Weak solutions for a degenerating PDE system for phase transitions and damage

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joint work with Riccarda Rossi (Università di Brescia, Italy)

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- The future perspectives: to apply the *entropic formulation* to damage phenomena
- Other possible applications of these formulations to: phase separation, liquid crystals, immiscible fluids.

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Concentrate on **damage** phenomena accounting for

- the absolute temperature θ
- the evolution of the (small) displacement variables $\mathbf{u}$
- the damage parameter $\chi \in [0,1]$: $\chi = 0$ (completely damaged), $\chi = 1$ (completely undamaged)

where the internal energy balance display nonlinear dissipation and the momentum equation contains χ-dependent elliptic operators, degenerating at the *pure phase* $\chi = 0$

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\begin{aligned}
&\mathrm{c}(\theta)\theta_t + \chi_t\theta + \rho\theta \operatorname{div} \mathbf{u}_t - \operatorname{div}(k(\theta)\nabla\theta)) = g + \chi|\varepsilon(\mathbf{u}_t)|^2 + |\chi_t|^2 \\
&\mathbf{u}_{tt} - \operatorname{div}(\chi\varepsilon(\mathbf{u}_t) + \chi\varepsilon(\mathbf{u}) - \rho\theta\mathbf{1}) = \mathbf{f} \\
&\chi_t + \partial I_{(-\infty,0]}(\chi_t) + A_s\chi + W'(\chi) \ni -\frac{|\varepsilon(\mathbf{u})|^2}{2} + \theta
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- **Unidirectional:** $I_{(-\infty,0]}(\chi_t) = 0$ if $\chi_t \in (-\infty, 0]$, $I_{(-\infty,0]}(\chi_t) = +\infty$ otherwise
- **Nonlocal:** A_s the $s-$Laplacian ($s = 1$: $-\Delta$)
- $W = \widehat{\beta} + \widehat{\gamma}$, $\widehat{\gamma} \in C^2(\mathbb{R})$, $\widehat{\beta}$ proper, convex, l.s.c., $\overline{\mathrm{dom}(\widehat{\beta})} = [0,1]$ (e.g. $\widehat{\beta} = I_{[0,1]}$)

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1. a suitable *energy conservation* and *entropy inequality* inspired by:

1.1. the works of E. Feireisl and co-authors ([Feireisl, Comput. Math. Appl. (2007)] and [Bulíček, Feireisl, & Málek, Nonlinear Anal. Real World Appl. (2009)]) for heat conduction in fluids $\Longrightarrow$ **weak** formulation of the **internal energy balance**

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2. a *generalization of the principle of virtual powers* inspired by:

2.1. a notion of *weak solution* introduced by [Heinemann, Kraus, WIAS preprint 1569 and Adv. Math. Sci. Appl. (2011)] for non-degenerating isothermal diffuse interface models for phase separation and damage $\Longrightarrow$ **weak** formulation of the **phase equation**

2.2. the notion of *energetic solution* - A. Mielke and co-authors ([Bouchitté, Mielke, Roubíček, ZAMP. Angew. Math. Phys. (2009) and [Mielke, Roubíček, Zeman, Comput. Methods Appl. Mech. Engrg. (2010)]) for rate-independent processes for complete damage $\Longrightarrow$ **weak** formulation of the **momentum balance**

The generalized principle of virtual powers: damage phenomena

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$\Rightarrow$ neglect the nonlinear terms $\chi|\varepsilon(\mathbf{u}_t)|^2 + |\chi_t|^2$ on the r.h.s (using the small perturbations assumption) in the internal energy balance

$$c(\theta)\theta_t + \chi_t\theta + \rho\theta\,\mathrm{div}\,\mathbf{u}_t - \mathrm{div}(k(\theta)\nabla\theta) = g \quad \underbrace{+\chi\varepsilon(\mathbf{u}_t)|^2 + |\chi_t|^2}_{=0}$$

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$\Rightarrow$ give a **weak formulation** of the (degenerate) momentum balance and the phase equation (principle of virtual powers)

$$\mathbf{u}_{tt} - \operatorname{div}(\chi\varepsilon(\mathbf{u}_t) + \chi\varepsilon(\mathbf{u}) - \rho\theta\mathbf{1}) = \mathbf{f}$$

$$\chi_t + \partial I_{(-\infty,0]}(\chi_t) + A_s\chi + W'(\chi) \ni -\frac{|\varepsilon(\mathbf{u})|^2}{2} + \theta$$

in order to handle the degeneracy and the nonlinearities in the $\mathbf{u}$+χ-equations.

