Cahn-Hilliard-Navier-Stokes systems with nonlocal interactions

Sergio Frigeri[1]

[1] Dipartimento di Matematica "F. Enriques"
Università degli Studi di Milano
www.mat.unimi.it/users/frigeri

Nona Giornata di Studio
Politecnico di Milano-Università di Pavia
Equazioni Differenziali e Calcolo delle Variazioni
Pavia, 23 maggio 2013

Supported by the FP7-IDEAS-ERC-StG Grant "EntroPhase"

Local model for multi-phase flow: a review

- Model H (Hohenberg and Halperin) Flow of viscous incompressible Newtonian macroscopically immiscible fluids (two phases A, B)
- Phase-field methods postulate the existence of a "diffuse interface" of partial mixing with thickness measured by $\epsilon > 0$ (*diffusive interface model*)
- An order parameter φ (concentration of A-component) and a mixing energy *E* in terms of φ and its spatial gradient are introduced
- State variables

$$\varphi = \text{ order parameter}$$
$$\mathbf{u} = \text{ velocity field}$$

Local Cahn-Hilliard-Navier-Stokes systems

In $\Omega \times (0,\infty)$, $\Omega \subset \mathbb{R}^d$, $d = 2,3$

$$
\begin{aligned}
&\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla\pi = \mu\nabla\varphi + \mathbf{h}\\
&\mathrm{div}(\mathbf{u}) = 0\\
&\varphi_t + \mathbf{u}\cdot\nabla\varphi = \mathrm{div}\,(m(\varphi)\nabla\mu)\\
&\mu = -\epsilon\Delta\varphi + \epsilon^{-1}F'(\varphi)
\end{aligned}
$$

μ chemical potential, first variation of the (total Helmholtz) free energy

$$
E(\varphi) = \int_\Omega \Big(\frac{\epsilon}{2}|\nabla\varphi|^2 + \frac{1}{\epsilon}F(\varphi)\Big)dx
$$

Free energy of a nonuniform system introduced by J.W. Cahn & J.E. Hilliard (1958)

- Rigorous derivation by Gurtin, Polignone and Viñals '96
- $m(\varphi)$ non-constant mobility

Local Cahn-Hilliard-Navier-Stokes systems

- $(\epsilon/2)|\nabla\varphi|^2$ free energy increase due to presence of two components
- F double-well potential: Helmoltz free energy density of A-component
  - Regular
    $$F(s) = (1 - s^2)^2 \qquad \forall s \in \mathbb{R}$$
  - Singular (J.W. Cahn & J.E. Hilliard '58)
    $$F(s) = -\frac{\theta_c}{2}s^2 + \frac{\theta}{2}((1+s)\log(1+s) + (1-s)\log(1-s))$$
    for all $s \in (-1, 1)$, with $0 < \theta < \theta_c$
- Math. results by Starovoitov ('97), Boyer ('99), Abels '09, Abels & Feireisl '08 ($\exists$ weak and strong sols, uniqueness and regularity) and by Abels '09, Gal & Grasselli '09 , Zhao, Wu & Huang '09 (convergence to single equilibria), Abels '09, Gal & Grasselli '09, '10 and '11 (attractors)

Nonlocal model for binary fluid motion

- **Nonlocal free energy** (van der Waals) suggested by Giacomin and Lebowitz ('97 & '98) and rigorously justified as macroscopic limit of microscopic phase segregation models (lattice gas with long range Kac potentials: interaction en. between $x, y \in \mathbb{Z}^d$ is $\gamma^d J(\gamma(x-y))$, $\gamma \to 0$)

$$E(\varphi) = \frac{1}{4}\int_\Omega \int_\Omega J(x-y)(\varphi(x)-\varphi(y))^2 dxdy + \int_\Omega F(\varphi(x))dx$$

  $J : \mathbb{R}^d \to \mathbb{R}$ is an interaction kernel s.t. $J(x) = J(-x)$ (usually nonnegative and radial). E.g. $J(x) = j_3|x|^{-1}$ in 3D, $J(x) = -j_2 \log|x|$ in 2D

- **Nonlocal chemical potential**

$$\mu = a\varphi - J * \varphi + F'(\varphi)$$

$$(J * \varphi)(x) := \int_\Omega J(x-y)\varphi(y)dy, \quad a(x) := \int_\Omega J(x-y)dy$$

Nonlocal Cahn-Hilliard-Navier-Stokes systems

Consider in $\Omega \times (0, \infty)$ ($\Omega \subset \mathbb{R}^d$ bounded, $d = 2, 3$)

$$
\begin{aligned}
&\varphi_t + \mathbf{u} \cdot \nabla \varphi = \operatorname{div}\, (m(\varphi) \nabla \mu) \\
&\mu = a\varphi - J * \varphi + F'(\varphi) \\
&\mathbf{u}_t - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla \pi = \mu \nabla \varphi + \mathbf{h} \\
&\operatorname{div}(\mathbf{u}) = 0
\end{aligned}
$$

subject to

$$
\frac{\partial \mu}{\partial n} = 0 \qquad \mathbf{u} = 0 \qquad \text{on} \quad \partial\Omega \times (0, \infty)
$$

$$
\mathbf{u}(0) = \mathbf{u}_0 \qquad \varphi(0) = \varphi_0 \qquad \text{in} \quad \Omega
$$

- Mass is conserved

$$
\overline{\varphi(t)} := |\Omega|^{-1} \int_\Omega \varphi(x, t) dx = \overline{\varphi}_0
$$

Some literature on nonlocal models

- Cahn-Hilliard equation: Giacomin & Lebowitz '97 and '98; Chen & Fife '00; Gajewski '02; Gajewski & Zacharias '03; Han '04; Bates & Han '05 ; Colli, Krejčí, Rocca & Sprekels '07; Londen & Petzeltová '11
- Navier-Stokes-Korteweg systems (liquid-vapour phase transitions): Rohde '05
- several other contributions on nonlocal Allen-Cahn equations and phase-field systems (notably by Bates et al. and Sprekels et al.)

