Weierstrass Institute for
Applied Analysis and Stochastics

“Entropic” solutions to a thermodynamically consistent PDE system for phase transitions and damage

Elisabetta Rocca – in collaboration with Riccarda Rossi (University of Brescia)

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State variables:

- the absolute temperature θ
- the (small) displacement variables $\mathbf{u}$ ($\varepsilon_{ij}(\mathbf{u}) := (u_{i,j} + u_{j,i})/2, i, j = 1, 2, 3$)
- the **damage**/**phase** parameter $\chi \in [0, 1]$: $\chi = 0$ (completely damaged/non-viscous phase), $\chi = 1$ (completely undamaged/viscous phase)

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$$\theta_t + \chi_t\theta + \rho\theta \operatorname{div} \mathbf{u}_t - \operatorname{div}(\mathsf{K}(\theta)\nabla\theta)) = g + a(\chi)\varepsilon(\mathbf{u}_t)\mathbb{V}\varepsilon(\mathbf{u}_t) + |\chi_t|^2$$

$$\mathbf{u}_{tt} - \operatorname{div}(a(\chi)\mathbb{V}\varepsilon(\mathbf{u}_t) + b(\chi)\mathbb{E}\varepsilon(\mathbf{u}) - \rho\theta\mathbf{1}) = \mathbf{f}$$

$$\chi_t + \mu\partial I_{(-\infty,0]}(\chi_t) \underbrace{-\operatorname{div}(|\nabla\chi|^{p-2}\nabla\chi)}_{A_p\chi} + W'(\chi) \ni -\frac{1}{2}b'(\chi)\varepsilon(\mathbf{u})\mathbb{E}\varepsilon(\mathbf{u}) + \theta$$

and the boundary conditions

$$\mathsf{K}(\theta)\nabla\theta \cdot n = h, \quad \mathbf{u} = 0, \quad \partial_n\chi = 0 \quad \text{on } \partial\Omega \times (0, T).$$

In the PDEs

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we have

- $\rho \rightsquigarrow$ thermal expansion coefficient;
- $\mathsf{K} \rightsquigarrow$ heat conductivity;
- $\mathbb{E} \rightsquigarrow$ elasticity tensor and $\mathbb{V} \rightsquigarrow$ viscosity tensor;
- $a,\, b \in \mathrm{C}^1([0,1]; [0,+\infty))$;
- $W \sim I_{[0,1]} + \gamma$, with $\gamma \in \mathrm{C}^1([0,1])$ and $I_{[0,1]}$ indicator of $[0,1]$;
- $\mu \in \{0,1\} \rightsquigarrow \mu = 1$ in **damage** (unidirectional/irreversible) and $\mu = 0$ in **phase transitions** (reversible);
- $\mathbf{f}$ volume force and g heat source.

- **GLOBAL - in time - existence result for the FULL PDE system** displaying the high order dissipative terms on the right hand in side in the temperature equation:

$$\theta_t + \chi_t\theta + \rho\theta \operatorname{div} \mathbf{u}_t - \operatorname{div}(\mathsf{K}(\theta)\nabla\theta)) = g + a(\chi)\varepsilon(\mathbf{u}_t)\mathbb{V}\varepsilon(\mathbf{u}_t) + |\chi_t|^2$$

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⇒ The **quadratic RHS** challenging: **neglected** or considered only in the **1D case** or for **local - in time - existence** (cf., e.g., [Bonetti-Bonfanti (2003), (2007)], [Krečí-Sprekels-Stefanelli (2003)], [Luterotti-Stefanelli (2002)], [R.-Rossi (2008), (2014)], [R.-Heinemann (2014)])

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◇ need enhanced estimates on $\mathbf{u}$ in

$$\mathbf{u}_{tt} - \mathrm{div}(a(\chi)\mathbb{V}\varepsilon(\mathbf{u}_t) + b(\chi)\mathbb{E}\varepsilon(\mathbf{u}) - \rho\theta\mathbf{1}) = \mathbf{f}$$

based on the higher regularity of $\chi \Longrightarrow A_p\chi$ needed with $p > d$