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$\Longrightarrow$ We shall approximate the system with a non-degenerating one, where we replace the momentum equation with

$$\mathbf{u}_{tt} - \mathrm{div}((\chi + \delta)\varepsilon(\mathbf{u}_t) + (\chi + \delta)\varepsilon(\mathbf{u}) - \rho\theta\mathbf{1}) = \mathbf{f} \qquad \text{for } \delta > 0$$

It seems to us that *both* the coefficients need to be truncated when taking the degenerate limit in the momentum equation. Indeed, on the one hand the truncation in front of $\varepsilon(\mathbf{u}_t)$ allows us to deal with the *main part* of the elliptic operator. On the other hand, in order to pass to the limit in the quadratic term on the right-hand side of χ-eq., we will also need to truncate the coefficient of $\varepsilon(\mathbf{u})$.

Our results

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[FIRST RESULT.] **Local in time well-posedness** for a suitable formulation of the reversible problem, using in

$$\theta_t + \chi_t\theta - \Delta\theta = g \quad \underbrace{+\chi\varepsilon(\mathbf{u}_t)|^2 + |\chi_t|^2}_{=0} \quad .$$

the *small perturbations assumption* in the **3D** (in space) setting [E.R., ROSSI, J. DIFFERENTIAL EQUATIONS, 2008]

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[Second result.] **Global well-posedness** in the **1D** case without *small perturbations assumption* [E.R, Rossi, Appl. Math., Special Volume (2008)]

Note: in both these results we assumed χ_0 separated from the thresholds 0 and 1 and we prove (via coercivity condition on W at the thresholds 0 and 1) that the solution χ of

$$\chi_t - \Delta\chi + W'(\chi) \ni -\frac{|\varepsilon(\mathbf{u})|^2}{2} + \theta$$

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The last result [E.R., R. Rossi, preprint arXiv:1205.3578v1 (2012)]: **global existence result in 3D** using a suitable notion of solution and without enforcing the separation property, i.e. **allowing for degeneracy** $\Longrightarrow$ we need a s-Laplacian or a p-Laplacian

The free-energy $\mathcal{F}$:

cf. [M. Frémond, Phase Change in Mechanics, Lecture Notes of UMI, Springer-Verlag, 2012]

$$\mathcal{F} = \int_\Omega \Big(f(\theta) + \chi \frac{|\varepsilon(\mathbf{u})|^2}{2} + \frac{a_s(\chi,\chi)}{2} + W(\chi) - \theta\chi \Big)\,\mathrm{d}x$$

- **the stiffness of the material decreases as $\chi \searrow 0$**
- f is a concave function, a primitive of the specific heat is $\hat{c}(\theta) = f(\theta) - \theta f'(\theta)$
- $a_s(z_1, z_2) := \int_\Omega \int_\Omega \frac{(\nabla z_1(x) - \nabla z_1(y)) \cdot (\nabla z_2(x) - \nabla z_2(y))}{|x-y|^{d+2(s-1)}}\,\mathrm{d}x\,\mathrm{d}y$ is the bilinear form associated to the nonlocal **fractional $s-$Laplacian A_s** (or $a_p(\chi,\chi) = |\nabla\chi|^p/p$)
- $s > d/2$ (or $p > d$): we need the embedding of $H^s(\Omega)$ into $C^0(\overline{\Omega})$
- $W = \widehat{\beta} + \widehat{\gamma}$, $\widehat{\gamma} \in C^2(\mathbb{R})$, $\widehat{\beta}$ proper, convex, l.s.c., $\overline{\mathrm{dom}(\widehat{\beta})} = [0,1]$ (e.g. $\widehat{\beta} = I_{[0,1]}$)
- we can include the thermal expansion term $-\rho\theta\mathrm{tr}(\varepsilon(\mathbf{u}))$ (neglect it in this presentation)

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The pseudo-potential $\mathcal{P}$:

$$\mathcal{P} = \frac{k(\theta)}{2}|\nabla\theta|^2 + \frac{1}{2}|\chi_t|^2 + \chi\frac{|\varepsilon(\mathbf{u}_t)|^2}{2} + I_{(-\infty,0]}(\chi_t)$$

- k is the heat conductivity
- $I_{(-\infty,0]}(\chi_t) = 0$ if $\chi_t \in (-\infty, 0]$, $I_{(-\infty,0]}(\chi_t) = +\infty$ otherwise