First mathematical results on nonlocal CHNS

- Existence of dissipative global weak sols in 2D-3D with regular (polynomial growth of arbitrary order) potentials and constant mobility (Colli, F. & Grasselli, J. Math. Anal. Appl. '12)
- Asymptotic behavior of weak sols in 2D (global attractor for the associated generalized semiflow) and in 3D (trajectory attractor) with regular potential and constant mobility (F. & Grasselli, J. Dynam Differential Equations '12)
- Singular potentials: existence of weak sols in 2D-3D with constant mobility and asymptotic behavior, i.e., global attractor in 2D and trajectory attractor in 3D (F. & Grasselli, Dyn. Partial Differ. Equ. '12)

$\exists$ weak sols (regular potential, constant mobility)

- **Assumptions on kernel and external force**

$$J \in W^{1,1}(\mathbb{R}^d) \qquad a(x) = \int_\Omega J(x-y)dy \geq 0$$

$$\mathbf{h} \in L^2_{loc}(\mathbb{R}^+; H^1_{div}(\Omega)') \qquad \mathbb{R}^+ := [0, \infty)$$

- **Notion of weak sol**
  Let $\mathbf{u}_0 \in L^2_{div}(\Omega)^d$, $\varphi_0 \in L^2(\Omega)$ with $F(\varphi_0) \in L^1(\Omega)$ and $0 < T < +\infty$ be given. Then a couple $[\mathbf{u}, \varphi]$ is a weak sol to the nonlocal CHNS system on $[0, T]$ if

$$\mathbf{u} \in L^\infty(0, T; L^2_{div}(\Omega)^d) \cap L^2(0, T; H^1_{div}(\Omega)^d)$$

$$\mathbf{u}_t \in L^{4/d}(0, T; H^1_{div}(\Omega)'),$$

$$\varphi \in L^\infty(0, T; L^4(\Omega)) \cap L^2(0, T; H^1(\Omega))$$

$$\varphi_t \in L^2(0, T; H^1(\Omega)')$$

$$\mu \in L^2(0, T; H^1(\Omega))$$

$\exists$ weak sols (regular potential, constant mobility)

and for all $\psi \in H^1(\Omega)$, for all $\mathbf{v} \in H^1_{div}(\Omega)^d$ and for a.e. $t \in (0, T)$

$$\langle \varphi_t, \psi \rangle + (\nabla \mu, \nabla \psi) = (\mathbf{u}, \varphi \nabla \psi)$$
$$\langle \mathbf{u}_t, \mathbf{v} \rangle + \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) = -(\mathbf{v}, \varphi \nabla \mu) + \langle \mathbf{h}, \mathbf{v} \rangle$$

with

$$\mathbf{u}(0) = \mathbf{u}_0, \qquad \varphi(0) = \varphi_0$$

where

$$\mu = a\varphi - J * \varphi + F'(\varphi)$$

and

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) := \int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{v} \cdot \mathbf{w} \qquad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in H^1_{div}(\Omega)^d$$

$\exists$ weak sols (regular potential, constant mobility)

Theorem (Colli, F. & Grasselli '11)

Assume $\boldsymbol{u}_0 \in L^2_{div}(\Omega)^d$, $\varphi_0 \in L^2(\Omega)$ *with* $F(\varphi_0) \in L^1(\Omega)$. *Then, for every* $T > 0$ $\exists$ *a weak sol* $[\boldsymbol{u}, \varphi]$ *on* $[0, T]$ *which satisfies the energy inequality (identity if* $d = 2$*) for all* $t > 0$

$$\mathcal{E}(\boldsymbol{u}(t), \varphi(t)) + \int_0^t (\nu\|\nabla \boldsymbol{u}(\tau)\|^2 + \|\nabla\mu(\tau)\|^2) d\tau$$
$$\leq \mathcal{E}(\boldsymbol{u}_0, \varphi_0) + \int_0^t \langle \boldsymbol{h}, \boldsymbol{u}(\tau)\rangle d\tau$$

where we have set

$$\mathcal{E}(\boldsymbol{u}(t), \varphi(t)) = \frac{1}{2}\|\boldsymbol{u}(t)\|^2$$
$$+ \frac{1}{4}\int_\Omega\int_\Omega J(x-y)(\varphi(x,t) - \varphi(y,t))^2 dxdy + \int_\Omega F(\varphi(t))$$

Remarks (regular potential, constant mobility)

- All results hold for more general double-well regular potentials F, i.e., for F **with polynomial growth of arbitrary order**
- **Main difficulty**: the nonlocal term implies that φ is not as regular as for the standard (local) CHNS system

$$\varphi \in L^2(H^1) \;\; (\textit{nonlocal}), \;\; \text{instead of } \varphi \in L^\infty(H^1) \;\; (\textit{local})$$