The free-energy

$$\mathcal{F} = \int_\Omega \Big(\theta(1-\log\theta) + b(\chi)\frac{\varepsilon(\mathbf{u})\mathbb{E}\varepsilon(\mathbf{u})}{2} + \frac{|\nabla\chi|^p}{p} + W(\chi) - \theta\chi - \rho\theta\mathrm{tr}(\varepsilon(\mathbf{u}))\Big)\,\mathrm{d}x$$

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The pseudo-potential of dissipation

$$\mathcal{P} = \frac{\mathsf{K}(\theta)}{2}|\nabla\theta|^2 + \frac{1}{2}|\chi_t|^2 + \frac{1}{2}a(\chi)\varepsilon(\mathbf{u}_t)\mathbb{V}\varepsilon(\mathbf{u}_t) + \mu I_{(-\infty,0]}(\chi_t)$$

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- **e.g.** $a(\chi) = \chi$ in damage and phase transitions: no viscosty when the material is completely damaged or when we are in the solid phase i.e. when $\chi = 0$

The momentum equation

$$\mathbf{u}_{tt} - \operatorname{div} \sigma = \mathbf{f} \quad \left(\sigma = \sigma^d + \sigma^{nd} = \frac{\partial \mathcal{P}}{\partial \varepsilon(\mathbf{u}_t)} + \frac{\partial \mathcal{F}}{\partial \varepsilon(\mathbf{u})} \right) \quad \text{becomes}$$

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The principle of virtual powers

$$B - \operatorname{div}\mathbf{H} = 0 \quad \left(B = \frac{\partial \mathcal{P}}{\partial \chi_t} + \frac{\partial \mathcal{F}}{\partial \chi}, \mathbf{H} = \frac{\partial \mathcal{F}}{\partial \nabla \chi}\right) \quad \text{becomes}$$

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The internal energy balance

$$e_t + \operatorname{div}\mathbf{q} = g + \sigma : \varepsilon(\mathbf{u}_t) + B\chi_t + \mathbf{H}\cdot\nabla\chi_t \quad \left(e = \mathcal{F} - \theta\frac{\partial \mathcal{F}}{\partial \theta}, \quad \mathbf{q} = \frac{\partial \mathcal{P}}{\partial \nabla \theta}\right)$$

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$$\boxed{\theta_t + \chi_t\theta + \rho\theta\operatorname{div}\mathbf{u}_t - \operatorname{div}(\mathsf{K}(\theta)\nabla\theta) = g + a(\chi)\varepsilon(\mathbf{u}_t)\mathbb{V}\varepsilon(\mathbf{u}_t) + |\chi_t|^2}$$

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- global existence result in 3D in the degenerate case $a(\chi) = b(\chi) = \chi$
- within the **small perturbations assumption, i.e. neglecting the quadratic terms** $a(\chi)\varepsilon(\mathbf{u}_t)\mathbb{V}\varepsilon(\mathbf{u}_t) + |\chi_t|^2$ on the r.h.s. in the internal energy balance

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[The most recent result] **[E.R., R. Rossi, WIAS preprint no. 1931 (2014)]**:

- global existence result in 3D in the non-degenerate case $\Longrightarrow$ replace a and b by $a + \delta$, $b + \delta$ in the momentum balance:

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- **without the small perturbations assumption**: we can handle the high order dissipative terms $a(\chi)\varepsilon(\mathbf{u}_t)\mathbb{V}\varepsilon(\mathbf{u}_t) + |\chi_t|^2$ on the r.h.s. in the internal energy balance

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We need here a suitable **weak formulation** of the problem to have **GLOBAL solutions in 3D**

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  - **EXISTENCE OF ENTROPIC SOLUTIONS**
  - improved regularity of θ under enhanced growth of K
  - limit as $p \downarrow 2$ (from the p-Laplacian to the Laplacian) $\longrightarrow$ a weaker notion of solution

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Uniqueness:

Not known even in the isothermal case, due to doubly nonlinear character of χ-eq.