The modelling

The momentum equation

$$\mathbf{u}_{tt} - \operatorname{div}\sigma = \mathbf{f} \quad \left(\sigma = \sigma^d + \sigma^{nd} = \frac{\partial \mathcal{P}}{\partial \varepsilon(\mathbf{u}_t)} + \frac{\partial \mathcal{F}}{\partial \varepsilon(\mathbf{u})}\right) \quad \text{becomes}$$

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The "standard" principle of virtual powers

$$B - \operatorname{div}\mathbf{H} = 0 \quad \left(B = \frac{\partial \mathcal{P}}{\partial \chi_t} + \frac{\partial \mathcal{F}}{\partial \chi}, \mathbf{H} = \frac{\partial \mathcal{F}}{\partial \nabla \chi}\right) \quad \text{becomes}$$

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The internal energy balance

$$e_t + \operatorname{div}\mathbf{q} = g + \sigma : \varepsilon(\mathbf{u}_t) + B\chi_t + \mathbf{H}\cdot\nabla\chi_t \quad \left(e = \mathcal{F} - \theta\frac{\partial \mathcal{F}}{\partial \theta}, \quad \mathbf{q} = \frac{\partial \mathcal{P}}{\partial \nabla\theta}\right)$$

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- We replace the momentum equation with a non-degenerating one

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$$\mathrm{c}(\theta)\theta_t + \chi_t\theta - \mathrm{div}(k(\theta)\nabla\theta) = g$$

and in the phase equation

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- A major difficulty stems from the simultaneous presence in (2) of $\partial I_{(-\infty,0]}(\chi_t)$ and $W'(\chi)$ and from the low regularities of $-\frac{|\varepsilon(\mathbf{u})|^2}{2} + \theta$ on the r.h.s. $\Longrightarrow$ follow the approach of [Heinemann, Kraus, Adv. Math. Sci. Appl. (2011)] and consider a suitable weak formulation of (2) consisting of a one-sided variational inequality + an energy inequality $\Longrightarrow$ **generalized principle of virtual powers**

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- For the analysis of the degenerate limit $\delta \searrow 0$ we have carefully adapted techniques from [Bouchitté, Mielke, Roubíček, ZAMP (2009)] and [Mielke, Roubíček, Zeman, Comput. Methods Appl. Mech. Engrg. (2011)] to the case of a *rate-dependent* equation for χ, also coupled with the temperature equation.

Energy vs Enthalpy

In order to deal with the low regularity of θ, rewrite the internal energy equation

$$c(\theta)\theta_t + \chi_t\theta - \mathrm{div}(k(\theta)\nabla\theta) = g$$

as the enthalpy equation

$$w_t + \chi_t\Theta(w) - \mathrm{div}(K(w)\nabla w) = g \quad \text{where}$$

$$w = h(\theta) := \int_0^\theta c(s)\,\mathrm{d}s, \quad \Theta(w) := \begin{cases} h^{-1}(w) & \text{if } w \geq 0, \\ 0 & \text{if } w < 0, \end{cases} \quad K(w) := \frac{k(\Theta(w))}{c(\Theta(w))}$$

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We assume that

- $\mathrm{c} \in C^0([0,+\infty);[0,+\infty))$
- $\exists\, \sigma_1 \geq \sigma > \frac{2d}{d+2}$: $\quad c_0(1+\theta)^{\sigma-1} \leq \mathrm{c}(\theta) \leq c_1(1+\theta)^{\sigma_1-1} \Longrightarrow h$ is strictly increasing
- the function $k : [0,+\infty) \to [0,+\infty)$ is continuous, and

$$\exists c_2,\ c_3 > 0 \ \ \forall\theta \in [0,+\infty) : \quad c_2\mathrm{c}(\theta) \leq k(\theta) \leq c_3(\mathrm{c}(\theta)+1)$$

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$$\exists c_2,\, c_3 > 0 \;\; \forall\theta \in [0,+\infty) : \quad c_2 c(\theta) \leq k(\theta) \leq c_3(c(\theta)+1)$$

$\implies \exists\, \bar{c} > 0 \;\; \forall\, w \in \mathbb{R} : \; c_2 \leq K(w) \leq \bar{c}$