- **Consequence**: regularity results (higher order estimates in 2D and 3D) and uniqueness of weak sols in 2D difficult issues

$\exists$ weak sols (singular potential, constant mobility)

Theorem (F. & Grasselli '12)

Let $\boldsymbol{u}_0 \in L^2_{div}(\Omega)^d$, $\varphi_0 \in L^\infty(\Omega)$ with $F(\varphi_0) \in L^1(\Omega)$. In addition, assume that $|\overline{\varphi_0}| < 1$. Then, for every $T > 0$ $\exists$ a weak sol $[\boldsymbol{u}, \varphi]$ on $[0, T]$ corresponding to $[\boldsymbol{u}_0, \varphi_0]$ s.t. $\overline{\varphi}(t) = \overline{\varphi_0}$ for all $t \in [0, T]$ and

$$\varphi \in L^\infty(Q), \quad |\varphi(x,t)| < 1 \quad a.e\,(x,t) \in Q := \Omega \times (0,T)$$
$$\varphi \in L^\infty(0,T; L^p(\Omega))$$

where $p \leq 6$ if $d = 3$ and $p < \infty$ if $d = 2$. Furthermore, the energy inequality holds and, if $d = 2$, every weak sol satisfies the energy identity

- **Idea of the proof**
  - Approximate problem with regular potential F_ϵ
  - Uniform (w.r.t. ϵ) estimates for the approximate sol $z_\epsilon = [\mathbf{u}_\epsilon, \varphi_\epsilon]$
  - Use $|\overline{\varphi_0}| < 1$ to control the averages $\{\overline{\mu}_\epsilon\}$
  - Pass to the limit $z_\epsilon \to z$
  - Use $F'(s) \to \pm\infty$ as $s \to \pm 1$ to show that $|\varphi| < 1$ in $\Omega \times (0, T)$ and hence that $z = [\mathbf{u}, \varphi]$ is indeed a sol
- **Remarks**
  - All results hold for more general double-well singular potentials satisfying $F'(s) \to \pm\infty$ as $s \to \pm 1$
  - No pure phases are admitted

Asymptotic behavior in 2D

Regular or singular potentials (constant mobility): by relying on the energy identity

$$\frac{d}{dt}\mathcal{E}(z) + \nu\|\nabla \mathbf{u}\|^2 + \|\nabla\mu\|^2 = \langle \mathbf{h}, \mathbf{u}\rangle \qquad \forall t > 0$$

Corollary (Colli, F. & Grasselli '11)

If $\boldsymbol{h} \in L^2_{tb}(\mathbb{R}^+; H^1_{div}(\Omega)')$, *i.e.*

$$\|\boldsymbol{h}\|^2_{L^2_{tb}(\mathbb{R}^+;H^1_{div}(\Omega)')} := \sup_{t\geq 0}\int_t^{t+1} \|\boldsymbol{h}(\tau)\|^2_{H^1_{div}(\Omega)'} d\tau < \infty$$

then every weak sol $z = [\boldsymbol{u}, \varphi]$ *satisfies the dissipative estimate*

$$\mathcal{E}(z(t)) \leq \mathcal{E}(z_0)e^{-kt} + F(\bar{\varphi}_0)|\Omega| + K \qquad \forall t \geq 0$$

with $k, K \geq 0$ *independent of* $z_0 := [\boldsymbol{u}_0, \varphi_0]$

Generalized semiflows (Ball '97)

Definition

Let $(\mathcal{X}, d)$ be metric space, a family of maps $z : [0, +\infty) \to \mathcal{X}$ is a generalized semiflow $\mathcal{G}$ if

- existence: $\forall z_0 \in \mathcal{X}$, $\exists\, z \in \mathcal{G}$ s.t. $z(0) = z_0$
- translates of elements of $\mathcal{G}$ still belong to $\mathcal{G}$
- concatenation property holds
- upper semicontinuity w.r.t. initial data: if $z_j \in \mathcal{G}$ with $z_j(0) \to z_0$, then $\exists$ subsequence z_{j_k} and $z \in \mathcal{G}$ s.t. $z(0) = z_0$ and $z_{j_k}(t) \to z(t)$ for all $t \geq 0$

$$T(t)\Theta = \{z(t) \;:\; z \in \mathcal{G},\; z(0) \in \Theta\}, \quad \forall\, \Theta \subset \mathcal{X}$$

Definition

$\mathcal{A} \subset \mathcal{X}$ is the global attractor for $\mathcal{G}$ if it is compact, fully invariant and attracts all bounded subsets B of $\mathcal{X}$, i.e. dist $(T(t)B, \mathcal{A}) \to 0$

Asymptotic behavior in 2D

Existence of the global attractor (autonomous case)

- For $m_0 \geq 0$ given, introduce the *phase space*

$$\mathcal{X}_{m_0} = L^2_{div}(\Omega)^2 \times \mathcal{Y}_{m_0}$$

where, for regular potential

$$\mathcal{Y}_{m_0} = \{\varphi \in L^2(\Omega) \ : \ F(\varphi) \in L^1(\Omega), \ |\bar{\varphi}| \leq m_0\}$$

and for singular potential

$$\mathcal{Y}_{m_0} := \{\varphi \in L^\infty(\Omega) : |\varphi| < 1, \ \ F(\varphi) \in L^1(\Omega), |\overline{\varphi}| \leq m_0\}$$

