Weierstrass Institute for

Applied Analysis and Stochastics

Entropic solutions for systems of PDEs arising in complex fluids dynamics

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Outline

- The dynamics we are interested in: **non-isothermal models** for
  - Liquid Crystals flows
  - Mixtures of two viscous incompressible Newtonian fluids
  - Damage phoenomena in viscoelastic materials
- The common features of the PDEs
- The new notion of solution
- The analytical results
- Some open related problems

- Hydrodynamics of **liquid crystals flows**:
  - a liquid crystal may flow like a liquid, but its molecules may be oriented in a crystal-like way
  - **aim:** deal with the nematic liquid crystals in the Landau-de Gennes theory, in which the order parameter describing the orientation of molecules is a matrix, the so-called $\mathbb{Q}$-tensor and to include velocity and temperature dependence in the model

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- **Two-phase mixtures of fluids**:
  - avoid analytical problems of interface singularities: an alternative approach to the sharp interface models is the diffuse interface models (the H-model). The sharp interface is replaced by a thin interfacial region where a partial mixing of the fluids is allowed; a new variable φ represents the concentration difference of the fluids
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- **Damage** phenomena:
  - **aim:** deal with a non-isothermal diffuse interface models in thermoviscoelasticity accounting for the evolution of the displacement variables, the order (damage) parameter χ, indicating the local proportion of damage

Common features: the **nonlinearity** of the related PDEs

Liquid crystals

$$\theta_t + \mathbf{v}\cdot\nabla_x\theta + \operatorname{div}\mathbf{q} = \theta\big(\partial_t f(\mathbb{Q}) + \mathbf{u}\cdot\nabla_x f(\mathbb{Q})\big) + \sigma : \nabla_x\mathbf{v} + \Gamma(\theta)|\mathbb{H}|^2$$

$$\operatorname{div}\mathbf{v} = 0,\ \partial_t\mathbf{v} + \operatorname{div}(\mathbf{v}\otimes\mathbf{v}) + \nabla_x p = \operatorname{div}(\sigma + \mathbb{T}(\theta,\mathbb{Q})),\quad \sigma = \nu(\theta)(\nabla_x\mathbf{v} + \nabla_x^t\mathbf{v})$$

$$\mathbb{Q}_t + \mathbf{v}\cdot\nabla_x\mathbb{Q} - \mathbb{S}(\nabla_x\mathbf{v},\mathbb{Q}) = \Gamma(\theta)\mathbb{H},\quad \mathbb{H} = \Delta\mathbb{Q} - \theta\frac{\partial f(\mathbb{Q})}{\partial\mathbb{Q}} - \frac{\partial G(\mathbb{Q})}{\partial\mathbb{Q}}$$

Two-phase mixtures of fluids

$$\theta_t + \mathbf{v}\cdot\nabla_x\theta + \operatorname{div}\mathbf{q} = -\theta(\varphi_t + \mathbf{v}\cdot\nabla_x\varphi) + \sigma : \nabla_x\mathbf{v} + |\nabla_x\mu|^2$$

$$\operatorname{div}\mathbf{v} = 0\,,\ \mathbf{v}_t + \operatorname{div}(\mathbf{v}\otimes\mathbf{v}) + \nabla_x p = \operatorname{div}\sigma - \mu\nabla_x\varphi,\quad \sigma = \nu(\theta)\left(\nabla_x\mathbf{v} + \nabla_x^t\mathbf{v}\right)$$

$$\varphi_t + \mathbf{v}\cdot\nabla_x\varphi = \Delta\mu\,,\quad \mu = -\Delta\varphi + W'(\varphi) - \theta$$

Damage

$$\theta_t + \operatorname{div}\mathbf{q} = -\theta(\chi_t + \rho\operatorname{div}\mathbf{u}_t) + a(\chi)|\varepsilon(\mathbf{u}_t)|^2 + |\chi_t|^2$$

$$\mathbf{u}_{tt} - \operatorname{div}(a(\chi)\varepsilon(\mathbf{u}_t) + b(\chi)\varepsilon(\mathbf{u}) - \rho\theta\mathbf{1}) = \mathbf{f},\quad \varepsilon(\mathbf{u}) = (\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u})/2$$

$$\chi_t + \partial I_{(-\infty,0]}(\chi_t) - \operatorname{div}(|\nabla\chi|^{p-2}\nabla\chi) + W'(\chi) \ni -b'(\chi)|\varepsilon(\mathbf{u})|^2/2 + \theta$$

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1. a suitable *energy conservation* and *entropy inequality* inspired by:

1.1. the works of E. Feireisl and co-authors ([Feireisl, Comput. Math. Appl. (2007)] and [Bulíček, Feireisl, & Málek, Nonlinear Anal. Real World Appl. (2009)]) for heat conduction in fluids

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2. a *generalization of the principle of virtual powers* inspired by:

2.1. a notion of *weak solution* introduced by [Heinemann, Kraus, Adv. Math. Sci. Appl. (2011)] for non-degenerating isothermal diffuse interface models for phase separation and damage

Liquid crystals

- The motivations:
  - Theoretical studies of these types of materials are motivated by **real-world applications**: proper functioning of many practical devices relies on optical properties of certain liquid crystalline substances in the presence or absence of an electric field: **a multi-billion dollar industry**
  - At the molecular level, what marks the difference between a liquid crystal and an ordinary, isotropic fluid is that, while the centers of mass of LC molecules do not exhibit any long-range correlation, **molecular orientations do exhibit orientational correlations**

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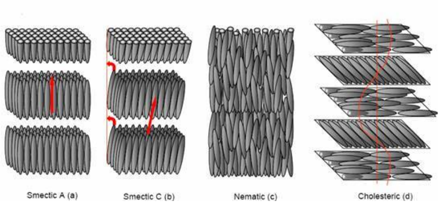
- The objective: include the **temperature dependence** in models describing the **evolution of nematic liquid crystal flows** within the Landau-De Gennes theories (cf. [De Gennes, Prost (1995)])

To the present state of knowledge, three main types of liquid crystals are distinguished, termed *smectic*, **nematic** and *cholesteric*



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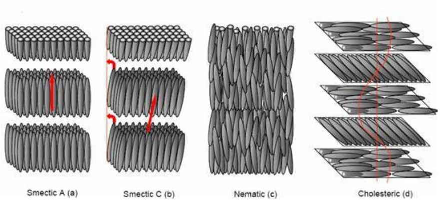