$\implies$ for every $s \in (1,\infty)$ $\exists\, C_s > 0$ $\forall\, w \in L^1(\Omega)$: $\|\Theta(w)\|_{L^s(\Omega)} \leq C_s(\|w\|_{L^{s/\sigma}(\Omega)}^{1/\sigma} + 1)$

The approximating non-degenerate Problem $[\mathrm{P}_\delta]$

Given $\delta > 0$, take $W' = \partial I_{[0,+\infty)} + \gamma$, $\gamma \in C^1(\mathbb{R})$, find (measurable) functions

$$w \in L^r(0,T;W^{1,r}(\Omega)) \cap L^\infty(0,T;L^1(\Omega)) \cap \mathrm{BV}([0,T];W^{1,r'}(\Omega)^*)$$
$$\mathbf{u} \in H^1(0,T;H^2(\Omega;\mathbb{R}^d)) \cap W^{1,\infty}(0,T;H^1_0(\Omega)) \cap H^2(0,T;L^2(\Omega;\mathbb{R}^d))$$
$$\chi \in L^\infty(0,T;H^s(\Omega)) \cap H^1(0,T;L^2(\Omega))$$

for every $1 \le r < \frac{d+2}{d+1}$, fulfilling the initial conditions

$$\mathbf{u}(0,x) = \mathbf{u}_0(x), \quad \mathbf{u}_t(0,x) = \mathbf{v}_0(x) \qquad \text{for a.e. } x \in \Omega$$
$$\chi(0,x) = \chi_0(x) \qquad \text{for a.e. } x \in \Omega$$

the equations (for every $\varphi \in \mathrm{C}^0([0,T];W^{1,r'}(\Omega)) \cap W^{1,r'}(0,T;L^{r'}(\Omega))$ and $t \in (0,T]$)

$$\int_\Omega \varphi(t)\,w(t)(\mathrm{d}x) - \int_0^t \int_\Omega w\varphi_t\,\mathrm{d}x + \int_0^t \int_\Omega \chi_t \Theta(w)\varphi\,\mathrm{d}x + \int_0^t \int_\Omega K(w)\nabla w \nabla\varphi\,\mathrm{d}x$$
$$= \int_0^t \int_\Omega g\varphi + \int_\Omega w_0\varphi(0)\,\mathrm{d}x$$
$$\mathbf{u}_{tt} - \mathrm{div}\,((\chi+\delta)\varepsilon(\mathbf{u}_t) + (\chi+\delta)\varepsilon(\mathbf{u})) = \mathbf{f} \text{ in } H^{-1}(\Omega;\mathbb{R}^d) \text{ a.e. in } (0,T)$$

and the subdifferential inclusion "in a suitable sense"

$$\chi_t + \partial I_{(-\infty,0]}(\chi_t) + A_s\chi + \partial I_{[0,+\infty)}(\chi) + \gamma(\chi) \ni -\frac{|\varepsilon(\mathbf{u})|^2}{2} + \Theta(w) \quad \text{in } H^{-s}(\Omega) \text{ and a.e. in } (0,T)$$

Generalized principle of virtual powers for $\delta > 0$ [E.R., R. Rossi, preprint arXiv:1205.3578v1 (2012)]

[Theorem 1] **($\delta > 0$)** Under the previous assumptions on the data ($W = I_{[0,+\infty)} + \widehat{\gamma}$), then,

[1.] Problem [P$_\delta$] admits a *weak solution* $(w, \mathbf{u}, \chi)$, which, beside fulfilling the enthalpy and momentum equations, satisfies $\chi_t(x, t) \le 0$ for almost all $t \in (0, T)$, and $(\forall \varphi \in L^2(0, T; H^s_+(\Omega)) \cap L^\infty(Q))$ the **one-sided inequality**

$$\int_0^T \int_\Omega \chi_t \varphi + a_s(\chi, \varphi) + \xi\varphi + \gamma(\chi)\varphi + \frac{|\varepsilon(\mathbf{u})|^2}{2}\varphi - \Theta(w)\varphi \le 0$$

with $\xi \in \partial I_{[0,+\infty)}(\chi)$ in the following sense:

$$\xi \in L^1(0, T; L^1(\Omega)), \ \langle \xi(t), \varphi - \chi(t) \rangle_{H^s(\Omega)} \le 0 \ \ \forall \varphi \in H^s_+(\Omega), \text{ a.e. } t \in (0, T)$$

