenerate Problem $[\mathrm{P}_\delta]$

Given $\delta > 0$, take $W' = \partial I_{[0,+\infty)} + \gamma$, $\gamma \in C^1(\mathbb{R})$, find (measurable) functions

$$w \in L^r(0,T;W^{1,r}(\Omega)) \cap L^\infty(0,T;L^1(\Omega)) \cap \mathrm{BV}([0,T];W^{1,r'}(\Omega)^*)$$
$$\mathbf{u} \in H^1(0,T;H^2(\Omega;\mathbb{R}^d)) \cap W^{1,\infty}(0,T;H^1_0(\Omega)) \cap H^2(0,T;L^2(\Omega;\mathbb{R}^d))$$
$$\chi \in L^\infty(0,T;H^s(\Omega)) \cap H^1(0,T;L^2(\Omega))$$

for every $1 \le r < \frac{d+2}{d+1}$, fulfilling the initial conditions

$$\mathbf{u}(0,x) = \mathbf{u}_0(x), \quad \mathbf{u}_t(0,x) = \mathbf{v}_0(x) \qquad \text{for a.e. } x \in \Omega$$
$$\chi(0,x) = \chi_0(x) \qquad \text{for a.e. } x \in \Omega$$

the equations (for every $\varphi \in \mathrm{C}^0([0,T];W^{1,r'}(\Omega)) \cap W^{1,r'}(0,T;L^{r'}(\Omega))$ and $t \in (0,T]$)

$$\int_\Omega \varphi(t)\, w(t)(\mathrm{d}x) - \int_0^t \int_\Omega w\varphi_t \,\mathrm{d}x + \int_0^t \int_\Omega \chi_t \Theta(w)\varphi \,\mathrm{d}x + \int_0^t \int_\Omega K(w)\nabla w \nabla\varphi \,\mathrm{d}x$$
$$= \int_0^t \int_\Omega g\varphi + \int_\Omega w_0\varphi(0)\,\mathrm{d}x$$

$$\mathbf{u}_{tt} - \mathrm{div}\left((\chi+\delta)\varepsilon(\mathbf{u}_t) + (\chi+\delta)\varepsilon(\mathbf{u})\right) = \mathbf{f} \text{ in } H^{-1}(\Omega;\mathbb{R}^d) \text{ a.e. in } (0,T)$$

and the subdifferential inclusion "in a suitable sense"

$$\chi_t + \partial I_{(-\infty,0]}(\chi_t) + A_s\chi + \partial I_{[0,+\infty)}(\chi) + \gamma(\chi) \ni -\frac{|\varepsilon(\mathbf{u})|^2}{2} + \Theta(w) \quad \text{in } H^{-s}(\Omega) \text{ and a.e. in } (0,T)$$

## Generalized principle of virtual powers for $\delta > 0$ [E.R., R. Rossi, preprint arXiv:1205.3578v1 (2012)]

**[Theorem 1] ($\delta > 0$)** Under the previous assumptions on the data, then,

[1.] Problem $[\mathrm{P}_\delta]$ admits a *weak solution* $(w, \mathbf{u}, \chi)$, which, beside fulfilling the enthalpy and momentum equations, satisfies $\chi_t(x,t) \leq 0$ for almost all $t \in (0,T)$, and $(\forall \varphi \in L^2(0,T; H^s_+(\Omega)) \cap L^\infty(Q))$ the *one-sided* inequality

$$\int_0^T \int_\Omega \chi_t \varphi + a_s(\chi, \varphi) + \xi\varphi + \gamma(\chi)\varphi + \frac{|\varepsilon(\mathbf{u})|^2}{2}\varphi - \Theta(w)\varphi \leq 0$$

with $\xi \in \partial I_{[0,+\infty)}(\chi)$ in the following sense:

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and the energy inequality for all $t \in (0,T]$, for $s = 0$, and for almost all $0 < s \le t$:

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$$\le \frac{1}{2} a_s(\chi(s), \chi(s)) + \int_\Omega W(\chi(s))\,\mathrm{d}x + \int_s^t \int_\Omega \chi_t \left( -\frac{|\varepsilon(\mathbf{u})|^2}{2} + \Theta(w) \right) \mathrm{d}x\,\mathrm{d}r$$

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Uniqueness of solutions for the irreversible system, even in the isothermal case, is still an open problem. This is mainly due to the doubly nonlinear character of the $\chi$ equation.