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The *smectic* phase forms well-defined layers that can slide one over another in a manner very similar to that of a soap

The **nematic** phase: the molecules have long-range orientational order, but no tendency to the formation of layers. Their center of mass positions all point in the same direction (within each specific domain)

Crystals in the *cholesteric* phase exhibit a twisting of the molecules perpendicular to the director, with the molecular axis parallel to the director

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- We consider the range of temperatures typical for the **nematic phase**

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- Most mathematical work has been done on the Oseen-Frank theory, in which the mean orientation of the rod-like molecules is described by a **vector field** d. However, more popular among physicists is the Landau-de Gennes theory, in which the order parameter describing the orientation of molecules is a matrix, the so-called $\mathbb{Q}$**-tensor**

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- The flow **velocity** $\mathbf{v}$ evidently disturbs the alignment of the molecules and also the converse is true: a change in the alignment will produce a perturbation of the velocity field $\mathbf{v}$. Moreover, we want to include in our model also the **changes of the temperature** θ

The Landau-de Gennes theory: the molecular orientation

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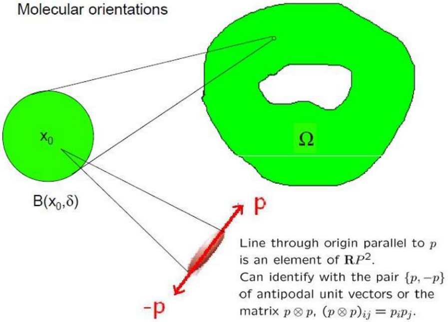
- The distribution of molecular orientations in a ball $B(x_0, \delta)$, $x_0 \in \Omega$ can be represented as a probability measure μ on the unit sphere $\mathbb{S}^2$ satisfying $\mu(E) = \mu(-E)$ for $E \subset \mathbb{S}^2$
- For a continuously distributed measure we have $d\mu(p) = \rho(p)dp$ where dp is an element of the surface area on $\mathbb{S}^2$ and $\rho \geq 0$, $\int_{\mathbb{S}^2} \rho(p)dp = 1$, $\rho(p) = \rho(-p)$



- The first moment $\int_{\mathbb{S}^2} p \, d\mu(p) = 0$, the second moment $M = \int_{\mathbb{S}^2} p \otimes p \, d\mu(p)$ is a symmetric non-negative 3×3 matrix (for every $\mathbf{v} \in \mathbb{S}^2$, $\mathbf{v} \cdot M \cdot \mathbf{v} = \int_{\mathbb{S}^2} (\mathbf{v} \cdot p)^2 \, d\mu(p) =< \cos^2 \theta >$, where θ is the angle between p and $\mathbf{v}$) satisfying $\mathrm{tr}(M) = 1$

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- If the orientation of molecules is equally distributed in all directions (the distribution is *isotropic*) and then $\mu = \mu_0$, where $d\mu_0(p) = \frac{1}{4\pi} dS$. In this case the second moment tensor is $M_0 = \frac{1}{4\pi} \int_{\mathbb{S}^2} p \otimes p\, dS = \frac{1}{3}\mathbf{1}$, because $\int_{\mathbb{S}^2} p_1 p_2\, dS = 0$, $\int_{\mathbb{S}^2} p_1^2\, dS = \int_{\mathbb{S}^2} p_2^2\, dS$, etc., and $\mathrm{tr}(M_0) = 1$

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- **The de Gennes $\mathbb{Q}$-tensor** measures the deviation of M from its isotropic value

$$\mathbb{Q} = M - M_0 = \int_{\mathbb{S}^2} \left(p \otimes p - \frac{1}{3}\mathbf{1} \right) d\mu(p)$$

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Note that (cf. [Ball, Majumdar, Molecular Crystals and Liquid Crystals (2010)])

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1.+2. implies $\mathbb{Q} = \lambda_1 \mathbf{n}_1 \otimes \mathbf{n}_1 + \lambda_2 \mathbf{n}_2 \otimes \mathbf{n}_2 + \lambda_3 \mathbf{n}_3 \otimes \mathbf{n}_3$, where $\{\mathbf{n}_i\}$ is an othonormal basis of eigenvectors of $\mathbb{Q}$ with corresponding eigenvalues λ_i such that $\lambda_1 + \lambda_2 + \lambda_3 = 0$

2.+3. implies $-\frac{1}{3} \leq \lambda_i \leq \frac{2}{3}$

- $\mathbb{Q} = 0$ does not imply $\mu = \mu_0$ (e.g. $\mu = \frac{1}{6} \sum_{i=1}^{3} (\delta_{e_i} + \delta_{-e_i})$)

The Ball-Majumdar singular potential

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- In order to naturally enforce the physical constraints in the eigenvalues of the symmetric, traceless tensors $\mathbb{Q}$, Ball and Majumdar have recently introduced in [Ball, Majumdar, Molecular Crystals and Liquid Crystals (2010)] a singular component

$$f(\mathbb{Q}) = \begin{cases} \inf_{\rho\in\mathcal{A}_{\mathbb{Q}}} \int_{S^2} \rho(\mathbf{p}) \log(\rho(\mathbf{p}))\, \mathrm{d}\mathbf{p} \text{ if } \lambda_i[\mathbb{Q}] \in (-1/3, 2/3),\ i = 1,2,3, \\ \infty \text{ otherwise,} \end{cases}$$

$$\mathcal{A}_{\mathbb{Q}} = \left\{ \rho : S^2 \to [0,\infty) \,\middle|\, \int_{S^2} \rho(\mathbf{p})\, \mathrm{d}\mathbf{p} = 1; \mathbb{Q} = \int_{S^2} \left(\mathbf{p}\otimes\mathbf{p} - \frac{1}{3}\mathbb{I} \right) \rho(\mathbf{p})\, \mathrm{d}\mathbf{p} \right\}.$$

to the bulk free-energy f_B enforcing the eigenvalues to stay in the interval $(-\frac{1}{3}, \frac{2}{3})$

[⇒] For the **Landau-de Gennes** free energy with “regular” potential, the hydrodynamic theory has been developed in [Paicu, Zarnescu, SIAM (2011) and ARMA (2012)] in the isothermal case

Our main contributions

We study the **non-isothermal** evolutionary system for nematic liquid crystals within the recent Ball-Majumdar $\mathbb{Q}$-tensorial model preserving the physical eigenvalue constraint on the traceless and symmetric matrices $\mathbb{Q}$:

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We work in the three-dimensional torus $\Omega \subset \mathbb{R}^3$ in order to avoid complications connected with boundary conditions. We consider the evolution of the following variables:

- the mean velocity field $\mathbf{v}$
- the tensor field $\mathbb{Q}$, representing preferred (local) orientation of the crystals
- the absolute temperature θ

- The free energy density takes the form

$$\mathcal{F} = \frac{1}{2}|\nabla\mathbb{Q}|^2 + f_B(\theta,\mathbb{Q}) - \theta\log\theta - a\theta^m$$

where

- $f_B(\theta,\mathbb{Q}) = \theta f(\mathbb{Q}) + G(\mathbb{Q})$ is bulk the configuration potential
- f is the convex l.s.c. and singular Ball-Majumdar potential, G is a smooth function of $\mathbb{Q}$
- $a\theta^m$ prescribes a power-like specific heat

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- The dissipation pseudo-potential is given by

$$\mathcal{P} = \frac{\nu(\theta)}{4}|\nabla \mathbf{v} + \nabla^t \mathbf{v}|^2 + I_{\{0\}}(\operatorname{div} \mathbf{v}) + \frac{\kappa(\theta)}{2\theta}|\nabla \theta|^2 + \frac{1}{2\Gamma(\theta)}|D_t \mathbb{Q}|^2$$

  - ν, κ and Γ are the smooth viscosity, the heat conductivity, and the collective rotational coefficients, $D_t\mathbb{Q}$ is a "generalized material derivative"
  - **Incompressibility**: I_0 the indicator function of $\{0\}$: $I_0 = 0$ if $\operatorname{div} \mathbf{v} = 0$, $+\infty$ otherwise)

We assume that the driving force governing the dynamics of the director $\mathbb{Q}$ is of **"gradient type"** $\partial_{\mathbb{Q}}\mathcal{F}$:

$$\partial_t \mathbb{Q} + \mathbf{v} \cdot \nabla \mathbb{Q} - \mathbb{S}(\nabla \mathbf{v}, \mathbb{Q}) = \Gamma(\theta)\mathbb{H}, \quad \text{(eq-Q)}$$

- The left hand side is the "generalized material derivative" $D_t\mathbb{Q} = \partial_t \mathbb{Q} + \mathbf{v} \cdot \nabla \mathbb{Q} - \mathbb{S}(\nabla \mathbf{v}, \mathbb{Q})$
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$\mathbb{Q}$-tensor equation

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- $\Gamma(\theta)$ represents a collective rotational viscosity coefficient
- The function f represents the convex part of a singular potential of **[Ball-Majumdar]** type

The Ball-Majumdar potential (cf. [Ball, Majumdar (2010)]) exhibit a logarithmic divergence as the eigenvalues of $\mathbb{Q}$ approaches $-\frac{1}{3}$ and $\frac{2}{3}$

$$f(\mathbb{Q}) = \begin{cases} \inf_{\rho \in \mathcal{A}_{\mathbb{Q}}} \int_{S^2} \rho(\mathbf{p}) \log(\rho(\mathbf{p}))\, \mathrm{d}\mathbf{p} \text{ if } \lambda_i[\mathbb{Q}] \in (-1/3, 2/3),\ i = 1,2,3, \\ \\ \infty \text{ otherwise,} \end{cases}$$

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$\Longrightarrow$ It explodes “logarithmically” as one of the eigenvalues of $\mathbb{Q}$ approaches the limiting values $-1/3$ or $2/3$.

Equation of momentum

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- The **coupling term (or "extra-stress")** $\mathbb{T}$ depends both on θ and $\mathbb{Q}$

$$\mathbb{T} = 2\xi\,(\mathbb{H}:\mathbb{Q})\left(\mathbb{Q}+\frac{1}{3}\mathbb{I}\right) - \xi\left[\mathbb{H}\left(\mathbb{Q}+\frac{1}{3}\mathbb{I}\right)+\left(\mathbb{Q}+\frac{1}{3}\mathbb{I}\right)\mathbb{H}\right] + (\mathbb{Q}\mathbb{H}-\mathbb{H}\mathbb{Q}) - \nabla\mathbb{Q}\odot\nabla\mathbb{Q}$$

where ξ is a fixed scalar parameter

Entropy inequality

The evolution of temperature is prescribed by stating the **entropy inequality**

$$s_t + \mathbf{v} \cdot \nabla s - \operatorname{div}\left(\frac{\kappa(\theta)}{\theta}\nabla\theta\right) \tag{eq-θ}$$

$$\geq \frac{1}{\theta}\left(\frac{\nu(\theta)}{2}\left|\nabla\mathbf{v} + \nabla^t\mathbf{v}\right|^2 + \Gamma(\theta)|\mathbb{H}|^2 + \frac{\kappa(\theta)}{\theta}|\nabla\theta|^2\right)$$

where $s" = -\partial_\theta \mathcal{F}'' = -f(\mathbb{Q}) + 1 + \log\theta + ma\theta^{m-1}$

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- $\kappa(r) = A_0 + A_k r^k$, $A_0,\ A_k > 0$, $\frac{3k+2m}{3} > 9$, $\frac{3}{2} < m \leq \frac{6k}{5}$
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- $\Gamma(r) = \Gamma_0 + \Gamma_1 r, \Gamma_0,\ \Gamma_1 > 0$
- The "heat" balance can be recovered by (formally) multiplying by θ
- Due to the **quadratic** terms, we can only interpret (eq-θ) as an **inequality**

Total energy balance

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$$\partial_t \left(\frac{1}{2}|\mathbf{v}|^2 + e\right) + \operatorname{div}\left((\frac{1}{2}|\mathbf{v}|^2 + e)\mathbf{v}\right) + \operatorname{div}\mathbf{q} \qquad \text{(eq-bal)}$$

$$= \operatorname{div}(\sigma\mathbf{v}) + \operatorname{div}\left(\Gamma(\theta)\nabla\mathbb{Q} : \left(\Delta\mathbb{Q} - \theta\frac{\partial f(\mathbb{Q})}{\partial\mathbb{Q}} + \lambda\theta\right)\right) + \boldsymbol{g}\cdot\mathbf{v}$$

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$$\sigma=\frac{\nu(\theta)}{2}(\nabla\mathbf{v}+\nabla^t\mathbf{v})-p\mathbb{I}+\mathbb{T}$$

- To control it, assuming periodic b.c.'s is essential

Theorem: existence of global in time “Entropic solutions”