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and the **energy inequality** for all $t \in (0,T]$, for $\tau = 0$, and for almost all $0 < \tau \leq t$:

$$\int_\tau^t \int_\Omega |\chi_t|^2 \,\mathrm{d}x\,\mathrm{d}r + \frac{1}{2} a_s(\chi(t),\chi(t)) + \int_\Omega W(\chi(t))\,\mathrm{d}x$$
$$\leq \frac{1}{2} a_s(\chi(\tau),\chi(\tau)) + \int_\Omega W(\chi(\tau))\,\mathrm{d}x + \int_\tau^t \int_\Omega \chi_t \left(-\frac{|\varepsilon(\mathbf{u})|^2}{2} + \Theta(w) \right) \mathrm{d}x\,\mathrm{d}r$$

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[2.] Suppose in addition that $g(x,t) \ge 0$, $\theta_0 > \underline{\theta}_0 \ge 0$ a.e. Then $\theta(x,t) := \Theta(w(x,t)) \ge \underline{\theta}_0 \ge 0$ a.e.

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Uniqueness of solutions for the irreversible system, even in the isothermal case, is still an open problem. This is mainly due to the doubly nonlinear character of the χ equation.

Generalized principle of virtual powers vs classical phase inclusion

Generalized principle of virtual powers vs classical phase inclusion

- If $(w, \mathbf{u}, \chi)$ are "more regular" and satisfy the notion of *weak solution*: the one-sided inequality $(\forall \varphi \in L^2(0,T; H^s_-(\Omega)) \cap L^\infty(Q))$:

$$\int_0^T \int_\Omega \chi_t \varphi + a_s(\chi, \varphi) + \xi\varphi + \gamma(\chi)\varphi + \frac{|\varepsilon(\mathbf{u})|^2}{2}\varphi - \Theta(w)\varphi \geq 0 \qquad \text{(one-sided)}$$

with $\xi \in \partial I_{[0,+\infty)}(\chi)$ and the energy inequality:

$$\int_\tau^t \int_\Omega |\chi_t|^2 \,\mathrm{d}x\,\mathrm{d}r + \frac{1}{2} a_s(\chi(t), \chi(t)) + \int_\Omega W(\chi(t))\,\mathrm{d}x$$
$$\leq \frac{1}{2} a_s(\chi(\tau), \chi(\tau)) + \int_\Omega W(\chi(\tau))\,\mathrm{d}x + \int_\tau^t \int_\Omega \chi_t \left(-\frac{|\varepsilon(\mathbf{u})|^2}{2} + \Theta(w) \right) \mathrm{d}x\,\mathrm{d}r \qquad \text{(energy)}$$

- "Differentiating in time" the energy inequality (energy) and using the chain rule, we conclude that $(w, \mathbf{u}, \chi, \xi)$ comply with

$$\left\langle \chi_t(t) + A_s(\chi(t)) + \xi(t) + \gamma(\chi(t)) + \frac{|\varepsilon(\mathbf{u})|^2}{2} - \Theta(w(t)), \chi_t(t) \right\rangle_{H^s(\Omega)} \leq 0 \text{ for a.e. } t \qquad \text{(ineq)}$$

(one-sided) − (ineq) + "$\chi_t \leq 0$ a.e." are equivalent to the usual phase inclusion

$$\chi_t + A_s\chi + \xi(t) + \gamma(\chi) + \frac{|\varepsilon(\mathbf{u})|^2}{2} - \Theta(w) \in -\partial I_{(-\infty,0]}(\chi_t) \text{ in } H^{-s}(\Omega)$$

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$$\int_\Omega w(t)(\mathrm{d}x) + \frac{1}{2}\int_\Omega |\mathbf{u}_t(t)|^2\,\mathrm{d}x + \int_\tau^t\int_\Omega |\chi_t|^2\,\mathrm{d}x\,\mathrm{d}r + \int_\tau^t\int_\Omega (\chi+\delta)|\varepsilon(\mathbf{u}_t)|^2\,\mathrm{d}x\,\mathrm{d}r$$
$$+ \frac{1}{2}\int_\Omega (\chi+\delta)|\varepsilon(\mathbf{u}(t))|^2\,\mathrm{d}x + a_s(\chi(t),\chi(t)) + \int_\Omega W(\chi(t))\,\mathrm{d}x$$
$$\leq \int_\Omega w(\tau)(\mathrm{d}x) + \frac{1}{2}\int_\Omega |\mathbf{u}_t(\tau)|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega (\chi+\delta)|\varepsilon(\mathbf{u}(\tau))|^2\,\mathrm{d}x + a_s(\chi(\tau),\chi(\tau))$$
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\begin{aligned}
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\end{aligned}
$$