# Generalized principle of virtual powers vs classical phase inclusion

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- Any *weak solution* $(w, \mathbf{u}, \chi)$ fulfills the total energy inequality for all $t \in (0, T]$, for $s = 0$, and for almost all $0 < s \leq t$

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$$+ \frac{1}{2}(\chi(t)+\delta)|\varepsilon(\mathbf{u}(t))|^2 + \frac{1}{2}a_s(\chi(t),\chi(t)) + \int_\Omega W(\chi(t))\,\mathrm{d}x$$
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- If $(w, \mathbf{u}, \chi)$ are "more regular" and satisfy the notion of *weak solution*, then, differentiating the energy inequality and using the chain rule, we conclude that $(w, \mathbf{u}, \chi, \xi)$ comply with

$$\left\langle \chi_t(t) + A_s(\chi(t)) + \xi(t) + \gamma(\chi(t)) + \frac{|\varepsilon(\mathbf{u})|^2}{2} - \Theta(w(t)), \chi_t(t) \right\rangle_{H^s(\Omega)} \leq 0 \text{ for a.e.} t$$

Using the *one-sided* inequality we obtain the classical phase inclusion:

$$\exists \zeta \in L^2(0,T;L^2(\Omega)) \text{ with } \zeta(x,t) \in \partial I_{(-\infty,0]}(\chi_t(x,t)) \text{ a.e. s.t.}$$
$$\chi_t + \zeta + A_s\chi + \xi + \gamma(\chi) = -\frac{|\varepsilon(\mathbf{u})|^2}{2} + \Theta(w) \text{ a.e.}$$

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- The presence of the $s$-Laplacian with $s > d/2 \Longrightarrow$ an estimate for $\chi$ in $L^\infty(0, T; H^s(\Omega))$ (from the total energy balance) $\Longrightarrow$ a suitable regularity estimate on the displacement variable $\mathbf{u} \Longrightarrow$ an $L^\infty(0, T; L^2(\Omega))$-bound on the quadratic nonlinearity $|\varepsilon(\mathbf{u})|^2$ on the right-hand side of

$$\chi_t + \partial I_{(-\infty,0]}(\chi_t) + A_s\chi + W'(\chi) \ni -\frac{|\varepsilon(\mathbf{u})|^2}{2} + \Theta(w)$$

## The techniques used in the proof of Thm. 1 ($\delta > 0$)

- We pass to the limit in a carefully designed time-discretization scheme
- The presence of the $s$-Laplacian with $s > d/2 \Longrightarrow$ an estimate for $\chi$ in $L^\infty(0,T;H^s(\Omega))$ (from the total energy balance) $\Longrightarrow$ a suitable regularity estimate on the displacement variable $\mathbf{u} \Longrightarrow$ an $L^\infty(0,T;L^2(\Omega))$-bound on the quadratic nonlinearity $|\varepsilon(\mathbf{u})|^2$ on the right-hand side of

$$\chi_t + \partial I_{(-\infty,0]}(\chi_t) + A_s\chi + W'(\chi) \ni -\frac{|\varepsilon(\mathbf{u})|^2}{2} + \Theta(w)$$