- Let $\mathcal{G}_{m_0}$ be the set of all weak sols corresponding to all initial data $z_0 = [\mathbf{u}_0, \varphi_0] \in \mathcal{X}_{m_0}$

Theorem (F. & Grasselli '11, '12)

Let $\boldsymbol{h} \in H^1_{div}(\Omega)'$. Then $\mathcal{G}_{m_0}$ is a generalized semiflow on $\mathcal{X}_{m_0}$ which possesses the global attractor

Remark: true for other sing. pots. F, provided F bdd.on $(-1, 1)$

Asymptotic behavior in 3D

Regular or singular potentials (constant mobility)

- the above dissipative estimate still holds for all weak sols satisfying the energy inequality between s and t, for $t \geq s$ (F. & Grasselli '11)

Trajectory attractor approach (Chepyzhov & Vishik)

- phase space is a space of trajectories $\mathcal{K}^+_{\mathcal{H}_+(\mathbf{h}_0)}$: all weak sols satisfying the energy inequality. On $\mathcal{K}^+_{\mathcal{H}_+(\mathbf{h}_0)}$ translation semigroup $\{T(t)\}$ acts
- the attraction of the trajectory attractor $\mathcal{A}_{\mathcal{H}_+(\mathbf{h}_0)}$ is w.r.t. a suitable weak topology Θ^+_{loc} for the family of bounded (in a suitable norm or metric) subsets of $\mathcal{K}^+_{\mathcal{H}_+(\mathbf{h}_0)}$

Theorem (F. & Grasselli '11, '12)

$\{T(t)\}$ acting on $\mathcal{K}^+_{\mathcal{H}_+(h_0)}$ possesses the uniform (w.r.t. $h \in \mathcal{H}_+(h_0)$) trajectory attractor $\mathcal{A}_{\mathcal{H}_+(h_0)}$. This set is strictly invariant, compact in Θ^+_{loc}. In addition, $\mathcal{K}^+_{\mathcal{H}_+(h_0)}$ is closed in Θ^+_{loc}, and $\mathcal{A}_{\mathcal{H}_+(h_0)} \subset \mathcal{K}^+_{\mathcal{H}_+(h_0)}$

Strong sols in 2D (reg. pot., const. mob.)

Theorem (F., Grasselli & Krejčí '13)

Let $\boldsymbol{h} \in L^2_{loc}(\mathbb{R}^+; L^2_{div}(\Omega)^2)$ *and in addition* $J \in W^{2,1}(\mathbb{R}^2)$*. If*

$$\boldsymbol{u}_0 \in H^1_{div}(\Omega)^2 \qquad \varphi_0 \in H^2(\Omega)$$

then, for every given $T > 0$, $\exists$ *unique strong sol* $z := [\boldsymbol{u}, \varphi]$ *s.t.*

$$\boldsymbol{u} \in L^\infty(0,T; H^1_{div}(\Omega)^2) \cap L^2(0,T; H^2(\Omega)^2)$$
$$\boldsymbol{u}_t \in L^2(0,T; L^2_{div}(\Omega)^2)$$
$$\varphi \in L^\infty(0,T; H^2(\Omega))$$
$$\varphi_t \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))$$

Moreover, a continuous dependence estimate w.r.t. data $\boldsymbol{u}_0, \varphi_0, \boldsymbol{h}$ *in* $L^2_{div}(\Omega)^2 \times H^1(\Omega)' \times L^2_{loc}(\mathbb{R}^+; L^2_{div}(\Omega)^2)$ *holds*

Strong sols in 2D (reg. pot., const. mob.)

An idea of the proof

1) The fact that $\varphi \in L^\infty(\Omega \times (0,T))$ and NS regularity in 2D $\Rightarrow$ regularity for $\mathbf{u}$
2) (Nonlocal CH)$\times \mu_t$ in $L^2(\Omega)$ and use the above regularity to get

$$\|\nabla \mu\|^2 + \int_0^t \|\varphi_t\|^2 d\tau \leq \|\nabla \mu_0\|^2 + C + \int_0^t \alpha(\tau)\|\nabla \mu(\tau)\|^2 d\tau$$

where $\alpha \in L^1(0,T)$ and C depend on $\|\nabla \mathbf{u}_0\|$, $\|\varphi_0\|_{H^2}$, T. Hence

$$\varphi \in L^\infty(0,T;H^1(\Omega)) \qquad \varphi_t \in L^2(0,T;L^2(\Omega))$$

Strong sols in 2D (reg. pot., const. mob.)

3) (Nonlocal CH)$_t \times \mu_t$ in $L^2(\Omega)$ and use regularity at point 1). By means of *some technical arguments* (Gagliardo-Nirenberg in 2D) we deduce

$$\frac{d}{dt}\int_\Omega (a + F''(\varphi))\varphi_t^2 + \frac{1}{4}\|\nabla \mu_t\|^2 \le \beta(t)\|\varphi_t\|^2 + C\|\varphi_t\|^4 + \gamma(t)$$

with $\beta, \gamma \in L^1(0, T)$. Then, use a nonlinear Gronwall lemma

$$\left.\begin{array}{r} w'(t) \le C_1\left(1 + w^2(t)\right) \\ \int_0^T w(\tau)d\tau \le C_2 \end{array}\right\} \Rightarrow w(t) \le C_3 = C_3(w(0), C_1, C_2, T)$$

and the improved regularity at point 2) to get

$$\varphi_t \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$$

4) By comparison in the nonlocal CH we get $\mu \in L^\infty(0, T; H^2(\Omega))$ and finally, using assumption $J \in W^{2,1}(\mathbb{R}^2)$, we get

$$\varphi \in L^\infty(0, T; H^2(\Omega))$$

Strong sols in 2D (reg. pot., const. mob.)