- The presence of the s-Laplacian with $s > d/2$ $\Longrightarrow$ an estimate for χ in $L^\infty(0,T;H^s(\Omega))$ (from the total energy balance) $\Longrightarrow$ we can now test the momentum balance by $-\mathrm{div}(\varepsilon(\mathbf{u}_t))$ $\Longrightarrow$ an $L^\infty(0,T;L^2(\Omega))$-bound on the quadratic nonlinearity $|\varepsilon(\mathbf{u})|^2$ on the right-hand side of

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- A Boccardo-Gallouët-type estimate + Gagliardo-Nirenberg inequality lead to an $L^r(0,T;W^{1,r}(\Omega))$-estimate on the enthalpy w (and hence on $\Theta(w)$) .

The total energy inequality in the degenerating case $\delta \searrow 0$

Rewrite the momentum equation

$$\partial_t^2 \mathbf{u}_\delta - \mathrm{div}((\chi+\delta)\varepsilon(\partial_t \mathbf{u}_\delta)) - \mathrm{div}((\chi+\delta)\varepsilon(\mathbf{u}_\delta)) = \mathbf{f}$$

using the new variables (*quasi-stresses*) $\boldsymbol{\mu}_\delta := \sqrt{\chi_\delta + \delta}\,\varepsilon(\partial_t \mathbf{u}_\delta)$, and $\boldsymbol{\eta}_\delta := \sqrt{\chi_\delta + \delta}\,\varepsilon(\mathbf{u}_\delta)$:

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The total energy inequality for $(w_\delta, \mathbf{u}_\delta, \chi_\delta)$ is

$$\begin{aligned}
&\int_\Omega w_\delta(t)(\mathrm{d}x) + \frac{1}{2}\int_\Omega |\partial_t \mathbf{u}_\delta(t)|^2\,\mathrm{d}x + \int_\tau^t \int_\Omega |\partial_t \chi_\delta|^2\,\mathrm{d}x + \frac{1}{2}\int_\tau^t \int_\Omega |\boldsymbol{\mu}_\delta|^2\,\mathrm{d}x \\
&\quad + \frac{|\boldsymbol{\eta}_\delta(t)|^2}{2} + \frac{1}{2}a_s(\chi_\delta(t),\chi_\delta(t)) + \int_\Omega W(\chi_\delta(t))\,\mathrm{d}x \\
&\le \int_\Omega w_\delta(\tau)(\mathrm{d}x) + \frac{1}{2}\int_\Omega |\partial_t \mathbf{u}_\delta(\tau)|^2\,\mathrm{d}x + \frac{|\boldsymbol{\eta}_\delta(\tau)|^2}{2} + \frac{1}{2}a_s(\chi_\delta(\tau),\chi_\delta(\tau)) \\
&\quad + \int_\Omega W(\chi_\delta(\tau))\,\mathrm{d}x + \int_\tau^t \int_\Omega \mathbf{f}\cdot\partial_t \mathbf{u}_\delta\,\mathrm{d}x + \int_\tau^t \int_\Omega g\,\mathrm{d}x
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The degenerate problem ($\delta = 0$): the existence theorem [E.R., R. Rossi, preprint arXiv:1205.3578v1 (2012)]

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[Theorem 2] **($\delta = 0$)** Under the previous assumptions, there exist

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such that it holds true (a.e. in any open set $A \subset \Omega \times (0,T)$: $\chi > 0$ a.e. in A)

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the weak enthalpy equation and the *weak* momentum and phase relations

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for all $\varphi \in L^2(0,T;H^s_+(\Omega)) \cap L^\infty(Q)$ with $\mathrm{supp}(\varphi) \subset \{\chi > 0\}$,

together with the **degenerate total energy inequality** (for almost all $t \in (0,T]$)

$$\int_\Omega w(t)(\mathrm{d}x) + \int_0^t \int_\Omega |\chi_t|^2\,\mathrm{d}x + \frac{1}{2}\int_0^t \int_\Omega |\boldsymbol{\mu}|^2\,\mathrm{d}x + \int_\Omega W(\chi(t))\,\mathrm{d}x + \mathcal{J}(t) \leq \int_\Omega w_0\,\mathrm{d}x$$