- A Boccardo-Gallouët-type estimates combined with the Gagliardo-Nirenberg inequality applied to the enthalpy equation in order to obtain an $L^r(0,T;W^{1,r}(\Omega))$-estimate on the enthalpy $w$ (and hence on $\Theta(w)$)

$$w_t + \chi_t\Theta(w) - \mathrm{div}(K(w)\nabla w) = g$$

# The total energy inequality in the degenerating case $\delta \searrow 0$

Rewrite the momentum equation

$$\partial_t^2 \mathbf{u}_\delta - \operatorname{div}((\chi + \delta)\varepsilon(\partial_t \mathbf{u}_\delta)) - \operatorname{div}((\chi + \delta)\varepsilon(\mathbf{u}_\delta)) = \mathbf{f}$$

using the new variables (*quasi-stresses*) $\boldsymbol{\mu}_\delta := \sqrt{\chi_\delta + \delta}\,\varepsilon(\partial_t \mathbf{u}_\delta)$, and $\boldsymbol{\eta}_\delta := \sqrt{\chi_\delta + \delta}\,\varepsilon(\mathbf{u}_\delta)$:

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The total energy inequality for $(w_\delta, \mathbf{u}_\delta, \chi_\delta)$ is

$$\begin{aligned}
&\int_\Omega w_\delta(t)(\mathrm{d}x) + \frac{1}{2}\int_\Omega |\partial_t \mathbf{u}_\delta(t)|^2\,\mathrm{d}x + \int_s^t \int_\Omega |\partial_t \chi_\delta|^2\,\mathrm{d}x + \frac{1}{2}\int_s^t |\boldsymbol{\mu}_\delta(r)|^2 \\
&+ \frac{|\boldsymbol{\eta}_\delta(t)|^2}{2} + \frac{1}{2} a_s(\chi_\delta(t),\chi_\delta(t)) + \int_\Omega W(\chi_\delta(t))\,\mathrm{d}x \\
&\le \int_\Omega w_\delta(s)(\mathrm{d}x) + \frac{1}{2}\int_\Omega |\partial_t \mathbf{u}_\delta(s)|^2\,\mathrm{d}x + \frac{|\boldsymbol{\eta}_\delta(s)|^2}{2} + \frac{1}{2} a_s(\chi_\delta(s),\chi_\delta(s)) \\
&+ \int_\Omega W(\chi_\delta(s))\,\mathrm{d}x + \int_s^t \int_\Omega \mathbf{f}\cdot \partial_t \mathbf{u}_\delta\,\mathrm{d}x + \int_s^t \int_\Omega g\,\mathrm{d}x
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## The degenerate problem ($\delta = 0$): the existence theorem [E.R., R. Rossi, preprint arXiv:1205.3578v1 (2012)]

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$$w \in L^r(0,T;W^{1,r}(\Omega)) \cap L^\infty(0,T;L^1(\Omega)) \cap \mathrm{BV}([0,T];W^{1,r'}(\Omega)^*)$$

$$\chi \in L^\infty(0,T;H^s(\Omega)) \cap H^1(0,T;L^2(\Omega)), \quad \chi(x,t) \geq 0, \quad \chi_t(x,t) \leq 0 \text{ a.e.}$$

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such that it holds true (a.e. in any open set $A \subset \Omega \times (0,T)$: $\chi > 0$ a.e. in $A$)

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the weak enthalpy equation and the *weak* momentum and phase relations

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for all $\varphi \in L^2(0,T;H^s_+(\Omega)) \cap L^\infty(Q)$ with $\mathrm{supp}(\varphi) \subset \{\chi > 0\}$,

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for all $\varphi \in L^2(0,T;H^s_+(\Omega)) \cap L^\infty(Q)$ with $\mathrm{supp}(\varphi) \subset \{\chi > 0\}$,

together with the **total energy inequality** (for almost all $t \in (0,T]$)

$$\int_\Omega w(t)(\mathrm{d}x) + \int_0^t \int_\Omega |\chi_t|^2\,\mathrm{d}x + \frac{1}{2}\int_0^t |\boldsymbol{\mu}(r)|^2 + \int_\Omega W(\chi(t))\,\mathrm{d}x + \mathcal{J}(t) = \int_\Omega w_0\,\mathrm{d}x$$