We can prove existence of at least one “Entropic solution” to system (eq-v)+(eq-Q)+(eq-θ)+(eq-bal) for finite-energy initial data , namely

$$\theta_0 \in L^\infty(\Omega), \ \operatorname{essinf}_{x\in\Omega} \theta_0(x) = \underline{\theta} > 0,$$

$$\mathbb{Q}_0 \in H^1(\Omega), \ f(\mathbb{Q}_0) \in L^1(\Omega),$$

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- Notice that, if the solution is more regular, the **entropy inequality** becomes an **equality** and, multiplying it by θ we just get the standard **internal energy balance**

$$\theta_t + \mathbf{v}\cdot\nabla_x\theta + \operatorname{div}\mathbf{q} = \theta\big(\partial_t f(\mathbb{Q}) + \mathbf{u}\cdot\nabla_x f(\mathbb{Q})\big) + \nu(\theta)\big|\nabla_x\mathbf{v} + \nabla_x^t\mathbf{v}\big|^2 + \Gamma(\theta)|\mathbb{H}|^2$$

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- However, this regularity is out of reach for this model: that is why this solution notion is significative

Two-phase mixtures of fluids

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  - viscous
  - incompressible
  - Newtonian fluids
  - of equal density

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  $\Longrightarrow$ use a **diffuse interface model**

  $\Longrightarrow$ the classical sharp interface replaced by a **thin interfacial region**
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- A partial mixing of the macroscopically immiscible fluids is allowed

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- The original idea of diffuse interface model for fluids: HOHENBERG and HALPERIN, '77

  $\Longrightarrow$ **H-model**

  Later, GURTIN ET AL., '96: continuum mechanical derivation based on microforces
- Models of two-phase or two-component fluids are receiving growing attention (e.g., ABELS, BOYER, GARCKE, GRÜN, GRASSELLI, LOWENGRUB, TRUSKINOVSKI, ...)

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- **Including temperature dependence is a widely open issue**
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- Our idea: a weak formulation of the system as a combination of *total energy balance* plus *entropy production inequality* $\Longrightarrow$ **"Entropic formulation"**

- This method has been recently proposed by [BULÍČEK-MÁLEK-FEIREISL, '09] for the Navier-Stokes-Fourier system and has been proved to be effective to study e.g.
  - nonisothermal models for **phase transitions** ([FEIREISL-PETZELTOVÁ-R., '09]) and
  - the evolution of **nematic liquid crystals** ([FRÉMOND, FEIREISL, R., SCHIMPERNA, ZARNESCU, '12,'13])

- We want to describe the behavior of a mixture of two incompressible fluids of the same density in terms of the following state variables
  - $\mathbf{v}$: macroscopic **velocity** (Navier-Stokes),
  - p: **pressure** (Navier-Stokes),
  - φ: **order parameter** (Cahn-Hilliard),
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  - θ: **absolute temperature** (Entropic formulation).
- We do not neglect convection and capillarity effects. We assume constant mobility and smooth configuration potential in Cahn-Hilliard. We take temperature dependent coefficients wherever possible. We assume the system being insulated from the exterior.

- We start by specifying two functionals:
  - the **free energy** Ψ, related to the equilibrium state of the material, and
  - the **dissipation pseudo-potential** Φ, describing the processes leading to dissipation of energy (i.e., transformation into heat)

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- Then we impose the balances of **momentum**, **configuration energy**, and both of **internal energy** and of **entropy**, in terms of these functionals
- The **thermodynamical consistency** of the model is then a direct consequence of the solution notion

The **total free energy** is given as a function of the state variables $E = (\theta, \varphi, \nabla_x \varphi)$

$$\Psi(E) = \int_\Omega \psi(E)\,\mathrm{d}x, \quad \psi(E) = f(\theta) - \theta\varphi + \frac{\varepsilon}{2}|\nabla_x \varphi|^2 + \frac{1}{\varepsilon}F(\varphi)$$

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- $\varepsilon > 0$ is related to the interfacial thickness
- we need $F(\varphi)$ to be the classical smooth double well potential $F(\varphi) \sim \frac{1}{4}(\varphi^2 - 1)^2$

The **dissipation potential** is taken as function of $\delta E = (D\mathbf{u}, \frac{D\varphi}{Dt}, \nabla_x \theta)$ and E

$$\Phi(\delta E, E) = \int_\Omega \left(\frac{\nu(\theta)}{2} |D\mathbf{v}|^2 + I_{\{0\}}(\mathrm{div}\,\mathbf{v}) + \frac{\kappa(\theta)}{2\theta} |\nabla_x \theta|^2 + \frac{|\nabla_x \mu|^2}{2} \right) \mathrm{d}x$$

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- the chemical potential μ is defined as $\frac{D\varphi}{Dt} = \Delta\mu$

- It is obtained (at least for no-flux b.c.'s) as the following **gradient-flow problem**

$$\partial_{L^2_\#(\Omega),\frac{D\varphi}{Dt}}\Phi + \partial_{L^2_\#(\Omega),\varphi_\#}\Psi = 0$$

where $L^2_\#(\Omega) = \{\xi \in L^2(\Omega) \,:\, \overline{\xi} := |\Omega|^{-1}\int_\Omega \xi \,\mathrm{d}x = 0\}$, $\varphi_\# = \varphi - \overline{\varphi_0}$

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- Using the form of the Free energy

$$\Psi = \int_\Omega \left(f(\theta) - \theta\varphi + \frac{\varepsilon}{2}|\nabla_x\varphi|^2 + \frac{1}{\varepsilon}F(\varphi)\right) \mathrm{d}x$$

and of the Pseudopotential of dissipation

$$\Phi = \int_\Omega \left(\frac{\nu(\theta)}{2}|D\mathbf{v}|^2 + I_{\{0\}}(\operatorname{div}\mathbf{v}) + \frac{\kappa(\theta)}{2\theta}|\nabla_x\theta|^2 + \frac{|\nabla_x\mu|^2}{2}\right) \mathrm{d}x$$

we then arrive at **the Cahn-Hilliard system with Neumann hom. b.c. for μ and φ**

$$\frac{D\varphi}{Dt} = \Delta\mu, \quad \mu = -\varepsilon\Delta\varphi + \frac{1}{\varepsilon}F'(\varphi) - \theta, \quad \frac{\partial\varphi}{\partial\mathbf{n}} = \frac{\partial\mu}{\partial\mathbf{n}} = 0 \text{ on } \Gamma \qquad \text{(CahnHill)}$$