$$+ \frac{1}{2}\int_\Omega |\mathbf{v}_0|^2\,\mathrm{d}x + \frac{1}{2}\chi_0|\varepsilon(\mathbf{u}_0)|^2 + \frac{1}{2}a_s(\chi_0,\chi_0) + \int_\Omega W(\chi_0)\,\mathrm{d}x + \int_0^t \int_\Omega \mathbf{f}\cdot\mathbf{u}_t\,\mathrm{d}x\mathrm{d}r + \int_0^t \int_\Omega g\,\mathrm{d}x$$

with $\displaystyle\int_0^t \mathcal{J}(r)\,\mathrm{d}r \geq \frac{1}{2}\int_0^t \left(\int_\Omega |\mathbf{u}_t(r)|^2\,\mathrm{d}x + |\boldsymbol{\eta}(r)|^2 + a_s(\chi(r),\chi(r))\right)$

A comparison between the solution notions

Weak solution to the *degenerating* irreversible full system ($\delta = 0$) $\Longleftrightarrow$ weak solution to the *non-degenerating* irreversible full system ($\delta > 0$)

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$$\text{for all } \varphi \in L^2(0, T; H^s_+(\Omega)) \cap L^\infty(Q) \text{ with } \mathrm{supp}(\varphi) \subset \{\chi > 0\},$$

coincides with the **one-sided inequality**

$$\int_0^T \int_\Omega \chi_t \varphi + a_s(\chi, \varphi) + \xi\varphi + \gamma(\chi)\varphi + \frac{|\varepsilon(\mathbf{u})|^2}{2}\varphi - \Theta(w)\varphi \leq 0$$

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$\forall \varphi \in L^2(0,T;H^s_+(\Omega)) \cap L^\infty(Q)$ and with $\xi \in \partial I_{[0,+\infty)}(\chi)$. Subtracting from the degenerate total energy inequality the weak enthalpy equation tested by 1, we recover (a.e. in $(0,T]$) **the energy inequality**:

$$\int_0^t \int_\Omega |\chi_t|^2\,\mathrm{d}x\,\mathrm{d}r + \frac{1}{2}a_s(\chi(t),\chi(t)) + \int_\Omega W(\chi(t))\,\mathrm{d}x$$
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Work in progress: avoid the small perturbation assumptions

We worked here with the small perturbation assumption, i.e. neglecting the quadratic contribution on the r.h.s in the internal energy balance:

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Our **next aim**: to couple the weak equations for $\mathbf{u}$ and χ with a suitable formulation of the internal energy balance:

a weak *energy conservation* and *entropy inequality*

$\Rightarrow$ inspired by the works of E. Feireisl and co-authors ([Feireisl, Comput. Math. Appl. (2007)] and [Bulíček, Feireisl, & Málek, Nonlinear Anal. Real World Appl. (2009)]) for heat conduction in fluids

Entropic formulation: a phase transitions model

A first application of the entropic formulation: solid-liquid phase transitions

In order to show the potential power of this idea we

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$\Longrightarrow$ a new weaker notion of solution is needed

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Finally, couple these relations to a suitable phase dynamics.

The entropy production

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$$\int_0^T \int_\Omega s_t \varphi - \int_0^T \int_\Omega \frac{\mathbf{q}}{\theta} \cdot \nabla \varphi = \int_0^T \int_\Omega r \varphi \quad \forall \varphi \in \mathcal{D}(\overline{Q}_T),$$

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r represents the entropy production rate. Then, in order to comply with the Clausius-Duhem inequality, we assume:

(i) r is a nonnegative measure on $[0, T] \times \overline{\Omega} =: \overline{Q}_T$;

(ii) $r \geq \frac{1}{\theta} \left(|\chi_t|^2 - \frac{\mathbf{q} \cdot \nabla \theta}{\theta} \right) \geq 0.$

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Taking $\mathbf{q} = -\nabla\theta$, $s = \log\theta + \chi$, we get

$$\int_0^T \int_\Omega \Big((\log\theta + \chi)\, \partial_t\varphi - \nabla\log\theta \cdot \nabla\varphi \Big) dxdt$$
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$\Rightarrow$ the total entropy is controlled by dissipation.