- **Regularization in finite time of weal sols**
  if $z_0 = [\mathbf{u}_0, \varphi_0] \in \mathcal{X}_{m_0}$, then for every $\tau > 0$ $\exists s_\tau \in (0, \tau]$ s.t. $z(s_\tau) \in \mathcal{X}^1_{m_0}$, where

$$\mathcal{X}^1_{m_0} := H^1_{div}(\Omega)^2 \times \mathcal{Y}^1_{m_0} \qquad \mathcal{Y}^1_{m_0} := \{\psi \in H^2(\Omega) : |\overline{\psi}| \le m_0\}$$

  Starting from s_τ the weak sol corresponding to z_0 becomes a (unique) strong sol $z \in C([s_\tau, \infty); \mathcal{X}^1_{m_0})$.
  The regularization is also uniform w.r.t. bdd in $\mathcal{X}_{m_0}$ sets of initial data. Indeed

Theorem (F., Grasselli & Krejčí '13)

$\exists \Lambda(m_0) > 0$ s.t. for every $z_0 \in H^1_{div}(\Omega)^2 \times H^2(\Omega)$ with $|\overline{\varphi}_0| \le m_0$ $\exists t^* = t^*(\mathcal{E}(z_0))$ s.t. the strong sol corresponding to z_0 satisfies

$$\|\nabla \boldsymbol{u}(t)\| + \|\varphi(t)\|_{H^2(\Omega)} + \int_t^{t+1} \|\boldsymbol{u}(s)\|_{H^2(\Omega)^2} \le \Lambda(m_0) \qquad \forall t \ge t^*$$

Strong sols in 2D (reg. pot., const. mob.)

Take $z_0 \in \mathcal{B}$ bdd subset of $\mathcal{X}_{m_0}$ and $\tau = 1$. Then $\exists t^* = t^*(\mathcal{B})$ s.t. $z(t) \in B_{\mathcal{X}^1_{m_0}}(0, \Lambda(m_0))$ for all $t \geq t^*$

$\Rightarrow$ **regularity of the global attractor**

$$\mathcal{A}_{m_0} \subset B_{\mathcal{X}^1_{m_0}}(0, \Lambda(m_0))$$

- **Convergence to equilibria of weak sols**

Theorem (F., Grasselli & Krejčí '13)

Take $z_0 \in \mathcal{X}_{m_0}$ and let $z \in C(\mathbb{R}^+; \mathcal{X}_{m_0})$ be a corresponding weak sol. Then

$$\emptyset \neq \omega(z) \subset \mathcal{E}_{m_0}$$

and $\exists t^* = t^*(z_0)$ s.t. the trajectory $\cup_{t \geq t^*} \{z(t)\}$ is precompact in $\mathcal{X}_{m_0}$. Moreover $\exists z_\infty \in \mathcal{E}_{m_0}$ s.t.

$$z(t) \to z_\infty \qquad \text{in } \mathcal{X}_{m_0} \qquad \text{as } t \to \infty$$

Strong sols in 2D (reg. pot., const. mob.)

Set of sationary sols

$$\mathcal{E}_{m_0} := \Big\{ z_\infty = [\mathbf{0}, \varphi_\infty] : \ \varphi_\infty \in L^2(\Omega), \quad F(\varphi_\infty) \in L^1(\Omega), \quad |\overline{\varphi}_\infty| \le m_0,$$
$$a\varphi_\infty - J * \varphi_\infty + F'(\varphi_\infty) = \mu_\infty, \ \ \mu_\infty = \overline{F'(\varphi_\infty)} \quad \text{a.e. in } \Omega \Big\}$$

- The result holds also for more general *analytic* potentials with polynomial growth of arbitrary order
- Main tool: generalized Łojasiewicz-Simon inequality: let $[\varphi_\infty, \mu_\infty] \in U \times \{\text{const}\}$ satisfy $DE(\varphi_\infty) = \mu_\infty$, where U is a neighbourhood of zero in $L^\infty(\Omega)$ and

$$E(\varphi) := \frac{1}{2}\|\sqrt{a}\varphi\|^2 - \frac{1}{2}(\varphi, J * \varphi) + \int_\Omega F(\varphi)$$

Then, $\exists \sigma, \lambda > 0$, $\theta \in (0, 1/2]$ s.t.

$$|E(\varphi) - E(\varphi_\infty)|^{1-\theta} \le \lambda \inf\{\|DE(\varphi) - \mu\|, \ \mu = \text{ const }\}$$

for all $\varphi \in U$ s.t. $\overline{\varphi} = \overline{\varphi}_\infty$ and $\|\varphi - \varphi_\infty\| < \sigma$

Non degenerate mobility, regular potential

Assumption: $m \in C^{0,1}_{loc}(\mathbb{R})$ and $\exists m_1, m_2 > 0$ s.t.