The **Navier-Stokes system** is obtained as a momentum balance by setting

$$\frac{D\mathbf{v}}{Dt} = \mathbf{v}_t + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) = \operatorname{div} \sigma, \qquad \text{(momentum)}$$

The **Navier-Stokes system** is obtained as a momentum balance by setting

$$\frac{D\mathbf{v}}{Dt} = \mathbf{v}_t + \mathrm{div}(\mathbf{v} \otimes \mathbf{v}) = \mathrm{div}\,(\sigma^d + \sigma^{nd}), \qquad \text{(momentum)}$$

where the stress σ is split into its

- **dissipative part**

$$\sigma^d := \frac{\partial \phi}{\partial D\mathbf{v}} = \nu(\theta) D\mathbf{v} - p\mathbb{I}, \quad \mathrm{div}\,\mathbf{v} = 0,$$

  representing kinetic energy which **dissipates** (i.e. is transformed into heat) due to viscosity, and its

- **non-dissipative part** $\sigma^{nd} = -\varepsilon \nabla_x \varphi \otimes \nabla_x \varphi$ which is determined in agreement with Thermodynamics

The balance of internal energy takes the form

$$(Q(\theta))_t + \mathbf{v} \cdot \nabla_x Q(\theta) + \theta \frac{D\varphi}{Dt} - \operatorname{div}(\kappa(\theta)\nabla_x \theta) = \nu(\theta)|D\mathbf{v}|^2 + |\nabla_x \mu|^2$$

where $Q(\theta) = f(\theta) - \theta f'(\theta)$ and $Q'(\theta) =: c_v(\theta)$

The internal energy balance

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where $Q(\theta) = f(\theta) - \theta f'(\theta)$ and $Q'(\theta) =: c_v(\theta)$

The **dissipation** terms on the right hand side are in perfect agreement with the Pseudopotential of dissipation

$$\Phi = \int_\Omega \left(\frac{\nu(\theta)}{2} |D\mathbf{v}|^2 + I_{\{0\}}(\mathrm{div}\, \mathbf{v}) + \frac{\kappa(\theta)}{2\theta} |\nabla_x \theta|^2 + \frac{|\nabla_x \mu|^2}{2} \right) \mathrm{d}x$$

Following [BULÍČEK, FEIREISL, & MÁLEK], we replace the pointwise internal energy balance by the **total energy balance**

$$(\partial_t + \mathbf{v} \cdot \nabla_x) \left(\frac{|\mathbf{v}|^2}{2} + e \right) + \operatorname{div} \big(p\mathbf{v} - \kappa(\theta) \nabla_x \theta - (\nu(\theta) D\mathbf{v}) \mathbf{v} \big)$$
$$= \operatorname{div} \big(\varphi_t \nabla_x \varphi + \mu \nabla_x \mu \big) \qquad \text{(energy)}$$

with the internal energy

$$e = F(\varphi) + \frac{1}{2} |\nabla_x \varphi|^2 + Q(\theta) \quad Q'(\theta) = c_v(\theta)$$

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$$= \operatorname{div}\left(\varphi_t\nabla_x\varphi + \mu\nabla_x\mu\right) \qquad \text{(energy)}$$

with the internal energy

$$e = F(\varphi) + \frac{1}{2}|\nabla_x\varphi|^2 + Q(\theta) \quad Q'(\theta) = c_v(\theta)$$

and the **entropy inequality**

$$(\Lambda(\theta)+\varphi)_t + \mathbf{v}\cdot\nabla_x(\Lambda(\theta)+\varphi) - \operatorname{div}\left(\frac{\kappa(\theta)\nabla_x\theta}{\theta}\right) \qquad \text{(entropy)}$$
$$\geq \frac{\nu(\theta)}{\theta}|D\mathbf{v}|^2 + \frac{1}{\theta}|\nabla_x\mu|^2 + \frac{\kappa(\theta)}{\theta^2}|\nabla_x\theta|^2, \quad \text{where} \quad \Lambda(\theta) = \int_1^\theta \frac{c_v(s)}{s}\,\mathrm{d}s \sim \theta^\delta$$

- a **weak form of the momentum balance** (in distributional sense)

$$\mathbf{v}_t + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x p = \operatorname{div}(\nu(\theta) D\mathbf{v}) - \operatorname{div}(\nabla_x \varphi \otimes \nabla_x \varphi), \quad \operatorname{div} \mathbf{v} = 0;$$

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- the **Cahn-Hilliard system** in $H^1(\Omega)'$

$$\varphi_t + \mathbf{v} \cdot \nabla_x \varphi = \Delta \mu, \quad \mu = -\Delta \varphi + F'(\varphi) - \theta;$$

- a **weak form of the momentum balance** (in distributional sense)

$$\mathbf{v}_t + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x p = \operatorname{div}(\nu(\theta) D\mathbf{v}) - \operatorname{div}(\nabla_x \varphi \otimes \nabla_x \varphi), \quad \operatorname{div} \mathbf{v} = 0;$$

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- a **weak form of the total energy balance** (in distributional sense)

$$\partial_t \left(\frac{1}{2} |\mathbf{v}|^2 + e \right) + \mathbf{v} \cdot \nabla_x \left(\frac{1}{2} |\mathbf{v}|^2 + e \right) + \operatorname{div} \left(p\mathbf{v} + \mathbf{q} - \mathbb{S}\mathbf{u} \right)$$

$$- \operatorname{div} \left(\varphi_t \nabla_x \varphi + \mu \nabla_x \mu \right) = 0 \quad \text{where} \quad e = F(\varphi) + \frac{1}{2} |\nabla_x \varphi|^2 + \int_1^\theta c_v(s) \, \mathrm{d}s;$$

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- the weak form of the **entropy production inequality**

$$(\Lambda(\theta) + \varphi)_t + \mathbf{v}\cdot\nabla_x(\Lambda(\theta)) + \mathbf{v}\cdot\nabla_x\varphi - \operatorname{div}\left(\frac{\kappa(\theta)\nabla_x\theta}{\theta}\right)$$

$$\geq \frac{\nu(\theta)}{\theta}|D\mathbf{v}|^2 + \frac{1}{\theta}|\nabla_x\mu|^2 + \frac{\kappa(\theta)}{\theta^2}|\nabla_x\theta|^2, \quad \text{where} \quad \Lambda(\theta) = \int_1^\theta \frac{c_v(s)}{s}\,\mathrm{d}s.$$