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Definition

We call $(\theta, \mathbf{u}, \chi)$ an *entropic solution* if it fulfills

- **entropic** formulation of **heat** eqn. = entropy ineq. + total energy ineq.
- momentum equation pointwise
- **weak** formulation of χ**-eqn.**: “generalized principle of virtual powers” if $\mu = 1$

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- momentum equation pointwise
- **χ-equation** holds **pointwise** if $\mu = 0$

Replace $\theta_t + \chi_t\theta + \rho\theta\,\mathrm{div}(\mathbf{u}_t) - \mathrm{div}(\mathsf{K}(\theta)\nabla\theta) = g + a(\chi)\varepsilon(\mathbf{u}_t)\mathbb{V}\varepsilon(\mathbf{u}_t) + |\chi_t|^2$ by

The **entropy inequality**

$$
\begin{aligned}
&\iint_{\Omega\times(0,T)} (\log(\theta) + \chi)\varphi_t - \iint_{\Omega\times(0,T)} \rho\mathrm{div}(\mathbf{u}_t)\varphi \\
&+ \iint_{\Omega\times(0,T)} \big(\mathsf{K}(\theta)|\nabla\log(\theta)|^2\varphi - \mathsf{K}(\theta)\nabla\log(\theta)\nabla\varphi\big) \\
&+ \iint_{\Omega\times(0,T)} (g + a(\chi)\varepsilon(\mathbf{u}_t)\mathbb{V}\varepsilon(\mathbf{u}_t) + |\chi_t|^2)\frac{\varphi}{\theta}\,\mathrm{d}x\,\mathrm{d}t + \iint_{\partial\Omega\times(0,T)} h\frac{\varphi}{\theta}\,\mathrm{d}S\,\mathrm{d}t \\
\leq &\left[\int_\Omega \log(\theta)\varphi\right]_0^T \qquad \text{for all } \varphi \in \mathrm{C}^\infty_{\mathrm{c}}(\Omega\times(0,T)) \ \&\ \varphi \geq 0;
\end{aligned}
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$$\iint_{\Omega\times(0,T)} (\log(\theta)+\chi)\varphi_t - \iint_{\Omega\times(0,T)} \rho\mathrm{div}(\mathbf{u}_t)\varphi$$
$$+ \iint_{\Omega\times(0,T)} \left(\mathsf{K}(\theta)|\nabla\log(\theta)|^2\varphi - \mathsf{K}(\theta)\nabla\log(\theta)\nabla\varphi\right)$$
$$+ \iint_{\Omega\times(0,T)} (g + a(\chi)\varepsilon(\mathbf{u}_t)\mathbb{V}\varepsilon(\mathbf{u}_t) + |\chi_t|^2)\frac{\varphi}{\theta}\,\mathrm{d}x\,\mathrm{d}t + \iint_{\partial\Omega\times(0,T)} h\frac{\varphi}{\theta}\,\mathrm{d}S\,\mathrm{d}t$$
$$\leq \left[\int_\Omega \log(\theta)\varphi\right]_0^T \qquad \text{for all } \varphi\in \mathrm{C}_{\mathrm{c}}^\infty(\Omega\times(0,T)) \ \&\ \varphi\geq 0;$$

The **total energy inequality**

$$\mathcal{E}(\theta,\mathbf{u},\mathbf{u}_t,\chi)(T) \leq \mathcal{E}(\theta,\mathbf{u},\mathbf{u}_t,\chi)(0) + \iint_{\Omega\times(0,T)} (\mathbf{f}\cdot\mathbf{u}_t + g) + \iint_{\partial\Omega\times(0,T)} h$$

where

$$\mathcal{E}(\theta,\mathbf{u},\mathbf{u}_t,\chi) = \int_\Omega \left(\theta + \tfrac{1}{2}b(\chi)\varepsilon(\mathbf{u})\mathbb{E}\varepsilon(\mathbf{u}) + \tfrac{1}{2}|\mathbf{u}_t|^2 + \tfrac{1}{p}|\nabla\chi|^p + W(\chi)\right)$$

- cf. **[Feireisl 2007]** (heat conduction in fluids)
- cf. **[Feireisl-Petzeltovà-R. 2009]** (full Frémond's model of phase transitions)

$$\left(\theta_t + \chi_t\theta + \rho\theta\,\mathrm{div}(\mathbf{u}_t) - \mathrm{div}(\mathsf{K}(\theta)\nabla\theta) = g + a(\chi)\varepsilon(\mathbf{u}_t)\mathbb{V}\varepsilon(\mathbf{u}_t) + |\chi_t|^2\right) \times \frac{\varphi}{\theta}$$

with $\varphi \in \mathrm{C}_c^\infty(\Omega \times (0,T))$

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Integrate by parts in **space**