The energy conservation and phase relation

The total energy has to be preserved. Hence

$$E(t) = E(0) \text{ for a.e. } t \in [0, T],$$

where

$$E \equiv \int_\Omega \left(\theta + W(\chi) + \frac{|\nabla \chi|^2}{2} \right) dx \,.$$

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Finally, the phase dynamics results as

$$\chi_t - \Delta\chi + W'(\chi) = \theta - \theta_c \quad \text{a.e. in } \Omega \times (0, T),$$

where W is a double well or double obstacle potential: $W = \widehat{\beta} + \widehat{\gamma}$ where

$\widehat{\beta} : \mathbb{R} \to [0, +\infty]$ is proper, lower semi-continuous, convex function

$\widehat{\gamma} \in C^2(\mathbb{R}), \widehat{\gamma}' \in C^{0,1}(\mathbb{R}) : \widehat{\gamma}''(r) \geq -K$ for all $r \in \mathbb{R}$, $W(r) \geq c_w r^2$ for all $r \in \mathrm{dom}(\widehat{\beta})$

Examples: $\widehat{\beta}(r) = r \ln(r) + (1-r)\ln(1-r)$ or $\widehat{\beta}(r) = I_{[0,1]}(r)$.

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$$\theta \in L^\infty(0,T;L^1(\Omega)) \cap L^s(Q_T), \quad \theta(x,t) > 0 \quad \text{a.e. in } Q_T$$

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- However, in this case and similarly in many other situations, to prove that the solution has this extra regularity is **out of reach**
- **It can be suitable also in different applications** such as the ones related to phase transitions in viscoelastic materials, SMA, liquid crystal flows, **damage phenomena**

Work in progress: an entropic formulation for the damage phenomena

We worked here with the small perturbation assumption, i.e. neglecting the quadratic contribution on the r.h.s in the internal energy balance:

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Finally, with C. Heinemann, C. Kraus and R. Rossi, we would like to study the case of **non-isothermal phase separation and damage**.

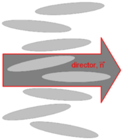
A further application to liquid crystals

- In [E. Feireisl, M. Frémond, E.R., G. Schimperna, ARMA 2012] we have coupled the incompressible Nevier-Stokes equation

$$\operatorname{div}\mathbf{v} = 0, \quad \mathbf{v}_t + \mathbf{v}\cdot\nabla_x\mathbf{v} + \nabla_x p = \operatorname{div}\mathbb{S} + \operatorname{div}\sigma^{nd} + \mathbf{g}$$

$$\mathbb{S} = \nu(\theta)\left(\nabla_x\mathbf{v} + \nabla_x^t\mathbf{v}\right), \quad \sigma^{nd} = -\nabla_x d \odot \nabla_x d + (\partial_d W(d) - \Delta d)\otimes d$$

and the evolution of the director field d, representing preferred orientation of molecules in a neighborhood of any point of a reference domain



$$d_t + \mathbf{v}\cdot\nabla_x d - d\cdot\nabla_x\mathbf{v} = \Delta d - \partial_d W(d)$$

with an **entropic formulation** of the inernal energy balance displaying higher order nonlinearities on the right hand side:

$$\theta_t + \mathbf{v}\cdot\theta + \operatorname{div}\mathbf{q} = \mathbb{S} : \nabla_x\mathbf{v} + |\Delta d - \partial_d W(d)|^2$$

- In [E. Feireisl, E.R., G. Schimperna, A. Zarnescu, preprint arXiv: 1207.1643v1 2012] we have extended it to the tensorial Ball-Majumdar model for liquid crystals

A further application to two phase fluids

- A fluid-mechanical theory for **two-phase mixtures of fluids** faces a well known mathematical difficulty:
  - the movement of the interfaces $\Longrightarrow$ Lagrangian description
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$$\operatorname{div} \mathbf{v} = 0\,, \quad \partial_t \mathbf{v} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) + \nabla p = \operatorname{div} \mathbb{S} - \mu \nabla_x \chi\,, \quad \mathbb{S} = \nu(\theta, \chi)\left(\nabla_x \mathbf{v} + \nabla_x^t \mathbf{v}\right) \tag{1}$$

$$\partial_t \theta + \operatorname{div}(\theta \mathbf{v}) + \operatorname{div} \mathbf{q} = \mathbb{S} : \nabla_x \mathbf{v} + |\nabla_x \mu|^2 \tag{2}$$

$$\partial_t \chi + \mathbf{v} \cdot \nabla_x \chi = \Delta \mu\,, \quad \mu = -\Delta \chi + W'(\chi) - \lambda(\theta) \tag{3}$$

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Entropic notion of solution is needed in order to interpret the internal energy balance (2).

Thanks for your attention!

cf. http://www.mat.unimi.it/users/rocca/