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- Concerning B.C.'s, our results are proved for **no-flux** conditions for θ, φ, and μ and **complete slip** conditions for $\mathbf{v}$

$$\mathbf{v} \cdot \mathbf{n}_{|\Gamma} = 0 \quad \text{(the fluid cannot exit } \Omega\text{, it can move tangentially to } \Gamma\text{)}$$

$$[\mathbb{S}\mathbf{n}] \times \mathbf{n}_{|\Gamma} = 0, \quad \text{where } \mathbb{S} = \nu(\theta) D\mathbf{v} \quad \text{(exclude friction effects with the boundary)}$$

They can be easily extended to the case of periodic B.C.'s for all unknowns

Theorem

We can prove existence of **at least** *one* **global in time "Entropic solution"** $(\mathbf{v}, \varphi, \mu, \theta)$

$$\mathbf{v} \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^3)) \cap L^2(0,T;\mathbf{V_n})$$
$$\varphi \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^3(\Omega)) \cap H^1(0,T;H^1(\Omega)')$$
$$\mu \in L^2(0,T;H^1(\Omega)) \cap L^{\frac{14}{5}}((0,T)\times\Omega)$$
$$\theta \in L^\infty(0,T;L^{\delta+1}(\Omega)) \cap L^\beta(0,T;L^{3\beta}(\Omega)) \cap L^2(0,T;H^1(\Omega))$$
$$\theta > 0 \text{ a.e. in } (0,T)\times\Omega, \quad \log\theta \in L^2(0,T;H^1(\Omega))$$

to system given by (momentum), (CahnHill), (entropy) *and* (energy), *in distributional sense and for finite-energy initial data*

$$\mathbf{v}_0 \in L^2(\Omega), \ \operatorname{div}\mathbf{v}_0 = 0, \quad \varphi_0 \in H^1(\Omega), \quad \theta_0 \in L^{\delta+1}(\Omega), \quad \theta_0 > 0 \text{ a.e.}$$

Damage phenomena

State variables:

- the absolute temperature θ
- the (small) displacement variables $\mathbf{u}$ ($\varepsilon_{ij}(\mathbf{u}) := (u_{i,j} + u_{j,i})/2, i, j = 1, 2, 3$)
- the **damage** parameter $\chi \in [0, 1]$: $\chi = 0$ (completely damaged), $\chi = 1$ (completely undamaged)

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$$\theta_t + \chi_t\theta + \rho\theta \operatorname{div} \mathbf{u}_t - \operatorname{div}(\mathsf{K}(\theta)\nabla\theta)) = g + a(\chi)|\varepsilon(\mathbf{u}_t)|^2 + |\chi_t|^2$$

$$\mathbf{u}_{tt} - \text{div}(a(\chi)\varepsilon(\mathbf{u}_t) + b(\chi)\varepsilon(\mathbf{u}) - \rho\theta\mathbf{1}) = \mathbf{f}$$

$$\chi_t + \partial I_{(-\infty,0]}(\chi_t) - \operatorname{div}(|\nabla\chi|^{p-2}\nabla\chi) + W'(\chi) \ni -b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} + \theta$$

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- **Unidirectional:** $I_{(-\infty,0]}(\chi_t) = 0$ if $\chi_t \in (-\infty, 0]$, $I_{(-\infty,0]}(\chi_t) = +\infty$ otherwise;
- p-Laplacian: $-\Delta_p : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ ($p > d$ for this presentation);
- **The double-obstacle:** $W = \widehat{\beta} + \widehat{\gamma}$, $\widehat{\gamma} \in C^2(\mathbb{R})$, $\widehat{\beta}$ proper, convex, l.s.c. (e.g. $\widehat{\beta} = I_{[0,1]}$)

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⇒ These terms were neglected in most of the past contribution in the literature or considered only in the 1D case or in the framework of local - in time - existence (cf., e.g., [E. Bonetti, G. Bonfanti (2007)], [P. Krečí, J. Sprekels, U. Stefanelli (2003)], [F. Luterotti and U. Stefanelli (2002)], [E.R., R. Rossi (2013)])

The free-energy cf. [M. Frémond, Phase Change in Mechanics, Lecture Notes of UMI, Springer-Verlag, 2012]

$$\mathcal{F} = \int_\Omega \Big(\theta(1 - \log\theta) + b(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} + \frac{|\nabla\chi|^p}{p} + W(\chi) - \theta\chi - \rho\theta \mathrm{tr}(\varepsilon(\mathbf{u})) \Big)\, \mathrm{d}x$$

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The pseudo-potential of dissipation

$$\mathcal{P} = \frac{\mathsf{K}(\theta)}{2}|\nabla\theta|^2 + \frac{1}{2}|\chi_t|^2 + a(\chi)\frac{|\varepsilon(\mathbf{u}_t)|^2}{2} + I_{(-\infty,0]}(\chi_t)$$

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- K is the heat conductivity, $\mathsf{K}(\theta) \geq c_1(1 + \nu\theta^k)$ for some $c_1,\ \nu > 0$, $k > 1$
- $I_{(-\infty,0]}(\chi_t) = 0$ if $\chi_t \in (-\infty, 0]$, $I_{(-\infty,0]}(\chi_t) = +\infty$ otherwise (**irreversibility** of the damage)

The momentum equation

$$\mathbf{u}_{tt} - \operatorname{div}\sigma = \mathbf{f} \quad \left(\sigma = \sigma^d + \sigma^{nd} = \frac{\partial \mathcal{P}}{\partial \varepsilon(\mathbf{u}_t)} + \frac{\partial \mathcal{F}}{\partial \varepsilon(\mathbf{u})}\right) \quad \text{becomes}$$

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The “standard” principle of virtual powers

$$B - \operatorname{div}\mathbf{H} = 0 \quad \left(B = \frac{\partial \mathcal{P}}{\partial \chi_t} + \frac{\partial \mathcal{F}}{\partial \chi}, \mathbf{H} = \frac{\partial \mathcal{F}}{\partial \nabla \chi}\right) \quad \text{becomes}$$

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The internal energy balance

$$e_t + \operatorname{div}\mathbf{q} = g + \sigma : \varepsilon(\mathbf{u}_t) + B\chi_t + \mathbf{H}\cdot\nabla\chi_t \quad \left(e = \mathcal{F} - \theta\frac{\partial \mathcal{F}}{\partial \theta}, \quad \mathbf{q} = \frac{\partial \mathcal{P}}{\partial \nabla\theta}\right)$$

becomes

$$\boxed{\theta_t + \chi_t\theta + \rho\theta \operatorname{div}\mathbf{u}_t - \operatorname{div}(\mathsf{K}(\theta)\nabla\theta) = g + a(\chi)|\varepsilon(\mathbf{u}_t)|^2 + |\chi_t|^2}$$