$$\iint_{\Omega\times(0,T)} (\partial_t \log(\theta) + \chi_t + \rho\mathrm{div}(\mathbf{u}_t))\,\varphi$$
$$+ \iint_{\Omega\times(0,T)} \left(\mathsf{K}(\theta)\nabla\log(\theta)\nabla\varphi - \mathsf{K}(\theta)\frac{\varphi}{\theta}\nabla\log(\theta)\nabla\theta\right)$$
$$= \iint_{\Omega\times(0,T)} (g + a(\chi)\varepsilon(\mathbf{u}_t)\mathbb{V}\varepsilon(\mathbf{u}_t) + |\chi_t|^2)\frac{\varphi}{\theta}\,\mathrm{d}x\,\mathrm{d}t + \iint_{\partial\Omega\times(0,T)} h\frac{\varphi}{\theta}\,\mathrm{d}S\,\mathrm{d}t$$

- cf. **[Feireisl 2007]** (heat conduction in fluids)
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with $\varphi \in \mathrm{C}^\infty_\mathrm{c}(\Omega \times (0,T))$

Integrate by parts in **time**

$$\begin{aligned}
&\iint_{\Omega\times(0,T)} (\log(\theta) + \chi)\varphi_t - \iint_{\Omega\times(0,T)} \rho\mathrm{div}(\mathbf{u}_t)\varphi \\
&\quad + \iint_{\Omega\times(0,T)} \Big(-\mathsf{K}(\theta)\frac{\varphi}{\theta}\nabla\log(\theta)\nabla\theta - \mathsf{K}(\theta)\nabla\log(\theta)\nabla\varphi\Big) \\
&\quad + \iint_{\Omega\times(0,T)} (g + a(\chi)\varepsilon(\mathbf{u}_t)\mathbb{V}\varepsilon(\mathbf{u}_t) + |\chi_t|^2)\frac{\varphi}{\theta}\,\mathrm{d}x\,\mathrm{d}t + \iint_{\partial\Omega\times(0,T)} h\frac{\varphi}{\theta}\,\mathrm{d}S\,\mathrm{d}t \\
&= \Big[\int_\Omega \log(\theta)\varphi\Big]_0^T
\end{aligned}$$

- cf. **[Feireisl 2007]** (heat conduction in fluids)
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with $\varphi \in \mathrm{C}^\infty_{\mathrm{c}}(\Omega \times (0,T))$, $\boxed{\varphi \geq 0}$

.. just as an **IN**equality

$$\iint_{\Omega\times(0,T)} (\log(\theta)+\chi)\varphi_t - \iint_{\Omega\times(0,T)} \rho\mathrm{div}(\mathbf{u}_t)\varphi$$
$$+ \iint_{\Omega\times(0,T)} \Big(\mathsf{K}(\theta)|\nabla\log(\theta)|^2\varphi - \mathsf{K}(\theta)\nabla\log(\theta)\nabla\varphi\Big)$$
$$+ \iint_{\Omega\times(0,T)} (g + a(\chi)\varepsilon(\mathbf{u}_t)\mathbb{V}\varepsilon(\mathbf{u}_t) + |\chi_t|^2)\frac{\varphi}{\theta}\,\mathrm{d}x\,\mathrm{d}t + \iint_{\partial\Omega\times(0,T)} h\frac{\varphi}{\theta}\,\mathrm{d}S\,\mathrm{d}t$$
$$\leq \left[\int_\Omega \log(\theta)\varphi\right]_0^T$$

The total energy inequality

$$\Big(\mathbf{u}_{tt} - \operatorname{div}(a(\chi)\mathbb{V}\varepsilon(\mathbf{u}_t) + b(\chi)\mathbb{E}\varepsilon(\mathbf{u}) - \rho\theta\mathbf{1}) = \mathbf{f}\Big) \qquad \times\ \mathbf{u}_t +$$

$$\Big(\theta_t + \chi_t\theta + \rho\theta\operatorname{div}(\mathbf{u}_t) - \operatorname{div}(\mathsf{K}(\theta)\nabla\theta) = g + a(\chi)\varepsilon(\mathbf{u}_t)\mathbb{V}\varepsilon(\mathbf{u}_t) + |\chi_t|^2\Big) \qquad \times\ 1 +$$

$$\Big(\chi_t + \mu\partial I_{(-\infty,0]}(\chi_t) + A_p\chi + W'(\chi) \ni -\frac{1}{2}b'(\chi)\varepsilon(\mathbf{u})\mathbb{E}\varepsilon(\mathbf{u}) + \theta\Big) \qquad \times\ \chi_t$$