$$m_1 \leq m(s) \leq m_2 \qquad \forall s \in \mathbb{R}$$

Weak formulation: $[\mathbf{u}, \varphi]$ weak sol if $\mathbf{u}$, φ have the same regularity properties as for the const. mobility nonlocal CHNS

$$\langle \varphi_t, \psi \rangle + \big(m(\varphi)\nabla\mu, \nabla\psi\big) = (\mathbf{u}\varphi, \nabla\psi) \qquad \forall \psi \in H^1(\Omega)$$

$$\langle \mathbf{u}_t, \mathbf{v} \rangle + \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) = -(\varphi\nabla\mu, \mathbf{v}) + \langle \mathbf{h}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in H^1_{div}$$

$$\mu = a\varphi - J * \varphi + F'(\varphi) \in L^2(0, T; H^1(\Omega)) \qquad \forall T > 0$$

Theorem (F., Grasselli & Rocca ’13)

Let $\boldsymbol{u}_0 \in L^2_{div}(\Omega)^d$, $\varphi_0 \in L^2(\Omega)$ s.t. $F(\varphi_0) \in L^1(\Omega)$. Then, $\exists$ weak sol $z = [\boldsymbol{u}, \varphi]$ corresponding to $z_0 = [\boldsymbol{u}_0, \varphi_0]$ and satisfying the energy inequality (equality if $d = 2$)

$$\mathcal{E}(z(t)) + \int_0^t \Big(\nu\|\nabla \boldsymbol{u}\|^2 + \|\sqrt{m(\varphi)}\nabla\mu\|^2\Big) d\tau \leq \mathcal{E}(z_0) + \int_0^t \langle \boldsymbol{h}, \boldsymbol{u} \rangle d\tau$$

Degenerate mobility, singular potential

Relevant case: mobility m degenerates at ± 1 and singular double-well potential F on $(-1, 1)$ (e.g. logarithmic like).

- $\varphi-$dependent mobility in the original derivation of CH eq. (J.W. Cahn & J.E. Hilliard, 1971). Thermodynamically reasonable choice

$$m(\varphi) = k(1 - \varphi^2)$$

- Other mobilities and singular potentials can be considered. Key assumption (cf. [Elliot & Garcke '96], [Gajewski & Zacharias '03], [Giacomin & Lebowitz '97,'98])

$$mF'' \in C([-1, 1])$$

Degenerate mobility, singular potential

- Another example

$$m(\varphi) = k(\varphi)(1 - \varphi^2)^n \qquad F(\varphi) = -k_1\varphi^2 + F_1''(\varphi)$$

where $k \in C([-1,1])$ bdd and non degenerate, F_1 convex s.t. $F_1''(\varphi) = l(\varphi)(1 - \varphi^2)^{-n}$, $n \geq 1$, $l \in C^1([-1,1])$

- Math. results on local/nonlocal CH/CHNS with variable mobility
  - local CH eq., non degenerate mobility: Barrett & Blowey '99 ($\exists$ and uniqueness in 2D, $\exists$ in 3D), Liu, Qi & Yin '06 (regularity in 2D), Schimperna '07 (global attractor in 3D)
  - local CH eq., degenerate mobility: Elliot & Garcke '96 ($\exists$), Schimperna & Zelik '13 ($\exists$, asymptotic behavior, separation)
  - nonlocal CH eq., degenerate mobility: Giacomin & Lebowitz '97,'98, Gajewski & Zacharias '03 ($\exists$ and uniqueness), Londen & Petzeltová '11, '11 (conv. to eq., separation)
  - local CHNS, degenerate mobility: Boyer '99 ($\exists$), Abels, Depner & Garcke '13 (unmatched densities, $\exists$)

Nonlocal CHNS with degenerate mobility: $\exists$ weak sols

Notion of weak sol: we are not able to control $\nabla\mu$ in some L^p space; hence we reformulate the definition of weak sol in such a way that μ does not appear any more.
Let $\mathbf{u}_0 \in L^2_{div}(\Omega)^d$, $\varphi_0 \in L^2(\Omega)$ with $F(\varphi_0) \in L^1(\Omega)$ and $0 < T < +\infty$ be given. A couple $[\mathbf{u}, \varphi]$ is a weak solution on $[0, T]$ corresponding to $[\mathbf{u}_0, \varphi_0]$ if

- $\mathbf{u}$, φ satisfy

$$
\begin{aligned}
&\mathbf{u} \in L^\infty(0, T; L^2_{div}(\Omega)^d) \cap L^2(0, T; H^1_{div}(\Omega)^d),\\
&\mathbf{u}_t \in L^{4/3}(0, T; H^1_{div}(\Omega)'), \qquad \text{if} \quad d = 3,\\
&\mathbf{u}_t \in L^2(0, T; H^1_{div}(\Omega)'), \qquad \text{if} \quad d = 2,\\
&\varphi \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)),\\
&\varphi_t \in L^2(0, T; H^1(\Omega)')
\end{aligned}
$$

and

$$\varphi \in L^\infty(Q_T), \qquad |\varphi(x,t)| \leq 1 \quad \text{a.e. } (x,t) \in Q_T := \Omega \times (0,T)$$