Our previous result and our goals

[The last result] [E.R., R. Rossi, preprint arXiv:1205.3578v2 (2012), to appear on M3AS]: **global existence result in 3D** using a suitable notion of solution and without enforcing the separation property, i.e. **allowing for degeneracy:** $a(\chi) = b(\chi) = \chi$, but always within the **small perturbations assumption, i.e. neglecting the quadratic terms** on the r.h.s. in the internal energy balance

[Our goals] We restric to the non-degenerate case $\Longrightarrow$ replace a and b by $a + \delta$, $b + \delta$ in the momentum balance:

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In order to handle

- the high order dissipative terms in the θ-equation
- the quadratic nonlinearity in the χ-equation

we need a suitable **weak formulation**

Hypothesis (I).

The function $\mathsf{K} : [0, +\infty) \to (0, +\infty)$ is continuous and

$$\exists c_0,\, c_1,\, \nu > 0,\, k > 1 \,:\, \forall \theta \in [0, +\infty) \quad c_0(1 + \theta^k) \leq \mathsf{K}(\theta) \leq c_1(1 + \nu\theta^k)$$

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Hypothesis (II). $a\in \mathrm{C}^1(\mathbb{R}),\ b\in\mathrm{C}^2(\mathbb{R})$ are such that $a(x),\ b(x)\geq 0,$ for all $x\in\mathbb{R}$

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Hypothesis (III). $W = \widehat{\beta} + \widehat{\gamma}$, where

$\widehat{\beta} : \mathrm{dom}(\widehat{\beta}) \to \mathbb{R}$ is proper, l.s.c., convex;, $\mathrm{dom}(\widehat{\beta}) \subseteq [0,+\infty)$ is bounded,

$$\widehat{\gamma} \in \mathrm{C}^2(\mathbb{R}), \quad \exists\, c_w,\, c_w' > 0 \;:\quad W(r) \ge c_w r^2 - c_w' \quad \forall r \in \mathrm{dom}(\widehat{\beta})$$

Hereafter, we shall denote by $\beta = \partial\widehat{\beta}$ the subdifferential of $\widehat{\beta}$, and set $\gamma := \widehat{\gamma}'$

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Hereafter, we shall denote by $\beta = \partial\widehat{\beta}$ the subdifferential of $\widehat{\beta}$, and set $\gamma := \widehat{\gamma}'$

Hypothesis (IV).

$$\mathbf{f} \in L^2(0, T; L^2(\Omega)),$$

$$g \in L^1(0, T; L^1(\Omega)) \cap L^2(0, T; H^1(\Omega)'), \quad g \ge 0 \quad \text{a.e. in } \Omega \times (0, T)\,,$$

and that the initial data comply with

$$\theta_0 \in L^1(\Omega), \quad \exists\, \theta_* > 0 \ : \quad \min_{\Omega} \theta_0 \ge \theta_* > 0\,, \quad \log\theta_0 \in L^1(\Omega),$$

$$\mathbf{u}_0 \in H^2_0(\Omega), \quad \mathbf{v}_0 \in L^2(\Omega; \mathbb{R}^d), \quad \chi_0 \in W^{1,p}(\Omega), \quad \widehat{\beta}(\chi_0) \in L^1(\Omega).$$

Given $\delta > 0$ there exists (measurable) functions

$$\theta \in L^\infty(0,T;L^1(\Omega)) \cap L^2(0,T;H^1(\Omega))\,,$$
$$\mathbf{u} \in H^1(0,T;H_0^2(\Omega)) \cap W^{1,\infty}(0,T;H_0^1(\Omega;\mathbb{R}^d)) \cap H^2(0,T;L^2(\Omega;\mathbb{R}^d))\,,$$
$$\chi \in L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega))\,,$$

fulfilling the initial conditions

$$\mathbf{u}(0,x) = \mathbf{u}_0(x), \quad \mathbf{u}_t(0,x) = \mathbf{v}_0(x) \qquad \text{for a.e. } x \in \Omega$$
$$\chi(0,x) = \chi_0(x) \qquad \text{for a.e. } x \in \Omega$$

together with

the *entropy inequality*

the *total energy inequality*

the weak momentum equation (a.e. in $\Omega \times (0,T)$)

the *generalized principle of virtual powers*

The entropy inequality + total energy inequality

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The *entropy inequality*

$$\int_0^T\int_\Omega(\log(\theta)+\chi)\varphi_t\,\mathrm{d}x\,\mathrm{d}t+\rho\int_0^T\int_\Omega\mathrm{div}(\mathbf{u}_t)\varphi\,\mathrm{d}x\,\mathrm{d}t-\int_0^T\int_\Omega\mathsf{K}(\theta)\nabla\log(\theta)\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t$$

$$\leq-\int_0^T\int_\Omega\mathsf{K}(\theta)\frac{\varphi}{\theta}\nabla\log(\theta)\cdot\nabla\theta\,\mathrm{d}x\,\mathrm{d}t-\int_0^T\int_\Omega\left((a(\chi)+\delta)|\varepsilon(\mathbf{u}_t)|^2+g+|\chi_t|^2\right)\frac{\varphi}{\theta}\,\mathrm{d}x\,\mathrm{d}t$$

for all $\varphi\in\mathcal{D}(\overline{\Omega}\times[0,T])$ with $\varphi\geq 0$;

The entropy inequality + total energy inequality

The *entropy inequality*

$$\int_0^T\int_\Omega (\log(\theta)+\chi)\varphi_t \,\mathrm{d}x\,\mathrm{d}t + \rho\int_0^T\int_\Omega \mathrm{div}(\mathbf{u}_t)\varphi\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Omega \mathsf{K}(\theta)\nabla\log(\theta)\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t$$

$$\leq -\int_0^T\int_\Omega \mathsf{K}(\theta)\frac{\varphi}{\theta}\nabla\log(\theta)\cdot\nabla\theta\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Omega \left((a(\chi)+\delta)|\varepsilon(\mathbf{u}_t)|^2 + g + |\chi_t|^2\right)\frac{\varphi}{\theta}\,\mathrm{d}x\,\mathrm{d}t$$

for all $\varphi \in \mathcal{D}(\overline{\Omega}\times[0,T])$ with $\varphi \geq 0$;