$\rightsquigarrow$ **total energy** balance

$$\mathcal{E}(\theta,\mathbf{u},\mathbf{u}_t,\chi)(T) = \mathcal{E}(\theta,\mathbf{u},\mathbf{u}_t,\chi)(0) + \iint_{\Omega\times(0,T)} (\mathbf{f}\cdot\mathbf{u}_t + g) + \iint_{\partial\Omega\times(0,T)} h$$

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$\rightsquigarrow$ **total energy INequality**

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$$\chi_t + \mu\partial I_{(-\infty,0]}(\chi_t) + A_p\chi + W'(\chi) \ni -\frac{1}{2}b'(\chi)\varepsilon(\mathbf{u})\mathbb{E}\varepsilon(\mathbf{u}) + \theta \qquad \text{in } \Omega\times(0,T), \quad (\mathrm{eq}_\chi)$$

$$\chi_t \qquad + A_p\chi + W'(\chi) \ni -\frac{1}{2}b'(\chi)\varepsilon(\mathbf{u})\mathbb{E}\varepsilon(\mathbf{u}) + \theta \qquad \textbf{a.e. in } \Omega \times (0,T), \tag{eq$_\chi$}$$

- If $\mu = 0$ (no unidirectionality, e.g. phase transitions) $\rightsquigarrow$ (eq$_\chi$) can be formulated **pointwise**

$$\chi_t + \partial I_{(-\infty,0]}(\chi_t) + A_p\chi + W'(\chi) \ni -\frac{1}{2}b'(\chi)\varepsilon(\mathbf{u})\mathbb{E}\varepsilon(\mathbf{u}) + \theta \qquad \text{in } \Omega \times (0,T), \quad (\mathrm{eq}_\chi)$$

- If $\mu = 0$ (no unidirectionality, e.g. phase transitions) $\rightsquigarrow$ (eq_χ) can be formulated **pointwise**

- If $\mu = 1$ **(unidirectional case, e.g. damage case)** for (eq_χ) to make sense **a.e. in $\Omega \times (0,T)$**, we need to estimate **separately** $A_p\chi$ and $\beta(\chi)$
  - Could be done by testing (eq_χ) by $\partial_t(A_p\chi + \beta(\chi))$ (cf. **[Bonfanti-Frémond-Luterotti 2000]**)
  - Would involve an integration by parts in time of

$$\iint \theta \partial_t (A_p\chi + \beta(\chi))$$

NOT doable, because of the low time-regularity of θ.

$$\chi_t + \partial I_{(-\infty,0]}(\chi_t) + A_p\chi + W'(\chi) \ni -\frac{1}{2}b'(\chi)\varepsilon(\mathbf{u})\mathbb{E}\varepsilon(\mathbf{u}) + \theta \qquad \text{in } \Omega \times (0,T)$$

$$\Updownarrow$$

$$-\left(\chi_t + A_p\chi + W'(\chi) + \frac{1}{2}b'(\chi)\varepsilon(\mathbf{u})\mathbb{E}\varepsilon(\mathbf{u}) - \theta\right) \in \partial I_{(-\infty,0]}(\chi_t)$$

$$\Updownarrow$$

$$\begin{cases} I_{(-\infty,0]}(v) \geq -\left(\chi_t + A_p\chi + W'(\chi) + \frac{1}{2}b'(\chi)\varepsilon(\mathbf{u})\mathbb{E}\varepsilon(\mathbf{u}) - \theta\right) v & \text{for all test functs } v \leq 0 \\ I_{(-\infty,0]}(\chi_t) \leq -\left(\chi_t + A_p\chi + W'(\chi) + \frac{1}{2}b'(\chi)\varepsilon(\mathbf{u})\mathbb{E}\varepsilon(\mathbf{u}) - \theta\right) \chi_t \end{cases}$$

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Definition **c.f. [Heinemann-Kraus 2010....]**

We call χ a **weak solution** to (eq_χ) if:

$\chi_t(x,t) \le 0$ for a.a. $(x,t) \in \Omega \times (0,T)$ **(unidirectionality), e.g. in damage** $+$

$$\int_\Omega \Big(\chi_t v + |\nabla\chi|^{p-2}\nabla\chi\cdot\nabla v + W'(\chi)v + \frac{1}{2}b'(\chi)\varepsilon(\mathbf{u})\mathbb{E}\varepsilon(\mathbf{u})v - \theta(t)v\Big)\,\mathrm{d}x \ge 0$$

for all test functs $v \le 0$ $+$

$$\iint_{\Omega\times(s,t)} |\chi_t|^2\,\mathrm{d}x\,\mathrm{d}r + \int_\Omega \left(\frac{1}{p}|\nabla\chi(t)|^p + W(\chi(t))\right)\mathrm{d}x$$

$$\le \int_\Omega \left(\frac{1}{p}|\nabla\chi(s)|^p + W(\chi(s))\right)\mathrm{d}x + \iint_{\Omega\times(s,t)} \chi_t\left(-\frac{1}{2}b'(\chi)\varepsilon(\mathbf{u})\mathbb{E}\varepsilon(\mathbf{u}) + \theta\right)\mathrm{d}x\,\mathrm{d}r$$

for all $0 \le s \le t \le T$.

Hypothesis (I).

The function $\mathsf{K} : [0, +\infty) \to (0, +\infty)$ is continuous and

$c_0(1 + \theta^\kappa) \leq \mathsf{K}(\theta) \leq c_1(1 + \nu\theta^\kappa)$ for $\kappa > 1$

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Hypothesis (II).

$$p > d, \quad a \in \mathrm{C}^1(\mathbb{R}),\ b \in \mathrm{C}^2(\mathbb{R}) \text{ are such that } a(x),\ b(x) \geq c_2 > 0, \text{ for all } x \in \mathbb{R}$$

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Hypothesis (III). The potential is $W = I_{[0,1]} + \widehat{\gamma}$, where $\widehat{\gamma} \in \mathrm{C}^1([0,1]), \quad \gamma = \widehat{\gamma}'$

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Hypothesis (IV).

$$
\begin{aligned}
&\mathbf{f} \in L^2(0,T;L^2(\Omega)),\\
&g \in L^1(0,T;L^1(\Omega)) \cap L^2(0,T;H^1(\Omega)'), \quad h \in L^1(0,T;L^2(\partial\Omega)) \quad g,\, h \geq 0,\\
&\theta_0 \in L^1(\Omega), \quad \inf_\Omega \theta_0 \geq \theta_* > 0\,, \quad \log\theta_0 \in L^1(\Omega),\\
&\mathbf{u}_0 \in H^2_0(\Omega), \quad \mathbf{v}_0 \in L^2(\Omega;\mathbb{R}^d), \quad \chi_0 \in W^{1,p}(\Omega), \quad \widehat{\beta}(\chi_0) \in L^1(\Omega)
\end{aligned}
$$

Under Hyp. (I)–(IV), there exists

$$\theta \in L^\infty(0,T;L^1(\Omega)) \cap L^2(0,T;H^1(\Omega)), \ \log\theta \in \mathrm{BV}([0,T];W^{1,d+\epsilon}(\Omega)^*), \ \theta \geq c > 0$$
$$\mathbf{u} \in H^1(0,T;H_0^2(\Omega)) \cap W^{1,\infty}(0,T;H_0^1(\Omega;\mathbb{R}^d)) \cap H^2(0,T;L^2(\Omega;\mathbb{R}^d))$$
$$\chi \in L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega))$$

solving the Cauchy problem for

- the **entropy inequality**
- the **total energy inequality**
- the weak momentum equation (a.e. in $\Omega \times (0,T)$)
- the **generalized principle of virtual powers** for the phase if $\mu = 1$ or the pointwise equation if $\mu = 0$

.. a trick from **[Feireisl-Petzeltovà-R. 2009]**

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- $\theta \in L^2(0,T;H^1(\Omega))$ derives from ($\alpha \in (0,1)$)

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- The **quadratic terms** go to the left-hand side (they are nonnegative on LHS)!