$$+ \frac{1}{2}\int_\Omega |\mathbf{v}_0|^2\,\mathrm{d}x + \frac{1}{2}\chi_0|\varepsilon(\mathbf{u}_0)|^2 + \frac{1}{2}a_s(\chi_0,\chi_0) + \int_\Omega W(\chi_0)\,\mathrm{d}x + \int_0^t\int_\Omega \mathbf{f}\cdot\mathbf{u}_t\,\mathrm{d}x\mathrm{d}r + \int_0^t\int_\Omega g\,\mathrm{d}x$$

with $\displaystyle\int_0^t \mathcal{J}(r)\,\mathrm{d}r \geq \frac{1}{2}\int_0^t \left(\int_\Omega |\mathbf{u}_t(r)|^2\,\mathrm{d}x + |\boldsymbol{\eta}(r)|^2 + a_s(\chi(r),\chi(r))\right)$

## A comparison between the solution notions

Weak solution to the *degenerating* irreversible full system ($\delta = 0$) $\Longleftrightarrow$ weak solution to the *non-degenerating* irreversible full system ($\delta > 0$)

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$$\text{for all } \varphi \in L^2(0,T;H^s_+(\Omega)) \cap L^\infty(Q) \text{ with } \mathrm{supp}(\varphi) \subset \{\chi > 0\},$$

coincides with

$$\int_0^T \int_\Omega \chi_t\varphi + a_s(\chi,\varphi) + \xi\varphi + \gamma(\chi)\varphi + \frac{|\varepsilon(\mathbf{u})|^2}{2}\varphi - \Theta(w)\varphi \leq 0$$

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$\forall \varphi \in L^2(0, T; H^s_+(\Omega)) \cap L^\infty(Q)$ and with $\xi \in \partial I_{[0,+\infty)}(\chi)$. Subtracting from the degenerate total energy inequality the weak enthalpy equation tested by 1, we recover (a.e. in $(0, T]$) the energy inequality:

$$\int_0^t \int_\Omega |\chi_t|^2 \,\mathrm{d}x\,\mathrm{d}r + \frac{1}{2} a_s(\chi(t), \chi(t)) + \int_\Omega W(\chi(t))\,\mathrm{d}x$$

$$\leq \frac{1}{2} a_s(\chi_0, \chi_0) + \int_\Omega W(\chi_0)\,\mathrm{d}x + \int_0^t \int_\Omega \chi_t \left( -\frac{|\varepsilon(\mathbf{u})|^2}{2} + \Theta(w) \right) \mathrm{d}x\,\mathrm{d}r$$

## Work in progress: an entropic formulation for the damage phenomena

We worked here with the small perturbation assumption, i.e. neglecting the quadratic contribution on the r.h.s in the internal energy balance:

$$\theta_t + \chi_t\theta - \Delta\theta = |\chi_t|^2 + \chi|\varepsilon(\mathbf{u}_t)|^2$$

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Our **next aim**: to couple the weak equations for $\mathbf{u}$ and $\chi$ with

✓ the entropy production

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for every test function $\varphi \in \mathcal{D}(\overline{Q}_T)$, $\varphi \geq 0$

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for every test function $\varphi\in\mathcal{D}(\overline{Q}_T)$, $\varphi\ge 0$ and

✓ the energy conservation

$$E(t) = E(0) \text{ for a.e. } t\in[0,T],$$

where

$$E \equiv \int_\Omega\left(\theta + W(\chi) + \frac{1}{2}a_s(\chi,\chi) + \frac{|\mathbf{u}_t|^2}{2} + \chi\frac{|\varepsilon(\mathbf{u})|^2}{2}\right)dx\,.$$

This is still a work in progress (with R. Rossi)...

# Possible further application

- A fluid-mechanical theory for **two-phase mixtures of fluids** faces a well known mathematical difficulty:
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$$\partial_t \theta + \operatorname{div}(\theta \mathbf{v}) + \operatorname{div} \mathbf{q} = \mathbb{S} : \nabla_x \mathbf{v} + |\nabla_x \mu|^2 \tag{2}$$

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**Entropic notion** of solution is needed in order to interpret the internal energy balance (2) ...

**Thanks for your attention!**

cf. http://www.mat.unimi.it/users/rocca/