The *total energy inequality* for almost all $t\in(0,T)$ and almost all $s\in(0,t)$, and for $s=0$

$$E(\theta(t),\mathbf{u}(t),\mathbf{u}_t(t),\chi(t)) \leq E(\theta(s),\mathbf{u}(s),\mathbf{u}_t(s),\chi(s)) + \int_s^t\int_\Omega g\,\mathrm{d}x\,\mathrm{d}r \int_s^t\int_\Omega \mathbf{f}\cdot\mathbf{u}_t\,\mathrm{d}x\,\mathrm{d}r$$

where

$$E(\theta,\mathbf{u},\mathbf{u}_t,\chi) := \int_\Omega \theta + \frac{1}{2}\int_\Omega |\mathbf{u}_t|^2 + \frac{1}{2}(b(\chi(t))+\delta)|\varepsilon(\mathbf{u})|^2(t) + \frac{1}{p}\int_\Omega |\nabla\chi|^p + \int_\Omega W(\chi)$$

The relations: $\chi_t(x,t) \leq 0$ for almost all $(x,t) \in \Omega \times (0,T)$, as well as

$$\int_\Omega \left(\chi_t(t)\varphi + |\nabla\chi(t)|^{p-2}\nabla\chi(t)\cdot\nabla\varphi + \xi(t)\varphi + \gamma(\chi(t))\varphi + b'(\chi(t))\frac{|\varepsilon(\mathbf{u}(t))|^2}{2}\varphi - \theta(t)\varphi \right) \geq 0$$

$$\text{for all } \varphi \in W^{1,p}_-(\Omega), \quad \text{for a.a. } t \in (0,T)$$

with $\xi \in \partial I_{[0,+\infty)}(\chi)$ in the following sense:

$$\xi \in L^1(0,T;L^1(\Omega)) \qquad \text{and} \qquad \langle \xi(t), \varphi - \chi(t)\rangle_{W^{1,p}(\Omega)} \leq 0 \ \forall\, \varphi \in W^{1,p}_+(\Omega), \text{ for a.a. } t \in (0,T)$$

and the energy inequality for all $t \in (0,T]$, for $s = 0$ and for almost all $0 < s \leq t$:

$$\int_s^t \int_\Omega |\chi_t|^2 \,\mathrm{d}x\,\mathrm{d}r + \int_\Omega \left(\frac{1}{p}|\nabla\chi(t)|^p + W(\chi(t)) \right) \mathrm{d}x$$
$$\leq \int_\Omega \left(\frac{1}{p}|\nabla\chi(s)|^p + W(\chi(s)) \right) \mathrm{d}x + \int_s^t \int_\Omega \chi_t \left(-b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} + \theta \right) \mathrm{d}x\,\mathrm{d}r$$

where

$$W^{1,p}_+(\Omega) := \left\{ \zeta \in W^{1,p}(\Omega) \ : \ \zeta(x) \geq 0 \quad \text{for a.a. } x \in \Omega \right\} \qquad \text{and analogously for } W^{1,p}_-(\Omega)$$

Generalized principle of virtual powers vs classical phase inclusion

- If $(\theta, \mathbf{u}, \chi)$ are "more regular" and satisfy the notion of *weak solution*:
  the one-sided inequality ($\forall \varphi \in L^2(0,T; W^{1,p}_{-}(\Omega)) \cap L^\infty(Q)$):

$$\int_0^T \int_\Omega \chi_t \varphi + |\nabla \chi|^{p-2} \nabla \chi \nabla \varphi + \xi \varphi + \gamma(\chi)\varphi + b'(\chi) \frac{|\varepsilon(\mathbf{u})|^2}{2} \varphi - \theta \varphi \geq 0 \qquad \text{(one-sided)}$$

  with $\xi \in \partial I_{[0,+\infty)}(\chi)$ and the energy inequality:

$$\int_s^t \int_\Omega |\chi_t|^2 \, \mathrm{d}x \, \mathrm{d}r + \frac{1}{p} |\nabla \chi(t)|^p + \int_\Omega W(\chi(t)) \, \mathrm{d}x$$

$$\leq \frac{1}{p} |\nabla \chi(s)|^p + \int_\Omega W(\chi(s)) \, \mathrm{d}x + \int_s^t \int_\Omega \chi_t \left(-b'(\chi) \frac{|\varepsilon(\mathbf{u})|^2}{2} + \theta \right) \mathrm{d}x \, \mathrm{d}r \qquad \text{(energy)}$$

- If $(\theta, \mathbf{u}, \chi)$ are "more regular" and satisfy the notion of *weak solution*:
  the one-sided inequality ($\forall \varphi \in L^2(0,T; W_-^{1,p}(\Omega)) \cap L^\infty(Q)$):

$$\int_0^T \int_\Omega \chi_t \varphi + |\nabla\chi|^{p-2}\nabla\chi\nabla\varphi + \xi\varphi + \gamma(\chi)\varphi + b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2}\varphi - \theta\varphi \geq 0 \qquad \text{(one-sided)}$$

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$$\leq \frac{1}{p}|\nabla\chi(s)|^p + \int_\Omega W(\chi(s))\,\mathrm{d}x + \int_s^t \int_\Omega \chi_t \left(-b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} + \theta\right)\mathrm{d}x\,\mathrm{d}r \qquad \text{(energy)}$$

- "Differentiating in time" the energy inequality (energy) and using the chain rule, we conclude that $(\theta, \mathbf{u}, \chi, \xi)$ comply with

$$\left\langle \chi_t(t) - \Delta_p\chi(t) + \xi(t) + \gamma(\chi(t)) + b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} - \theta(t), \chi_t(t)\right\rangle_{W^{1,p}(\Omega)} \leq 0 \text{ for a.e. } t \qquad \text{(ineq)}$$

  (one-sided) − (ineq) + "$\chi_t \leq 0$ a.e." are equivalent to the usual phase inclusion

$$\chi_t - \Delta_p\chi + \xi + \gamma(\chi) + b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} - \theta \in -\partial I_{(-\infty,0]}(\chi_t) \text{ in } W^{1,p}(\Omega)^*$$

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- **Some further developments:**
  - The study of the long-time behaviour of the **Liquid crystal model** WORK IN PROGRESS
  - Uniqueness and equilibria in 2D for the **Two-phase fluids model** WORK IN PROGRESS
  - Temperature-dependence in a model for **damage** and **phase separation** WORK IN PROGRESS