$$\iint_{\Omega\times(0,t)}\left(g+a(\chi)\varepsilon(\mathbf{u}_t)\mathbb{V}\varepsilon(\mathbf{u}_t)+|\chi_t|^2\right)\theta^{\alpha-1}\,\mathrm{d}x\mathrm{d}s \quad \underbrace{-\iint_{\Omega\times(0,t)}\mathsf{K}(\theta)\nabla\theta\nabla(\theta^{\alpha-1})\,\mathrm{d}x\mathrm{d}s}_{\sim\,\iint_{\Omega\times(0,t)}|\nabla\theta^{(\kappa+\alpha)/2}|^2\,\mathrm{d}x\mathrm{d}s}$$

$$= \underbrace{\int_\Omega\frac{1}{\alpha}\theta(t)^\alpha\,\mathrm{d}x}_{\text{estimate by l.h.s. via Gagliardo-Nirenberg}} \quad -\int_\Omega\frac{1}{\alpha}\theta(0)^\alpha\,\mathrm{d}x+\text{OK terms}$$

- Get $\iint_{\Omega\times(0,t)}|\nabla\theta^{(\kappa+\alpha)/2}|^2\,\mathrm{d}x\mathrm{d}s\le C$, hence $\iint_{\Omega\times(0,t)}|\nabla\theta|^2\,\mathrm{d}x\mathrm{d}s\le C$

- $\log(\theta) \in \mathrm{BV}([0,T]; W^{1,d+\epsilon}(\Omega)^*)$ derives from

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- Hence a comparison estimate for $\partial_t \log(\theta)$ tested with fcts. $v \in W^{1,d+\epsilon}(\Omega)$

$$\partial_t \log(\theta) = -\chi_t - \rho\,\mathrm{div}(\mathbf{u}_t) + \frac{1}{\theta}\left(g + a(\chi)\varepsilon(\mathbf{u}_t)\mathbb{V}\varepsilon(\mathbf{u}_t) + |\chi_t|^2\right)$$

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- $\theta \in L^2(0,T;H^1(\Omega))$ & $\partial_t \log(\theta) \in \mathrm{BV}([0,T]; W^{1,d+\epsilon}(\Omega)^*)$ give **strong compactness** for θ via Aubin-Lions Theorem

- $\mathbf{u} \in H^1(0,T; H_0^2(\Omega;\mathbb{R}^d)) \cap W^{1,\infty}(0,T; H_0^1(\Omega;\mathbb{R}^d))$ derives from

$$\iint \Big(\mathbf{u}_{tt} - \operatorname{div}((a(\chi)\mathbb{V}\varepsilon(\mathbf{u}_t) + b(\chi)\mathbb{E}\varepsilon(\mathbf{u}) - \rho\theta\mathbf{1}) = \mathbf{f}\Big) \times (-\operatorname{div}(\varepsilon(\mathbf{u}_t)))$$

where

in $\iint \operatorname{div}(a(\chi)\mathbb{V}\varepsilon(\mathbf{u}_t))\operatorname{div}(\varepsilon(\mathbf{u}_t))$ we calculate $\operatorname{div}(a(\chi)\mathbb{V}\varepsilon(\mathbf{u}_t))$

$\rightsquigarrow$ need for $\nabla\chi$ bded in $L^p(\Omega)$, $p > d$

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- Limit passage as $p \downarrow 2$ possible, convergence to a weaker notion of sol. (**worst** energy inequalities), **only if** $\mu = 1$

- All the estimates can be made rigorous via time-discretization
- Time-discrete scheme carefully tailored to **nonlinear** estimates of heat equation
  - **fully implicit** $\rightsquigarrow$ essential for strict positivity
  - eqns. tightly coupled $\Rightarrow$ existence via fixed point theorem
  - discrete versions of **total energy inequality** & entropy inequality hold $\longrightarrow$ estimates & passage to the limit $\Rightarrow$ conclusion of existence proof
- Compactness
- Limit passage via lower semicontinuity + maximal monotone operator techniques
- Note that the fact that the inqualities can be proved at a discrete level could be useful for numerics

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  - it can be applied in different context: liquid crystals [Feireisl-R.-Schimperna-Zarnescu], two phase fluids [Eleuteri-R.-Schimperna], etc.

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Advantages and further developments

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  - Temperature-dependent **damage** processes with **phase separation** in thermoviscoelasticity [Heinemann-Kraus-R.-Rossi] 

Many thanks

to all of you for the attention

and

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