- for every $\psi \in H^1(\Omega)$, every $\mathbf{v} \in H^1_{div}(\Omega)^d$ and for almost any $t \in (0,T)$ we have

$$\langle \varphi_t, \psi \rangle + \int_\Omega m(\varphi) F''(\varphi) \nabla\varphi \cdot \nabla\psi + \int_\Omega m(\varphi) a \nabla\varphi \cdot \nabla\psi$$
$$+ \int_\Omega m(\varphi)(\varphi \nabla a - \nabla J * \varphi) \cdot \nabla\psi = (\mathbf{u}\varphi, \nabla\psi)$$
$$\langle \mathbf{u}_t, \mathbf{v} \rangle + \nu(\nabla\mathbf{u}, \nabla\mathbf{v}) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) = \big((a\varphi - J * \varphi)\nabla\varphi, \mathbf{v}\big) + \langle \mathbf{h}, \mathbf{v} \rangle$$
$$\mathbf{u}(0) = \mathbf{u}_0, \qquad \varphi(0) = \varphi_0$$

Theorem (F., Grasselli & Rocca '13)

Let $M \in C^2(-1,1)$ *s.t.* $m(s)M''(s) = 1$, $M(0) = M'(0) = 0$. *Let*

$$\boldsymbol{u}_0 \in L^2_{div}(\Omega)^d, \quad \varphi_0 \in L^\infty(\Omega) \quad F(\varphi_0) \in L^1(\Omega) \quad M(\varphi_0) \in L^1(\Omega).$$

Then, for every $T > 0$ ∃ *a weak sol* $z := [\boldsymbol{u}, \varphi]$ *on* $[0, T]$ *corresponding to* $[\boldsymbol{u}_0, \varphi_0]$ *s.t.* $\overline{\varphi}(t) = \overline{\varphi}_0$ *for all* $t \in [0, T]$ *and* $\varphi \in L^\infty(0, T; L^p(\Omega))$, *with* $p \leq 6$ *for* $d = 3$ *and* $2 \leq p < \infty$ *for* $d = 2$. *In addition,* z *satisfies the energetic inequality (identity if* $d = 2$*)*

$$\frac{1}{2}(\|\boldsymbol{u}(t)\|^2 + \|\varphi(t)\|^2) + \int_0^t \int_\Omega m(\varphi)F''(\varphi)|\nabla\varphi|^2 + \int_0^t \int_\Omega am(\varphi)|\nabla\varphi|^2$$
$$+ \nu \int_0^t \|\nabla \boldsymbol{u}\|^2 \leq \frac{1}{2}(\|\boldsymbol{u}_0\|^2 + \|\varphi_0\|^2) + \int_0^t \int_\Omega (a\varphi - J * \varphi)\boldsymbol{u} \cdot \nabla\varphi$$
$$+ \int_0^t \int_\Omega m(\varphi)(\nabla J * \varphi - \varphi\nabla a) \cdot \nabla\varphi + \int_0^t \langle \boldsymbol{h}, \boldsymbol{u} \rangle \qquad \forall t > 0$$

Nonlocal CHNS with degenerate mobility: remarks

A comparision with the constant mobility case

- Condition $|\overline{\varphi}_0| < 1$ not required (only less strict condition $|\overline{\varphi}_0| \leq 1$): this is due to the different weak sol formulation w.r.t. the case of constant mobility
- Therefore, if F is bounded (e.g. F is the log pot) and at $t = 0$ the fluid is in a pure phase, i.e. $\varphi_0 = 1$ a.e. in Ω, and furthermore $\mathbf{u}_0 = \mathbf{u}(0)$ is given in $L^2(\Omega)^d_{div}$, then the couple

  $$\mathbf{u} = \mathbf{u}(x,t) \qquad \varphi = \varphi(x,t) = 1 \qquad \text{a.e. in } \Omega \quad \text{a.a. } t,$$

  where $\mathbf{u}$ is a sol of NS with non-slip b.c. explicitly satisfies the weak formulation
- This possibility is excluded in the model with constant mobility, since in such model the chemical potential μ (and hence $F'(\varphi)$) appears explicitly

Nonlocal CHNS with degenerate mobility: remarks

The degenerate vs. the strongly degenerate mobility

- If $m'(1) \neq 0$ and $m'(-1) \neq 0$, then F and M are bdd in $[-1, 1] \Rightarrow$ conditions $F(\varphi_0) \in L^1(\Omega)$ and $M(\varphi_0) \in L^1(\Omega)$ satisfied by every φ_0 s.t. $|\varphi_0| \leq 1$ in $\Omega \Rightarrow$ *existence of pure phases allowed*
- If $m'(1) = m'(-1) = 0$ (*strongly degenerate mobility*), then conditions $F(\varphi_0) \in L^1(\Omega)$ and $M(\varphi_0) \in L^1(\Omega)$ imply that $\{x \in \Omega : \varphi_0(x) = 1\}$ and $\{x \in \Omega : \varphi_0(x) = -1\}$ have both measure zero ($\Rightarrow |\overline{\varphi}_0| < 1$). Furthermore, also $\{x \in \Omega : \varphi(x,t) = 1\}$ and $\{x \in \Omega : \varphi(x,t) = -1\}$ have both measure zero for a.a. $t > 0 \Rightarrow$ *existence of pure phases not allowed* (even on subsets of Ω of positive measure)

More regular chemical potential

Theorem (F., Grasselli & Rocca '13)

Assume that

$$\sup_{x\in\Omega}\int_\Omega |\nabla J(x-y)|^\kappa dy < \infty$$

where $\kappa = 6/5$ *if* $d = 3$ *and* $\kappa > 1$ *if* $d = 2$*. Let* φ_0 *be s.t.*

$$F'(\varphi_0) \in L^2(\Omega)$$

Then, $\exists$ *weak sol* $z = [\boldsymbol{u}, \varphi]$ *that also satisfies*

$$\mu \in L^\infty(0,T;L^2(\Omega)), \qquad \nabla\mu \in L^2(0,T;L^2(\Omega)^d)$$

As a consequence, $z = [\boldsymbol{u}, \varphi]$ *also satisfies the weak formulation and the energy inequality (identity for* $d = 2$*) of the non degenerate mobility case.*

- Let $\mathcal{G}_{m_0}$ be the set of all weak sols corresponding to all initial data $z_0 = [\mathbf{u}_0, \varphi_0] \in \mathcal{X}_{m_0}$, where, for $m_0 \in [0,1]$

$$\mathcal{X}_{m_0} = L^2_{div}(\Omega)^d \times \mathcal{Y}_{m_0}$$

$$\mathcal{Y}_{m_0} := \{\varphi \in L^\infty(\Omega) : |\varphi| \le 1, F(\varphi), M(\varphi) \in L^1(\Omega), |\overline{\varphi}| \le m_0\}$$

The metric on $\mathcal{X}_{m_0}$ is

$$\mathbf{d}(z_1, z_2) = \|\mathbf{u}_1 - \mathbf{u}_2\| + \|\varphi_1 - \varphi_2\| \quad \forall z_i = [\mathbf{u}_i, \varphi_i] \in \mathcal{X}_{m_0}, i = 1, 2$$

Theorem (F., Grasselli & Rocca '13)

Let $\mathbf{h} \in H^1_{div}(\Omega)'$. Then $\mathcal{G}_{m_0}$ is a generalized semiflow on $\mathcal{X}_{m_0}$ which possesses the global attractor

Remark: existence of the global attractor established without the restriction $|\overline{\varphi}| < 1$ on the generalized semiflow. In particular this result does not require the separation property

The convective nonlocal CH with degenerate mobility

Consider in $\Omega \times (0,\infty)$ ($\Omega \subset \mathbb{R}^d$ bounded, $d = 2,3$)

$$\varphi_t + \mathbf{u} \cdot \nabla\varphi = \text{ div } (m(\varphi)\nabla\mu)$$
$$\mu = a\varphi - J * \varphi + F'(\varphi)$$

Theorem (F., Grasselli & Rocca '13)

Let $\mathbf{u} \in L^2_{loc}(\mathbb{R}^+; H^1_{div}(\Omega)^d \cap L^\infty(\Omega)^d)$ be given and let $\varphi_0 \in L^\infty(\Omega)$ s.t. $F(\varphi_0), M(\varphi_0) \in L^1(\Omega)$. Then, $\exists$ weak sol φ s.t. $\overline{\varphi}(t) = \overline{\varphi}_0$. Furthermore $\varphi \in L^\infty(\mathbb{R}^+; L^p(\Omega))$, with $p \le 6$ for $d = 3$ and $2 \le p < \infty$ for $d = 2$. In addition, the following energy identity holds

$$\frac{1}{2}\frac{d}{dt}\|\varphi\|^2 + \int_\Omega m(\varphi)F''(\varphi)|\nabla\varphi|^2 + \int_\Omega am(\varphi)|\nabla\varphi|^2$$
$$+ \int_\Omega m(\varphi)(\varphi\nabla a - \nabla J * \varphi) \cdot \nabla\varphi = 0$$

The convective nonlocal CH with degenerate mobility

Theorem (F., Grasselli & Rocca '13)

The weak sol is unique

Hence, we can define a semiflow $S(t)$ on $\mathcal{Y}_{m_0}$, $m_0 \in [0,1]$, endowed with the metric induced by the L^2-norm and the arguments used in the proofs of the previous results can be adapted. In particular

Theorem (F., Grasselli & Rocca '13)

Assume $\boldsymbol{u} \in L^\infty(\Omega)^d$ is given independent of time. Then, the dynamical system $(\mathcal{Y}_{m_0}, S(t))$ possesses a connected global attractor

Remark: uniqueness of sol and existence of the global attractor for the local CH with degenerate mobility are open issues

Some developments and open issues

In progress

- *uniqueness of the weak sol in 2D* , regular potentials, constant mobility and viscosity; strong-weak uniqueness with variable viscosity in 2D and exp. attractors in 2D (with Grasselli and Gal)
- deep quench limit as $\theta \to 0$: the sol $z_\theta \to$ sol of nonlocal CHNS with degenerate mobility and double-obstacle potential ($\theta = 0$)

$$F(s) = \begin{cases} -(\theta_c/2)s^2 & \text{if } |s| \leq 1 \\ \infty & \text{otherwise} \end{cases}$$

Local vs. nonlocal CH/CHNS: nonlocal CH/CHNS physically more realistic and more satisfactory results then local CH/CHNS $\Rightarrow$ nonlocal CH/CHNS maybe "a better" phenomenological model to describe two-phase fluids??

Open issues

- unmatched densities
- non-isothermal nonlocal CH-NS model
- compressible models